QUASISYMMETRIC PROPERTY FOR CONJUGACIES BETWEEN ANOSOV DIFFEOMORPHISMS OF THE TWO-TORUS

HUYI HU AND YUNPING JIANG

Abstract. We prove that the restrictions of the conjugacy between two Anosov diffeomorphisms of the two-torus to the stable and unstable manifolds are quasisymmetric homeomorphisms.

(This paper is dedicated to Professor Lo Yang for his 70^{th} birthday)

1. Introduction

The study of the quasisymmetric property for a conjugacy between two one-dimensional maps has led to solutions of many important problems in one-dimensional dynamical systems and in complex dynamical systems. We give a partial list of references in this direction [9, 10, 15, 11, 21, 5, 19, 18].

A quasisymmetric homeomorphism could be very singular, that is, it could map a set of positive Lebesgue measure to a set of zero Lebesgue measure or vice versa. Generally speaking, as a conjugacy between certain two one-dimensional dynamical systems, a homeomorphism must be either totally singular or smooth (see, for examples, [9, 12, 13, 14, 15, 16]). However, a quasisymmetric homeomorphism has many important geometric properties. For example, a quasisymmetric homeomorphism of the real line can be extended to the whole complex plan as a quasiconformal homeomorphism (refer to [2]).

We would like to push the study of the quasisymmetric property into higher dimensional dynamical systems but with either one-dimensional stable manifolds or one-dimensional unstable manifolds. In [3], Cawley did a similar study and more emphasized on the geometric structure of the space of Anosov diffeomorphisms of the two-torus parametrized by potentials on stable and unstable manifolds. In this paper, we study the quasisymmetric property of a conjugacy between two Anosov diffeomorphisms of the two-torus when the conjugacy is restricted to stable and unstable manifolds. The main technique we use in this paper is the Markov partition method (see [22]) which has been used in the study of the quasisymmetric property of one-dimensional dynamical systems (see [9]).

2. Notations and the Main Theorem

Let $\mathbb{T}^2$ be the two-torus. Let f be an Anosov diffeomorphism of the two-torus. Suppose f is at least $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. By the definition, f is an Anosov diffeomorphism if there is a invariant splitting of the tangent bundle $T\mathbb{T}^2 = E^s \oplus E^u$, where the subbundle E^s is contracted by f, and the subbundle E^u is expanded by f. That is, by considering the Lebesgue metric $\|\cdot\|$ on $\mathbb{T}^2$, there are two constants $0 < \mu < 1$ and $C_0 > 0$ such that for all $n \geq 0$,

$$\|Df^n v\| \leq C_0 \mu^n \|v\|, \quad \forall v \in E^s,$$

and

$$\|Df^{-n} v\| \leq C_0 \mu^n \|v\|, \quad \forall v \in E^u.$$

The only 2-dimensional smooth manifold that support an Anosov diffeomorphism is the two-torus.

The stable and unstable manifold theorem [6] says that for an Anosov diffeomorphism f, $\mathbb{T}^2$ can be foliated by two transversal $C^{1+\alpha}$ submanifolds W^s and W^u such that $TW^s = E^s$ and $TW^u = E^u$. Here W^s and W^u are called the stable and unstable manifolds for f. For each x in $\mathbb{T}^2$, the stable manifold $W^s(x)$ and the unstable manifold $W^u(x)$ passing x are

$$W^s(x) = \{y \in \mathbb{T}^2 \mid d(f^n(x), f^n(y)) \to 0, \ n \to \infty\}$$

and

$$W^u(x) = \{y \in \mathbb{T}^2 \mid d(f^{-n}(x), f^{-n}(y)) \to 0, \ n \to \infty\}.$$

Each $W^s(x)$ or $W^u(x)$ is a connecting $C^{1+\alpha}$ immersed submanifold.

