# Polynomial loss of memory for maps of the interval with a neutral fixed point

Romain Aimino[*] Huyi Hu [†] Matt Nicol [‡]
Andrew Török [§] Sandro Vaienti [¶]

February 18, 2014

**Abstract**

We give an example of a sequential dynamical system consisting of intermittent-type maps which exhibits loss of memory with a polynomial rate of decay. A uniform bound holds for the upper rate of memory loss. The maps may be chosen in any sequence, and the bound holds for all compositions.

## 0 Introduction

The notion of loss of memory for non-equilibrium dynamical systems was introduced in the 2009 paper by Ott, Stenlund and Young [2]; they wrote:

Let $\rho_0$ denote an initial probability density w.r.t. a reference measure $m$, and suppose its time evolution is given by $\rho_t$. One may ask if these probability distributions retain memories of their past. We will say a system **loses its memory in the statistical sense** if for two initial distributions $\rho_0$ and $\hat{\rho_0}$, $\int |\rho_t - \hat{\rho}_t| dm \to 0$.

In [2] the rate of convergence of the two densities was proved to be exponential for certain sequential dynamical systems composed of one-dimensional

*Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France. e-mail:<aimino@cpt.univ-mrs.fr>.

†Mathematics Department, Michigan State University, East Lansing, MI 48824, USA. e-mail: <hu@math.msu.edu>.

‡Department of Mathematics, University of Houston, Houston Texas, USA. e-mail: <nicol@math.uh.edu>.

§Department of Mathematics, University of Houston, Houston Texas, USA. e-mail: <torok@math.uh.edu>.

¶Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France. e-mail:<vaienti@cpt.univ-mrs.fr>.

piecewise expanding maps. Coupling was the technique used for the proof. The same technique was successively applied to time-dependent Sinai billiards with moving scatterers by Stenlund, Young, and Zhang [5] and it gave again an exponential rate. A different approach, using the Hilbert projective metric, allowed Gupta, Ott and Török [4] to obtain exponential loss of memory for time-dependent multidimensional piecewise expanding maps.

All the previous papers prove an exponential loss of memory in the strong sense, namely

$$\int |\rho_t - \hat{\rho}_t| dm < Ce^{-\alpha t}.$$

In the invertible setting, Stenlund [1] proves loss of memory in the weak-sense for random composition of Anosov diffeomorphisms, namely

$$|\int f \circ \mathcal{T}_n d\mu_1 - \int f \circ \mathcal{T}_n d\mu_2| < Ce^{-\alpha t}$$

where $f$ is a Hölder observable, $\mathcal{T}_n$ denotes the composition of $n$ maps and $\mu_1$ and $\mu_2$ are two probability measures absolutely continuous with respect to the Riemannian volume whose densities are Hölder. It is easy to see that loss of memory in the strong sense implies loss of memory in the weak sense, for densities in the corresponding function spaces and $f \in L^\infty$.

A natural question is: are there examples of time-dependent systems exhibiting loss of memory with a slower rate of decay, say polynomial, especially in the strong sense? We will construct such an example in this paper as a (modified) Pomeau-Manneville map:

$$T_\alpha(x) = \begin{cases} x + \frac{3^\alpha}{2^{1+\alpha}} x^{1+\alpha}, \ 0 \le x \le 2/3 \\ 3x - 2, \ 2/3 \le x \le 1 \end{cases} \qquad 0 < \alpha < 1. \tag{0.1}$$

We use this version of the Pomeau-Manneville intermittent map because the derivative is increasing on [0,1), where it is defined, and this allows us to simplify the exposition. We believe the result remains true for time-dependent systems comprised of the *usual* Pomeau-Manneville maps, for instance the version studied in [3]. We will refer quite often to [3] in this note. As in [3], we will identify the unit interval $[0, 1]$ with the circle $S^1$, in such a way the map becomes continuous.

We will see in a moment how an initial density evolves under composition with maps which are slight perturbations of (0.1). To this purpose we will define the perturbations of the usual Pomeau-Manneville map that we will consider.

The perturbation will be defined by considering maps $T_\beta(x)$ as above with $0 < \beta^* \le \beta \le \alpha^*$. Note that $T_\beta = T_\alpha$ on $2/3 \le x \le 1$. The reference measure

*The strictly positive lower bound $\beta^*$ is necessary to prevent the growth to infinity of the will be Lebesgue ($m$). If $\beta^* \leq \beta_k \leq \alpha$ is chosen, we denote by $P_{\beta_k}$ the *Perron-Frobenius* (PF) transfer operator associated to the map $T_{\beta_k}$.

Let us suppose $\phi$, $\psi$ are two observables in an appropriate (soon to be defined) functional space; then the basic quantity that we have to control is

$$\int |P_{\beta_n} \circ \cdots P_{\beta_1}(\phi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)| dm. \tag{0.2}$$

Our goal is to show that it decays polynomially fast and independently of the sequence $P_{\beta_n} \circ \cdots \circ P_{\beta_1}$: we stress that there is no probability vector to weight the $\beta_k$. Note that, by the results of [12], one cannot have in general a faster than polynomial decay, because that is the (sharp) rate when iterating a single map $T_\beta$, $0 < \beta < 1$.

In order to prove our result, Theorem 1.6, we will follow the strategy used in [3] to get a polynomial upper bound (up to a logarithmic correction) for the correlation decay. We introduced there a perturbation of the transfer operator which was, above all, a technical tool to recover the loss of dilation around the neutral fixed point by replacing the observable with its conditional expectation to a small ball around each point. It turns out that the same technique allows us to control the evolution of two densities under concatenation of maps if we can control the distortion of this sequence of maps. The control of distortion will be, by the way, the major difficulty of this paper.

Note that the convergence of the quantity (0.2) implies the decay of the non-stationary correlations, with respect to $m$:

$$\left| \int \psi(x) \phi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1}(x) dm - \int \psi(x) dm \int \phi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1}(x) dm \right| \leq$$

$$||\phi||_\infty ||P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\mathbf{1}(\int \psi dm))||_1$$

provided $\phi$ is essentially bounded and $\mathbf{1}(\int \psi dm)$ remains in the functional space where the convergence of (0.2) takes place. In particular, this holds for $C^1$ observables, see Theorem 1.6.

Conze and Raugy [7] call the decorrelation described above *decorrelation* for the *sequential dynamical system* $T_{\beta_n} \circ \cdots \circ T_{\beta_1}$. Estimates on the rate of decorrelation (and the function space in which decay occurs) are a key ingredient in the Conze-Raugy theory to establish central limit theorems for the sums $\sum_{k=0}^{n-1} \phi(T_{\beta_k} \circ \cdots \circ T_{\beta_1} x)$, after centering and normalisation. The question could be formulated in this way: does the ratio

$$\frac{\sum_{k=0}^{n-1} [\phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1}(x) - \int \phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1} dm]}{|| \sum_{k=0}^{n-1} \phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1} ||_2}$$

second derivative in (2.7); on the other hand several estimates are true for any $0 < \beta \leq \alpha$ and we will follow that when no confusion arises.

converge in distribution to the normal law $\mathcal{N}(0,1)$?

It would be interesting to establish such a limit theorem for the sequential dynamical system constructed with our intermittent map (0.1). Besides the central limit theorem, other interesting questions could be considered for our sequential dynamical systems, for instance the existence of *concentration inequalities* (see the recent work [9] in the framework of the Conze-Raugy theory), and the existence of *stable laws*, especially for perturbations of maps $T_\alpha$ with $\alpha > 1/2$, which is the range for which the unperturbed map exhibits stable laws [10].

We said above that we did not choose the sequence of maps $T_\beta$ according to some probability distribution. A random dynamical system has been considered in the recent paper [8] for similar perturbations of the usual Pomeau-Manneville map. To establish a correspondence with our work, let us say that those authors perturbed the map $T_\alpha$ by modifying again the slope, but taking this time finitely many values $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq 1$, with a finite discrete law. This random transformation has a unique stationary measure, and the authors consider annealed correlations on the space of Hölder functions. They prove in [8] that such annealed correlations decay polynomially at a rate bounded above by $n^{1-\frac{1}{\alpha_1}}$.

As a final remark, we would like to address the question of proving the loss of memory for intermittent-like maps, but with the sequence given by adding a varying constant to the original map, considered to act on the unit circle (additive noise). This problem seems much harder and a possible strategy would be to consider induction schemes, as it was done recently in [11] to prove stochastic stability in the strong sense.

**NOTATIONS.** We will index the perturbed maps and transfer operators respectively as $T_{\beta_k}$ and $P_{\beta_k}$ with $0 < \beta^* \leq \beta_k \leq \alpha$, the number $\beta^* > 0$ being arbitrary. Since we will be interested in concatenations like $P_{\beta_n} \circ P_{\beta_{n-1}} \circ \cdots \circ P_{\beta_m}$ we will use equivalently the following notations

$$P_{\beta_n} \circ P_{\beta_{n-1}} \circ \cdots \circ P_{\beta_m} = P_n \circ P_{n-1} \circ \cdots \circ P_m.$$

We will see that very often the choice of $\beta_k$ will be not important in the construction of the concatenation; in this case we will adopt the useful notations, where the exponent of the $P$'s is the number of transfer operators in the concatenation:

$$P_{\beta_n} \circ P_{\beta_{n-1}} \circ \cdots \circ P_{\beta_m} := P_m^{n-m+1}$$

$$P_k^n = P_{k+n-1} \circ P_{k+n-2} \circ \cdots \circ P_k$$

In the same way, when we concatenate maps we will use the notations $T_n \circ T_{n-1} \circ \cdots \circ T_m$ instead of $T_{\beta_n} \circ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_m}$. We let $\overline{T}^k$ denote the concatenation of $k$ (possibly) different maps $T_l$, whenever the sequence of this concatenation does not matter.

Finally, for any sequences of numbers $\{a_n\}$ and $\{b_n\}$, we will write $a_n \approx b_n$ if $c_1 b_n \leq a_n \leq c_2 b_n$ for some constants $c_2 \geq c_1 > 0$. The first derivative will be denoted as either $T'$ or $DT$ and the value of $T$ on the point $x$ as either $Tx$ or $T(x)$.

# 1 The cone, the kernel, the decay

Thanks to a general theory by Hu [6], we know that the density $f$ of the absolutely continuous invariant measure of $T_\alpha$ in the neighborhood of 0 satisfies $f(x) \leq$ constant $x^{-\alpha}$, where the value of the constant has an expression in terms of the derivative of $T$ at 0 and of the value of $f$ in the pre-image of 0 different from 0. We will construct a cone which is preserved by the transfer operator of each $T_\beta, 0 < \beta \leq \alpha$, and the density of each $T_\beta$ will be the only fixed point of a suitable subset of that cone.