Suppose f and g are two Anosov diffeomorphisms of the two-torus. We say that f and g are topologically conjugate if there is a homeomorphism h of the two-torus such that

$$h \circ f = g \circ h.$$

Frank [4] and Manning [20] showed that every Anosov diffeomorphism f of the two-torus is topologically conjugate to a linear example; that is, to an automorphism defined by a hyperbolic element A of $GL(2, \mathbb{Z})$ whose determinant has absolute value one. Thus every Anosov diffeomorphism f of the two-torus has a fixed point, which we always take it as 0. It is known that the conjugacy h between any two Anosov diffeomorphisms is Hölder continuous (this will also be a corollary of our main theorem in this paper).

There is another very important geometric concept for a homeomorphism of the real line called *quasisymmetry* in complex analysis. A homeomorphism H of the real line $\mathbb{R}$ is called quasisymmetric if there is a constant $M \geq 1$ such that

$$\frac{1}{M} \leq \Big| \frac{H(x+t)-H(x)}{H(x)-H(x-t)} \Big| \leq M, \quad \forall x \in \mathbb{R}, \ \forall 0 < t \leq 1.$$

Suppose $W_f^s(0)$ and $W_f^u(0)$ and $W_g^s(0)$ and $W_g^u(0)$ are the stable and unstable manifolds for f and g. Since they are all connecting $C^{1+\alpha}$ submanifolds of the two-torus, we have $C^{1+\alpha}$ embeddings $\rho_{s,f}$, $\rho_{u,f}$, $\rho_{s,g}$, and $\rho_{u,g}$ from $\mathbb{R}$ onto $W_f^s(0)$, $W_f^u(0)$, $W_g^s(0)$, and $W_g^u(0)$, respectively. We assume that $\rho_{s,f}$, $\rho_{u,f}$, $\rho_{s,g}$, and $\rho_{u,g}$ preserve the arc-length.

For the conjugacy h from f to g, define

$$H_s = \rho_{s,g}^{-1} \circ h \circ \rho_{s,f} \quad \text{and} \quad H_u = \rho_{u,g}^{-1} \circ h \circ \rho_{u,f}.$$

Then they are two homeomorphisms of the real lines. We say $h|W_f^s(0)$ and $h|W_f^u(0)$ are quasisymmetric if H_s and H_u are quasisymmetric. We will prove is the following:

Theorem 1. *Suppose f and g are two conjugated Anosov diffeomorphisms of the two-torus and h is a conjugacy between f and g, that is, $h \circ f = g \circ h$. Then*

$$h|W_f^s(0) : W_f^s(0) \to W_g^s(0) \quad \textit{and} \quad h|W_f^u(0) : W_f^u(0) \to W_g^u(0)$$

are both quasisymmetric homeomorphisms.

It is known that a quasisymmetric homeomorphism of the real line is Hölder continuous [2]. So the Hölder continuity property of h, which is a known result for a long time, is a corollary of the above theorem.

Corollary 1. *A conjugacy h between Anosov diffeomorphisms f and g of the two-torus is Hölder continuous.*

3. Markov Partitions

Suppose f is an Anosov diffeomorphism of $\mathbb{T}^2$. Then f has a local product structure, that is, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) \leq \delta$, $W_\epsilon^s(x) \cap W_\epsilon^y(y)$ contains exact one point, denoted by $[x, y]$, where $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ are the local stable and unstable manifold at x given by

$$W_\epsilon^s(x) = \{y \in \mathbb{T}^2 \mid d(f^n(x), f^n(y)) \leq \epsilon, \ \forall n \geq 0\},$$

and

$$W_\epsilon^u(x) = \{y \in \mathbb{T}^2 \mid d(f^{-n}(x), f^{-n}(y)) \leq \epsilon, \ \forall n \geq 0\}.$$

A set R whose diameter is less than δ is called a *rectangle* if $x, y \in R$ implies $[x, y] \in R$. A rectangle R is proper if it is the closure of its interior. It is easy to check that if R is a rectangle, so is $f(R)$, and if R and S are rectangles, so is $R \cap S$, provided the diameters of the rectangles involved are all small.