We define the cone of functions

$$\mathcal{C}_1 := \{f \in C^0(]0,1]);\ f \geq 0;\ f \text{ decreasing};\ X^{\alpha+1} f \text{ increasing}\}$$

where $X(x) = x$ is the identity function.

**Lemma 1.1.** *The cone* $\mathcal{C}_1$ *is* invariant *with respect to the operators* $P_\beta,\ 0 < \beta \leq \alpha < 1$.

*Proof.* Put $T_\beta^{-1}(x) = \{y_1, y_2\}, y_1 < y_2$; put also $\chi_\beta = \frac{3^\beta y_1^\beta}{2^{1+\beta}}$. Then a direct computation shows that

$$X^{\alpha+1} P_\beta f(x) = \frac{f(y_1) y_1^{\alpha+1} (1+\chi_\beta)^{\alpha+1}}{1+(1+\beta)\chi_\beta} + f(y_2) \left(\frac{3y_2 - 2}{y_2}\right)^{\alpha+1} \frac{y_2^{\alpha+1}}{3}.$$

The result now follows since the maps $x \to x^{\alpha+1} f(x)$, $x \to \chi_\beta$, $x \to y_1$, $x \to y_2$ are increasing. The fact that $\alpha \geq \beta$ implies the monotonicity of $\chi \to \frac{(1+\chi)^{\alpha+1}}{1+(1+\beta)\chi}$. $\square$

We now denote $m(f) = \int_0^1 f(x)dx$ and recall that for any $0 < \beta < 1$ we have $m(P_\beta f) = m(f)$.

**Lemma 1.2.** *Given* $0 < \alpha < 1$*, the cone*

$$\mathcal{C}_2 := \{f \in \mathcal{C}_1 \cap L^1_m;\ f(x) \leq a x^{-\alpha}\ m(f)\}$$

*is preserved by all the operators* $P_\beta,\ 0 < \beta \leq \alpha$*, provided* $a$ *is large enough.*

*Proof.* Let us suppose that $\int_0^1 f dx = 1$; then we look for a constant $a$ for which $P_\beta f(x) \leq ax^{-\alpha}$. Using the notations in the proof of the previous Lemma and remembering that $x^{\alpha+1} f(x) \leq f(1) \leq \int_0^1 f dx = 1$, we get

$$P_\beta f(x) = \frac{f(y_1)}{T'_\beta(y_1)} + \frac{f(y_2)}{T'_\beta(y_2)} \leq \frac{a y_1^{-\alpha}}{T'_\beta(y_1)} + \frac{y_2^{-\alpha-1}}{T'_\beta(y_2)} =$$

$$\left\{ \left(\frac{x}{y_1}\right)^\alpha \frac{1}{T'_\beta(y_1)} + \frac{1}{a} \frac{x^\alpha}{y_2^{\alpha+1} T'_\beta(y_2)} \right\} ax^{-\alpha},$$

but

$$\left(\frac{x}{y_1}\right)^\alpha \frac{1}{T'_\beta(y_1)} + \frac{1}{a} \frac{x^\alpha}{y_2^{\alpha+1} T'_\beta(y_2)} \leq \frac{(1+\chi_\beta)^\alpha}{1+(1+\beta)\chi_\beta} + \frac{1}{a}(\frac{3}{2} y_1^{\alpha-\beta} \chi_\beta (1+\chi_\beta)^\alpha \leq$$

$$\frac{(1+\chi_\beta)^\alpha}{1+(1+\beta)\chi_\beta} + \frac{1}{a}(\frac{3}{2})^\alpha \chi_\beta, \ (*)$$

where the last step is justified by the fact that $\beta \leq \alpha$ and $0 \leq \chi_\beta \leq 1/2$. By taking the common denominator one gets

$$(*) \leq \frac{1 + \{\beta + [(\alpha-\beta) + 2^\alpha a^{-1}(\beta+2)]\}\chi_\beta}{1+(1+\beta)\chi_\beta}.$$

We get the desired result if $(\alpha - \beta) + 2^\alpha a^{-1}(\beta + 2) \leq 1$, which is satisfied whenever

$$a \geq \frac{2^\alpha(2+\alpha)}{1-\alpha}.$$

□

**Remark 1.3.** *The preceding two lemmas imply the following properties which will be used later on.*

1. $\forall f \in \mathcal{C}_2, \ \inf_{x\in[0,1]} f(x) = f(1) \geq \min\{a; \ [\frac{\alpha(1+\alpha)}{a^\alpha}]^{\frac{1}{1-\alpha}}\} m(f)$.
2. *For any concatenation* $P_1^m = P_m \circ \cdots \circ P_1$ *we have* $P_1^m \mathbf{1}(x) \geq \min\{a; \ [\frac{\alpha(1+\alpha)}{a^\alpha}]^{\frac{1}{1-\alpha}}\}$.

See the proof of Lemma 2.4 in [3] for the proof of the first item, the second follows immediately from the first.

**Remark 1.4.** *Using the previous Lemmas it is also possible to prove the existence of the density in* $\mathcal{C}_2$ *for the unique a.c.i.m. by using the same argument as in Lemma 2.3 in [3].*

We now take $f \in \mathcal{C}_2$ and define the *averaging operator*:

$$\mathbb{A}_\varepsilon f(x) := \frac{1}{2\varepsilon} \int_{B_\varepsilon(x)} f dm$$

where $B_r(x)$ denotes the ball of radius $r$ centered at the point $x \in S^1$, and define a new *perturbed transfer operator* by

$$\mathbb{P}_{\varepsilon,m} := P_m^{n_\varepsilon} \mathbb{A}_\varepsilon = P_{\beta_{m+n\varepsilon-1}} \circ \cdots \circ P_{\beta_m} \mathbb{A}_\varepsilon$$

where $n_\varepsilon$ will be defined later on. It is very easy to see that

**Lemma 1.5.**

$$||\mathbb{P}_{\varepsilon,m} f - P_m^{n_\varepsilon} f||_1 \leq c||f||_1 \varepsilon^{1-\alpha}$$

*where $c$ is independent of $\beta$.*

*Proof.* By linearity and contraction of the operators $P_\beta$ we bound the left hand side of the quantity in the statement of the lemma by $\int |\mathbb{A}_\varepsilon f - f|dx$ and this quantity gives the prescribed bound as in Lemma 3.1 in [3]. □

It is straightforward to get the following representation for the operator $\mathbb{P}_{\varepsilon,m}$ :

$$\mathbb{P}_{\varepsilon,m} f(x) = \int_0^1 K_{\varepsilon,m}(x,z) f(z) dz$$

where

$$K_{\varepsilon,m}(x,z) := \frac{1}{2\varepsilon} P_m^{n_\varepsilon} \mathbf{1}_{B_\varepsilon(z)}(x).$$

We now observe that standard computations (see for instance Lemma 3.2 in [3]). It allows us to show that the preimages $a_n := T_{\alpha,1}^{-n} 1$ verify $a_n \approx \frac{1}{n^{\frac{1}{\alpha}}}$; here $T_{\alpha,1}^{-1}$ denotes the left pre-image of $T_\alpha^{-1}$, a notation which we will also use later on. Those points are the boundaries of a countable Markov partition and they will play a central role in the following computations; notice that the factors $c_1, c_2$ in the bounds $c_1 \frac{1}{n^{\frac{1}{\alpha}}} \leq a_n \leq c_2 \frac{1}{n^{\frac{1}{\alpha}}}$ depend on $\alpha$ and therefore on $\beta$, but we will only use the $a_n$ associated to the exponent $\alpha$; in particular we will denote by $c_\alpha$ the constant $c_2$ associated to $T_\alpha$; the dependance on $\alpha$, although implicit, will not play any role in the following.

We will prove in the next section the following important fact.

- *Property (**P**).* There exists $\gamma > 0$ such that for all $\varepsilon > 0$, $x, z \in [0,1]$ and for any sequence $\beta_m, \cdots, \beta_{m+n_\varepsilon-1}$, if $n_\varepsilon = [\frac{3c_\alpha}{2\varepsilon^\alpha}]$ then

$$K_{\varepsilon,m}(x,z) \geq \gamma.$$

We now show how the positivity of the kernel implies the main result of this paper.

**Theorem 1.6.** *Suppose $\psi, \phi$ are in $\mathcal{C}_2$ for some a with equal expectation $\int \phi dm = \int \psi dm$. Then for any $0 < \beta^* \le \alpha < 1$ and for any sequence $T_{\beta_1}, \cdots, T_{\beta_n}$, $n > 1$, of maps of Pomeau-Manneville type (0.1) with $\beta^* \le \beta_k \le \alpha$, $k \in [1, n]$, we have*

$$\int |P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\phi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)| dm \le C_\alpha(||\phi||_1 + ||\psi||_1) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}},$$

*where the constant $C_\alpha$ depends only on the map $T_\alpha$, and $|| \cdot ||_1$ denotes the $L^1_m$ norm.*

*A similar rate of decay holds for $C^1$ observables $\phi$ and $\psi$ on $S^1$; in this case the rate of decay has an upper bound given by*

$$C_\alpha \ \mathcal{F}(||\phi||_{C^1} + ||\psi||_{C^1}) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}$$

*where the function $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is affine.*

One can ask what happens if we relax the assumption that all $\beta_n$ must lie in an interval $[\beta_\star, \alpha]$ with $0 < \beta_\star < \alpha < 1$. For instance, if the sequence $\beta_n$ satisfies $\beta_n < 1$ and $\beta_n \to 1$, does the quantity $\|P_1^n \phi - P_1^n \psi\|_1$ go to 0 for all $\phi, \psi$ in $C^1$ with $\int \phi = \int \psi$? Similarly, what can we say when $\beta_n \to 0$? It follows from our main result that the decay rate of $\|P_1^n \phi - P_1^n \psi\|_1$ is superpolynomial, but can we get more precise estimates for particular sequences $\beta_n$, like $\beta_n = n^{-\theta}$ or $\beta_n = e^{-cn^\theta}$, $\theta > 0$? We can also ask whether there is, in the case where $\beta_n \in [\beta_\star, \alpha]$ covered by our result, an elementary proof for the decay of $\|P_1^n \phi - P_1^n \psi\|_1$.