For a rectangle R and a point $x \in R$, we denote $W^s(x, R) = W^s_\epsilon(x) \cap R$ and $W^u(x, R) = W^u_\epsilon(x) \cap R$. Note that if R is connected, then both $W^s(x, R)$ and $W^u(x, R)$ are connected curves.

A Markov partition for f is a set $\mathcal{R} = \{R_1, \cdots, R_n\}$ of proper connected rectangles satisfying:

(1) $\mathbb{T}^2 = \cup_{i=1}^n R_i$;
(2) $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $1 \leq i \neq j \leq n$;
(3) $fW^s(x, R_i) \subset W^s(f(x), R_j)$ if $x \in R_i$ and $f(x) \in R_j$;
(4) $fW^u(x, R_i) \supset W^u(f(x), R_j)$ if $x \in R_i$ and $f(x) \in R_j$.

Sinai proved that any Anosov diffeomorphism has a Markov partition of arbitrarily small diameter [22]. Since we only consider Anosov diffeomorphisms of the two-torus and since every such an Anosov diffeomorphism is topologically conjugate to a linear one, we can construct a canonical Markov partition for every f as follows. Note that diameters of rectangles in this canonical Markov partition may not be small.

Suppose A is a hyperbolic automorphism of $\mathbb{T}^2$ conjugating to f. We first construct a canonical Markov partition for A (refer to [1]). Note that A can be defined by a hyperbolic matrix whose absolute value of the determinant is 1. So the matrix has an eigenvalue whose absolute value is greater than 1 (called the unstable eigenvalue) and an eigenvalue whose absolute value is less than 1 (called the stable eigenvalue).

Suppose E^s and E^u are the stable and unstable eigenspaces of the matrix respectively. Then they are two transversal lines passing through the origin of $\mathbb{R}^2$. Suppose the unit square $[0, 1) \times [0, 1)$ is a copy of $\mathbb{T}^2$ on the plane. Project into the $\mathbb{T}^2$ a segment in E^s through the origin, and a segment in E^u through the origin. Extended these segments until they cut the $\mathbb{T}^2$ into parallelograms. The set of these parallelograms is our canonical Markov partition $\mathcal{R}_A$ for A. The reader may refer to [17, pp. 84-86] for more details and some pictures of a canonical Markov partition. Let h_A be the conjugacy from A to f, that is, $h_A \circ A = f \circ h_A$. Then $\mathcal{R}_f = h_A(\mathcal{R}_A)$ is our canonical Markov partition for f.

4. Nested sequence of partitions on $W^s(0)$ and $W^u(0)$.

For a canonical Markov partition $\mathcal{R} = \mathcal{R}_f = \{R_1, \cdots, R_n\}$ for f, we define

$$\kappa_0^s = \{W^s(x, R_i) \mid x \in W^s(0), 1 \leq i \leq n\}$$

and

$$\kappa_0^u = \{W^u(x, R_i) \mid x \in W^u(0), 1 \leq i \leq n\}.$$

So κ_0^s is a partition of $W^s(0)$ into countably many segments $W^s(x, R_i)$, $x \in W^s(0)$, $R_i \in \mathcal{R}$ and κ_0^u is a partition of $W^u(0)$ into countably many segments $W^u(x, R_i)$, $x \in W^u(0)$, $R_i \in \mathcal{R}$. Then we define $\kappa_n^s = f^n \kappa_0^s$ and $\kappa_n^u = f^{-n} \kappa_0^u$ for any $n \geq 1$. That is, κ_n^s consists of all segments l^s in $W^s(0)$ such that $f^{-n}(l) \in \kappa_0^s$ and κ_n^u consists of all segments l^u in $W^u(0)$ such that $f^n(l) \in \kappa_0^u$. By the condition (3) and (4) we know that each element of κ_n^s or κ_n^u is a union of some elements of κ_{n+1}^s or κ_{n+1}^u respectively.