*Proof of Theorem 1.6.* We begin to prove the first part of the theorem for $\mathcal{C}_2$ observables. We let $n_\varepsilon = [\frac{3c_\alpha}{2\varepsilon^\alpha}]$ and write $n = kn_\varepsilon + m$. We add and subtract to the difference in the integral a term composed by the product of the first $m$ usual PF operators and the product of $k$ averaged operator $\mathbb{P}_\varepsilon$, each composed by $n_\varepsilon$ random PF operators; precisely we use the notation introduced above to get:

$$(LM) := \int |P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\phi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)| dm =$$

$$\int |P_1^n(\phi) - \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\phi)$$
$$+ \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\phi)$$
$$- \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\psi)$$
$$+ \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\psi) - P_1^n(\psi)| dm.$$

Thus

$$(LM) \le ||P_1^n(\phi) - \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\phi)||_1 +$$
$$||P_1^n(\psi) - \mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\psi)||_1 +$$
$$||\mathbb{P}_{\varepsilon, m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon, m+1} P_1^m(\phi - \psi)||_1.$$

We now treat the first term $I$ in $\phi$ on the right hand side ( the terms in $\psi$ being equivalent), and we consider the last term $III$ after that. We thus have:

$$I = ||P^{n_\varepsilon}_{m+1+(k-1)n_\varepsilon} \cdots P^{n_\varepsilon}_{m+1} P^m_1(\phi) - \mathbb{P}_{\varepsilon,m+1+(k-1)n_\varepsilon} \circ \cdots \circ \mathbb{P}_{\varepsilon,m+1} P^m_1(\phi)||_1.$$

To simplify the notations we put

$$\begin{cases} \mathbb{R}_1 := \mathbb{P}_{\varepsilon,m+1}, \\ \vdots \\ \mathbb{R}_k := \mathbb{P}_{\varepsilon,m+1+(k-1)n_\varepsilon}, \end{cases}$$

and

$$\begin{cases} \mathbb{Q}_1 := P^{n_\varepsilon}_{m+1}, \\ \vdots \\ \mathbb{Q}_k := P^{n_\varepsilon}_{m+1+(k-1)n_\varepsilon}, \end{cases}$$

which reduce the above inequality to

$$I = ||(\mathbb{Q}_k \cdots \mathbb{Q}_1 - \mathbb{R}_k \cdots \mathbb{R}_1) P^m_1(\phi)||_1.$$

By induction we can easily see that

$$\mathbb{Q}_k \cdots \mathbb{Q}_1 - \mathbb{R}_k \cdots \mathbb{R}_1 = \sum_{j=1}^{k} \prod_{l=0}^{k-j-1} \mathbb{R}_{k-l} (\mathbb{R}_j - \mathbb{Q}_j) \prod_{l=0}^{j-1} \mathbb{Q}_{j-l-1}$$

with $\mathbb{R}_{-1} = \mathbf{1}$ and $\mathbb{Q}_0 = \mathbf{1}$; by setting $\phi_m := P^m_1(\phi)$ and $\tilde{\phi}_m = P^m_1(\phi - \psi)$, we have therefore to bound by the quantity

$$\sum_{j=1}^{k} || \prod_{l=0}^{k-j-1} \mathbb{R}_{k-l} (\mathbb{R}_j - \mathbb{Q}_j) \prod_{l=0}^{j-1} \mathbb{Q}_{j-l-1} \phi_m ||_1.$$

We now observe that $\mathbb{Q}_{j-l-1}\phi_m \in \mathcal{C}_2$; moreover $||\mathbb{R}_m g||_1 \leq ||g||_1 \ \forall g \in \mathcal{C}_2, \ 1 \leq m \leq k$. Then we finally get, by invoking also Lemma 1.5,

$$I \leq ||\mathbb{Q}_k \cdots \mathbb{Q}_1 \phi_m - \mathbb{R}_k \cdots \mathbb{R}_1 \phi_m||_1 \leq$$

$$\sum_{j=1}^{k} c||\phi_m||_1 \varepsilon^{1-\alpha} \leq ck||\phi||_1 \varepsilon^{1-\alpha}.$$

We now look at the third term $III$ which could be written as, by using the simplified notations introduced above: $III = ||\mathbb{R}_k \cdots \mathbb{R}_1 \tilde{\phi}_m||_1$. By using Property (**P**) and by applying the same arguments as in the footnote 6 in [3], one gets

$$||\mathbb{R}_k \cdots \mathbb{R}_1 \tilde{\phi}_m||_1 \leq e^{-\gamma k} ||\phi - \psi||_1.$$

In conclusion we get

$$(LM) \leq ck\varepsilon^{1-\alpha}(||\phi||_1 + ||\psi||_1) + e^{-\gamma k}(||\phi||_1 + ||\psi||_1) \leq$$

$$c\frac{n}{n_\varepsilon}\varepsilon^{1-\alpha} + e^{\gamma}\ e^{-\gamma\frac{n}{n\varepsilon}}(||\phi||_1 + ||\psi||_1) \leq C_\alpha\ (||\phi||_1 + ||\psi||_1)n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}}$$

having chosen $\varepsilon = n^{-\frac{1}{\alpha}}\left(\log n^{(\frac{1}{\alpha}-1)\frac{3^\alpha c_\alpha^\alpha}{\gamma 2^\alpha}}\right)^{\frac{1}{\alpha}}$.

In order to prove the second part of the theorem for $C^1$ observables, we invoke the same argument as at the end of the proof of Theorem 4.1 in [3]. We notice in fact that if $\psi \in C^1$ then we can choose $\lambda, \nu \in \mathbb{R}$ such that $\psi_{\lambda,\nu}(x) = \psi + \lambda x + \nu \in \mathcal{C}_2$, the dependance of the parameters with respect to the $C^1$ norm being affine. For instance $\lambda$ and $\nu$ could be chosen in such a way to verify the following constraints: $0 > \lambda > -||\psi'||_\infty$; $\nu > \max\{\frac{(1+\alpha)||\psi||_\infty + ||\psi'||_\infty - \lambda(2+\alpha)}{1+\alpha}, \frac{1+a}{a-1}||\psi||_\infty - \frac{a\lambda}{2}\}$.

□

## 2 Distortion: proof of Property (P)

The main technical problem is now to check the positivity of the kernel; we will follow closely the strategy of the proof of Proposition 3.3 in [3]. We recall that

$$2\varepsilon\ K_{\varepsilon,m}(x,\ \cdot\ ) = P_m^{n_\varepsilon}\mathbf{1}_J(x)$$

where $J$ is an interval which we will take later on as a ball of radius $\varepsilon$.

By iterating we get (we denote with $T_{l,k}^{-1},\ k = 1, 2$, the two inverse branches of $T_l$):

$$2\varepsilon\ K_{\varepsilon,m} = \sum_{l_{n\varepsilon}} \cdots \sum_{l_1} \frac{\mathbf{1}_J(T_{1,l_1}^{-1}\cdots T_{n_\varepsilon,l_{n\varepsilon}}^{-1}x)}{|T_1'(T_{1,l_1}^{-1}\cdots T_{n_\varepsilon,l_{n\varepsilon}}^{-1}x)T_2'(T_{2,l_2}^{-1}\cdots T_{n_\varepsilon,l_{n\varepsilon}}^{-1}x)\cdots T_{n_\varepsilon}'(T_{n_\varepsilon,l_{n\varepsilon}}^{-1}x)|} =$$

$$\sum_{l_{n\varepsilon}} \cdots \sum_{l_1} \frac{\mathbf{1}_J(x_{n_\varepsilon})}{|T_1'(x_{n_\varepsilon})T_2'(T_1x_{n_\varepsilon})\cdots T_{n_\varepsilon}'(T_{n_\varepsilon-1}\cdots T_1x_{n_\varepsilon})|}$$

where $x_{n_\varepsilon} = T_{1,l_1}^{-1}\cdots T_{n_\varepsilon,l_{n\varepsilon}}^{-1}x$ ranges over all points in the preimage of $x \in T_{n_\varepsilon} \circ \cdots \circ T_1 J$. The quantity on the right hand side is bounded from below by

$$2\varepsilon\ K_{\varepsilon,m} \geq \mathbf{1}_{T_{n\varepsilon}\circ\cdots\circ T_1(J)}(x)\ \inf_{z\in J}\frac{1}{|T_1'(z)T_2'(T_1z)\cdots T_{n_\varepsilon}'(T_{n_\varepsilon-1}\cdots T_1z)|}.$$

We also notice that for $0 \leq x \leq 2/3$, $T_\alpha x \leq T_\beta x$; moreover we observe that, as a function of $\alpha$, the first derivative of $T_\alpha$ is decreasing in some interval near zero. In fact, if we differentiate $T_\alpha'$ w.r.t. $\alpha$ and we impose that such a derivative be negative, we obtain the condition that $\log(3/2)(\alpha+1) + 1 + (\alpha+1)\log x < 0$, which is satisfied if we restrict to values of $x$ for which $x < \frac{2}{3}e^{-\frac{1}{\alpha+1}}$. We put $a_d$ the pre-image of $T_{\alpha,1}^{-d}1$ closest to $\frac{2}{3}e^{-\frac{1}{\alpha+1}}$ on the left.

Let us now take the number $\delta = \max\{a_d, a_{d-1} - a_d\}$. Any interval $J$ of length larger or equal to $\delta$ will cover all of the circle in a few steps or it will cross the point $2/3$. In the latter case, call $J'$ the image of $J$ with $|J'| > |J|$, where $|\cdot|$ denotes the length. We denote by $J'_r, J'_l$ the portion of $J'$ respectively on the right and on the left side of the point $2/3$ respectively. If $|J'_r| > a_{d,2} - 2/3$, where $a_{d,2} := T_{\alpha,1}^{-1}a_d$ (notice that the right branches are the same for all $\beta \leq \alpha$), then in a finite number of steps (uniform in $\beta$), the image of $J'_r$,, and therefore of $J$, will cover all the circle. Otherwise we have to wait again a finite number of steps, still independent of $\beta$, for which the image of $J'_l$ will have a length larger than $1/3$ and therefore its successive image will cover all the circle. We have thus shown that having fixed an interval $J$ of length $\geq \delta$, we can find a uniform $n_0$ (for the choice of the maps $T_\beta, \beta > 0$), for which $\mathbf{1}_{T_{n_0}\circ\cdots\circ T_1 J}(x) = 1, \forall x \in S^1$. Since the inverse of the derivative of all the $T_\beta$ are bounded from below by $1/3$, we could conclude that for any interval of length at least $\delta$, there are constants $n_0$ and $c_0$ such that $(P_{n_0} \circ \cdots \circ P_1\mathbf{1})(x) \geq c_0$ and therefore we have the same for *any* power $n \geq n_0$ thanks to item 2 of Lemma 1.3. We have therefore to control the ratio

$$\inf_{z\in J} \frac{1}{|T'_1(z)T'_2(T_1z)\cdots T'_m(T_{m-1}\cdots T_1z)|}$$

where $m$ is now the time needed for the interval $J$ to became an interval of length $\delta$. We proceed as in the proof of Proposition 3.3 in [3]; we call $I_d = (0, a_d]$ the intermittent region and $H_d$ the complementary set, the hyperbolic region.