5. Holonomy map

For any two segments l^s and $\tilde{l}^s$ of κ_0^s in a same rectangle $R \in \mathcal{R}$, a holonomy map $\theta^s(x) : l^s \to \tilde{l}^s$ is defined by sliding along the unstable curves, that is, for any $x \in l^s$, $\theta^s(x) = [z, x]$, the only point contained in the intersection $W^s(z) \cap W^u(x)$, where z is any point in $\tilde{l}^s$. Similarly, for any two segments l^u and $\tilde{l}^u$ of κ_0^u in a same rectangle $R \in \mathcal{R}$, a holonomy map $\theta^u(y) : l^u \to \tilde{l}^u$ is defined by sliding along the stable curves, that is, for any $y \in l^s$, $\theta^u(y) = [y, z]$, the only point contained in the intersection $W^s(y) \cap W^u(z)$, where z is any point in $\tilde{l}^u$. The proof of the following lemma can be founded in [6, 17] (also, refer to [7, Proposition 3.2]), using the facts that both stable and unstable foliations are codimension one.

Lemma 1. *All holonomies are Lipschitz continuous with a uniform Lipschitz constant. More precisely, there is a constant $C_1 > 0$ such that for any two segments l^s and $\tilde{l}^s$ of κ_0^s in a same rectangle $R \in \mathcal{R}$,*

$$d(\theta^s(x), \theta^s(x')) \leq C_1 d(x, x'), \quad \forall\, x,\ x' \in l^s,$$

and for any two segments l^u and $\tilde{l}^u$ of κ_0^s in a same rectangle $R \in \mathcal{R}$,

$$d(\theta^u(y), \theta^u(y')) \leq C_1 d(y, y'), \quad \forall\, y,\ y' \in l^u.$$

This lemma implies the following.

Lemma 2. *There is a constant $C_1 > 1$ such that*

$$\frac{1}{C_1} \leq \frac{|l^s|}{|m^s|}, \frac{|l^u|}{|m^u|} \leq C_1$$

for all $l^s, m^s \in \kappa_0^s$ and $l^u, m^u \in \kappa_0^u$, where $|\cdot|$ means the length of the segment.

Remark 1. *Following the method used in one-dimensional dynamical systems (see* [9]*), Cawley* [3] *studied the quasisymmetric property of holonomies.*

6. Distortions

For an Anosov diffeomorphism, we have that

Lemma 3 (Distortion). *For any $\epsilon > 0$, there is a constant $C_2 = C_2(\epsilon) > 0$ such that for any $x, y \in W^s(0)$ with $d^s(x, y) \leq \epsilon$ and $n > 0$,*

$$\frac{1}{C_2} \leq \frac{\|Df^n(y)|_{E_y^s}\|}{\|Df^n(x)|_{E_x^s}\|} \leq C_2$$

and for any $x, y \in W^u(0)$ with $d^u(x, y) \leq \epsilon$ and $n > 0$,

$$\frac{1}{C_2} \leq \frac{\|Df^{-n}(y)|_{E_y^s}\|}{\|Df^{-n}(x)|_{E_x^s}\|} \leq C_2,$$

where d^s and d^u are the distances along $W^s(0)$ and $W^u(0)$ respectively.

The proof of this lemma is the same as the proof of the naive distortion lemma in one-dimensional dynamical systems (see [9, Chapter 1]) and can be found in many books for hyperbolic dynamical systems, see, for example, [17].

7. Bounded nearby geometry

Definition 1. *The nested sequences of partitions $\kappa^s = \{\kappa_n^s\}$ or $\kappa^u = \{\kappa_n^u\}$ are said to have bounded nearby geometry if there is a constant $C > 0$ such that for any two adjacent segments $l^s, m^s \in \kappa_n^s$ or $l^u, m^u \in \kappa_n^u$, $n \geq 0$,*

$$\frac{1}{C} \leq \frac{|l^s|}{|m^s|} \leq C \quad \text{or} \quad \frac{1}{C} \leq \frac{|l^u|}{|m^u|} \leq C$$

respectively.

Theorem 2. *Suppose f is a $C^{1+\alpha}$ Anosov diffeomorphism for some $0 < \alpha \leq 1$. Then the nested sequences of partitions κ^s and κ^u have the bounded nearby geometry.*

Proof. By Lemma 2, there is a constant $C_2 = C_2(2\epsilon) > 0$ such that

$$\frac{1}{C_1} \leq \frac{|l^s|}{|m^s|}, \frac{|l^u|}{|m^u|} \leq C_1$$

for any two adjacent segments $l^s, m^s \in \kappa_0^s$ or $l^u, m^u \in \kappa_0^u$.

For any $n \geq 1$ and for any two adjacent segments l^s and m^s in κ_n^s or l^u and m^u in κ_n^u, $f^{-n}(l^s)$ and $f^{-n}(m^s)$ or $f^n(l^u)$ and $f^n(m^u)$ are two adjacent segments in κ_0^s or in κ_0^u respectively. Then we apply the distortion lemma, Lemma 3, to get the result. $\square$

Lemma 4. *For any $c > 0$, there exists $k = k(c) > 0$ such that for any $n > 0$, $l^s \in \kappa_n^s$ and $m^s \in \kappa_{n+k}^s$ with $m^s \subset l^s$,*

$$|m^s| \leq c|l^s|,$$

and for any $l^u \in \kappa_n^u$ and $m^u \in \kappa_{n+k}^u$ with $m^u \subset l^u$,

$$|m^u| \leq c|l^u|.$$

Proof. By the above theorem we know that $\{|\tilde{l}^s| \mid \tilde{l}^s \in \kappa_0^s\}$ and $\{|\tilde{l}^u| \mid \tilde{l}^u \in \kappa_0^u\}$ are bounded above and below. Since f is uniformly contracting along the stable direction and f^{-1} is uniformly contracting along the unstable direction, we can take $k > 0$ such that for any $\tilde{l}^s \in \kappa_0^s$ and $\tilde{m}^s \in \kappa_k^s$ with $\tilde{m}^s \subset \tilde{l}^s$,

$$|\tilde{m}^s| \leq cC_2^{-1}|\tilde{l}^s|,$$

and for any $\tilde{l}^u \in \kappa_0^u$ and $\tilde{m}^u \in \kappa_k^u$ with $\tilde{m}^u \subset \tilde{l}^u$,

$$|\tilde{m}^u| \leq cC_2^{-1}|\tilde{l}^u|.$$

Note that if $l^s \in \kappa_n^s$ and $m^s \in \kappa_{n+k}^s$ with $m^s \subset l^s$, then $f^{-n}(l^s) \in \kappa_0^s$ and $f^{-n}(m^s) \in \kappa_k^s$ with $f^{-n}(m^s)^s \subset f^{-n}(l^s)$. Hence, we have

$$|f^{-n}(m^s)| \leq cC_2^{-1}|f^{-n}(l^s)|,$$

and similarly if $l^u \in \kappa_n^u$ and $m^u \in \kappa_{n+k}^u$ with $m^u \subset l^u$, then

$$|f^n(m^u)| \leq cC_2^{-1}|f^n(l^u)|,$$

Now we apply Lemma 3 to get $|m^s| \leq c|l^s|$ and $|m^u| \leq c|l^u|$. □

8. Quasisymmetric property

Proof of Theorem 1. We adapted a technique in [9, 10] to prove from the bounded nearby geometry to the quasisymmetric property.

Suppose $W_f^s(0)$ and $W_f^u(0)$ and $W_g^s(0)$ and $W_g^u(0)$ are the stable and unstable manifolds for f and g at 0. Suppose $\rho_{s,f} : \mathbb{R} \to W_f^s(0)$, $\rho_{u,f} : \mathbb{R} \to W_f^u(0)$, $\rho_{s,g} : \mathbb{R} \to W_g^s(0)$, $\rho_{u,g} : \mathbb{R} \to W_g^u(0)$ are embedding maps preserving arc length.

We prove that $H_u = \rho_{u,g}^{-1} \circ h \circ \rho_{u,f} : \mathbb{R} \to \mathbb{R}$ is a quasisymmetric homeomorphism. The proof that $H_s = \rho_{s,g}^{-1} \circ h \circ \rho_{s,f} : \mathbb{R} \to \mathbb{R}$ is a quasisymmetric homeomorphism is the exactly same just by replacing u by s.