**Case $J \subset I_d$.**

We first compute such a distortion estimate when the interval $J$ is in the intermittent region $I_d$. Let us call $\Delta_k := (a_{k+1}, a_{k-1})$ the union of two adjacent elements of the Markov partition associated to $T_\alpha$. We suppose now that $J$ contains at most one $a_k$ for $k > 4$, so that $J \subset \Delta_k$. We will establish a one-to-one correspondence between the $T_\beta$ concatenations of $J$ and the $T_\alpha$ iterates of $\Delta_k$. Since $T_\alpha x \leq T_\beta x$ whenever $x \leq 2/3$, we have, provided we stay in the intermittent region:

$$\begin{cases} T_1J \cap \Delta_{k+1} = \emptyset, \\ T_2 \circ T_1J \cap \Delta_k = \emptyset, \\ \vdots \\ T_l \circ T_{l-1} \circ \cdots \circ T_1J \cap \Delta_{k-l+2} = \emptyset. \end{cases}$$

We now follow the itinerary of $J$ for $m$ times in the intermittent region; notice that if $a, b$ are two points in $J$:

$$\frac{D(T_m \circ \cdots \circ T_1)(a)}{D(T_m \circ \cdots \circ T_1)(b)} \leq$$

$$\exp \sum_{j=0}^{m-1} T''_{m-j}(\xi_{m-j}) |T_{m-j-1} \circ \cdots \circ T_1 a - T_{m-j-1} \circ \cdots \circ T_1 b| \tag{2.3}$$

where $\xi_{m-j} \in T_{m-j-1} \circ \cdots \circ T_1 J \subset T_{m-j-1} \circ \cdots \circ T_1 \Delta_k$. Going to the last iterate and coming back we have (we set for simplicity $|\Delta|_m = |T_{m-1} \circ \cdots \circ T_1 J|$):

$$(2.3) \leq \exp \sum_{j=0}^{m-1} \frac{T''_{m-j}(\xi_{m-j})|\Delta|_m}{D(T_{m-1} \circ \cdots \circ T_{m-j})(\eta_{m,j})} \tag{2.4}$$

where $\eta_{m,j}$ belongs to $(T_{m-1} \circ \cdots \circ T_1 J)$. Now we observe that the set $T_{m-j-1} \circ \cdots \circ T_1 J$, which is the $m-j-1$ *random* concatenation of $J$, is disjoint from the $m-j-1$ *deterministic* iterate of $T_\alpha J$, which is the interval $\Delta_{k-(m-j-1)+2} = T_\alpha^{m-j-1} \Delta_k = (a_{k+(m-j-1)+3}, a_{k+(m-j-1)+1})$. Since the second derivatives and the first derivatives are respectively decreasing and increasing w.r.t. the variable $x \in (0, 2/3)$, and by change of variable $l = k - m - j$, we have

$$(2.3) \leq \exp \sum_{l=k-1}^{k-m} \frac{T''_l(a_{l+2})|\Delta|_m}{DT_{l-1}(a_{l+2}) \cdots DT_1(a_{k-m})}.$$

By monotonicity of the first derivative of $T$ with respect to the parameter $\alpha$, we could substitute all the derivative of $T_\beta$ in the denominator of the previous inequality with $T'_\alpha$ computed in the same points. This plus the useful bound, for this kind of maps: $T'_\alpha(a_{l+1}) \geq \frac{|a_l - a_{l+1}|}{|a_{l+1} - a_{l+2}|}$, give us under iteration

$$T'_\alpha(a_{l+2})T'_\alpha(a_{l+1}) \cdots T'_\alpha(a_{k-m}) \geq c_3 |a_{l+2} - a_{l+3}|^{-1} \tag{2.5}$$

where $c_3 = |a_d - a_{d-1}|$. By substituting into (2.3) we get

$$(2.3) \leq \exp\{\sum_{l=k-1}^{k-m} c_3 \frac{T''_l(a_{l+2})|\Delta|_m}{|a_{l+2} - a_{l+3}|^{-1}}\}.$$

Since $|a_{l+2} - a_{l+3}|^{-1} \approx l^{\frac{1}{\alpha}+1}$ and $T''_\beta(a_l) \approx l^{-\frac{\beta-1}{\alpha}}$ we have that the series above is summable with sums $c_4$, so that

$$\frac{D(T_m \circ \cdots \circ T_1)(a)}{D(T_m \circ \cdots \circ T_1)(b)} \leq \exp\{c_5 |T_{m-1} \circ \cdots \circ T_1 J|\} \tag{2.6}$$

with $c_5 = c_4 c_3$.

**Case** $J \subset H_d$.

We now take $J \subset H_d$ and follow its orbit until it enters the intermittent region. Since we are going to use distortion arguments and the mean value theorem, we should take care of the situation when $J$ or one of its iterates crosses the point $2/3$ where the maps are not anymore differentiable. Let us call $\tilde{J}$ the iterate $T_k \circ \cdots \circ T_1 J$ (possibly with $k = 0$ which reduces to consider simply $J$), which crosses the point $2/3$. If the right portion of $\tilde{J}$, call it $\tilde{J}_r$, has length

$|\tilde{J}_r| > a_{d,2} - 2/3$, then, by the previous argument above, a few more iterates of $\tilde{J}_r$, and therefore of $J$, will cover the entire circle.