Let $\xi_{n,f} = \rho_{u,f}^{-1}\kappa_{n,f}^u$ and $\xi_{n,g} = \rho_{u,g}^{-1}\kappa_{n,g}^u$ for $n \geq 0$. Then they are two sequences of nested partitions on the real line and $H_u\xi_{n,f} = \xi_{n,g}$.

Let Ω be the set of all endpoints of intervals $I \in \xi_{n,f}$, $n = 0, 1 \cdots, \infty$. It is a dense subset in $\mathbb{R}$.

For $x \in \Omega$. Consider the interval $[x-t, x]$. There is a largest integer $n \geq 0$ such that there is an interval $I = [a, x] \in \xi_{n,f}$ satisfying $[x-t, x] \subseteq I$. Suppose

$J = [b, x] \in \xi_{n+1,f}$. Then $J \subseteq [x-t, x]$. Let $J' = [x, c] \in \xi_{n+1,f}$. From Theorem 2 for f, there is a constant $C_1 > 0$ such that

$$C_1^{-1} \le \frac{|J'|}{|J|} \le C_1.$$

If $|J'| > t$, we have $|J'| \le C_1|J| \le C_1 t$. Take $k = k(C_1^{-1})$ as in Lemma 4, and let $J'_k = [x, c_k] \in \xi_{n+k,f}$. Then $J'_k \subset J'$ and by the lemma we have $|J'_k| \le C_1^{-1}|J'| \le t$. This implies that $J'_k \subseteq [x, x+t]$. So we have

$$\frac{|H(J'_k)|}{|H(I)|} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le \frac{|H(J')|}{|H(J)|},$$

where $H(I) \in \xi_{n,g}$, $H(J), H(J') \in \xi_{n+1,g}$, and $H(J'_k) \in \xi_{n+k+1,g}$. Now from Theorem 2 for g, we have a constant $C > 0$ such that

$$C^{-1} \le \frac{|H(J'_k)|}{|H(I)|} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le \frac{|H(J')|}{|H(J)|} \le C.$$

If $|J'| \le t$, we have $|J'| \ge C_1^{-1} t$. We take the same $k = k(C_1^{-1})$ as above, and let $J'_{-k} = [x, c_{-k}] \in \xi_{n-k,f}$. Then $J'_{-k} \supset J'$ and by Lemma 4 we have $|J'| \le C_1^{-1}|J'_{-k}|$ and therefore $|J'_{-k}| \ge C_1|J'| \ge t$. This implies that $J'_{-k} \supseteq [x, x+t]$. So we have

$$\frac{|H(J')|}{|H(I)|} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le \frac{|H(J'_{-k})|}{|H(J)|},$$

where $H(I) \in \xi_{n,g}$, $H(J), H(J') \in \xi_{n+1,g}$, and $H(J'_{-k}) \in \xi_{n-k+1,g}$. Now from Theorem 2 for g, we have a constant $C > 0$, such that

$$C^{-1} \le \frac{|H(J')|}{|H(I)|} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le \frac{|H(J'_{-k})|}{|H(J)|} \le C.$$

For any $x \in \mathbb{R}$, since Ω is dense in $[0, 1]$, we have a sequence $x_n \in \Omega$ such that $x_n \to x$ as $n \to \infty$. For any $t > 0$, we have that

$$C^{-1} \le \frac{|H(x_n+t) - H(x_n)|}{|H(x_n) - H(x_n-t)|} \le C.$$

Since H is continuous on $\mathbb{R}$, we get that

$$C^{-1} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le C.$$

We proved the theorem. $\square$

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Mathematics Department, Michigan State University, East Lansing, MI 48824, USA
E-mail address: `hu@math.msu.edu`

Department of Mathematics, the Graduate Center and Queens College of CUNY, Flushing, NY 11367
E-mail address: `yunping.jiang@qc.cuny.edu`