The other case, $|\tilde{J}_r| \leq a_{d,2} - 2/3$, will be treated later; actually it splits into two subcases. As we will see, in one of these two cases, which we will call the *easy* one, we could apply the same argument as below, so that we could restrict ourselves to use the mean value theorem until the image of $J$ meets the point $2/3$; suppose it happens for $n_1$ steps. By calling $a, b$ two points in $J$ we have by standard estimates:

$$\frac{D(T_{n_1} \circ \cdots \circ T_1)(a)}{D(T_{n_1} \circ \cdots \circ T_1)(b)} \leq$$

$$\exp \sum_{l=0}^{n_1-1} \frac{\sup_\xi T''_{n_1-l}\xi}{\inf_\xi T'_{n_1-l}\xi} |T_{n_1-l-1} \circ \cdots \circ T_1 a - T_{n_1-l-1} \circ \cdots \circ T_1 b|. \tag{2.7}$$

Since $0 < \beta^* \leq \beta \leq \alpha$, the ratio $\frac{\sup_\xi T''_\beta \xi}{\inf_\xi T'_\beta \xi}$ and the quantity $[T'_\beta(x)]^{-1}$ will be uniformly bounded, in $\beta$ and for $x \in H_d$, respectively by a positive constants $D$ and $0 < r < 1$. This immediately implies that

$$\frac{D(T_{n_1} \circ \cdots \circ T_1)(a)}{D(T_{n_1} \circ \cdots \circ T_1)(b)} \leq \exp\{\, c_2 |T_{n_1-1} \circ \cdots \circ T_1 J|\}$$

where $c_2 = \frac{D}{1-r}$ and finally

$$\inf_{z \in J} \frac{1}{|T'_1(z) \cdots T'_{n_1}(T_{n_1-1} \cdots T_1 z)|} \geq \frac{|J|}{|T_{n_1} \circ \cdots \circ T_1 J|} \, \exp\{-c_2 |T_{n-1} \circ \cdots \circ T_1 J|\}\,. \tag{2.8}$$

We now procced as in the last part of the proof of Proposition 3.3 in [3].

We shall first consider two cases not covered by the previous analysis. The first happens when some iterate of $J$, call it $\tilde{J}$, crosses the point $2/3$ and the initial interval $J$ was in the hyperbolic region. This was treated above. We were left with the situation when the right part of $\tilde{J}$, $\tilde{J}_r$ (we will similarly call $\tilde{J}_l$ the left part), had length smaller that $a_{d,2} - 2/3$. Suppose first that $\tilde{J}_l$ is a larger portion of $\tilde{J}$, for instance the length of $\tilde{J}_l$ is larger than $1/3$ of the length of $\tilde{J}$. Then by loosing just a factor $1/3$ we could continue the iteration by only considering the orbit of $\tilde{J}_l$. This is equivalent to consider the iterates of an interval of length $1/3|J|$ with the right hand point placed at the fixed point 1 and moving in the hyperbolic region: this is the *easy* case anticipated above since it completely fits with the distortion computations in the hyperbolic region. We then consider the case whenever $\tilde{J}_r$ has length larger than $1/3$ of the length of $\tilde{J}$. We first notice that this situation is equivalent to the orbit of an interval of the same length as $\tilde{J}_r$ with the left hand point placed again at the fixed point 0. We now treat this case together with the more general situation of some iterates of $J$, call it again $\tilde{J}$, which falls in the intermittent region and crosses at least two boundary points $a_k$. Notice first that since the first derivative of $T_\alpha(x)$ is a *decreasing* function of $\alpha$ (provided we remain in the region $(0, a_d)$), and an *increasing* function of $x$, whenever $T_\alpha^{k-d}(a_{k+1}, a_k) = (a_{d+1}, a_d)$, then $|T_{\beta_{k-d}} \circ \cdots \circ T_{\beta_1}(a_{k+1}, a_k)| \geq \delta$. We therefore cut $\tilde{J}$ into pieces $\Delta_{k_-}, \cdots, \Delta_{k_+}$, such that each of them contains two boundary points and the union of them is of size larger than $|\tilde{J}|/3$. For these intervals $\Delta_k$, the distortion in the intermittent region described above gives, for any choice of the composed transfer operators:

$$P_{k-d} \circ \cdots \circ P_1 \mathbf{1}_{\Delta_k} \geq \mathbf{1}_{\Delta_{1,\cdots,k-d}} e^{-c_5} |\Delta_k|$$

where $\Delta_{1,\cdots,k-d}$ is the $T_{k-d} \cdots \circ T_1$ image of $\tilde{J}$, of length larger than $\delta$. By taking now $l = n_0 + k_+ - d$ we have

$$P_l \circ \cdots \circ P_1 \mathbf{1}_{\tilde{J}} \geq \sum_{k=k_-}^{k_+} P_{l+d-k} \circ \cdots \circ P_{k-d+1} \circ P_{k-d} \cdots \circ P_1 \mathbf{1}_{\Delta_k} \geq$$

$$\sum_{k=k_-}^{k_+} c_0 e^{-c_5} |\Delta_k| \geq c_0 e^{-c_5} \frac{|\tilde{J}|}{3}.$$

*Putting it together.*

We have now a complete control of the distortion in both the intermittent and the chaotic regions: we call $I$ and $II$ the situations when the random iterates of the interval $J$ stay respectively in the hyperbolic region by spending there a time $n_j, j \geq 1$, and in the intermittent region by spending a time $m_j, j \geq 1$ and covering each time at most one boundary point of the $a_k$. We call $III$ the third situation described above where the iterate of $J$ covers more than one boundary point $a_k$: note that whenever the iterate of $J$ follows in this situation, it will surely grows more than $\delta$ before leaving the intermittent region. We therefore get after $t = n_1 + m_1 + ... + n_p + l$ iterations, where $l = n_0$ if the third case $III$ never occurs and $l = n_0$ if $III$ happens:

$$P_t \circ \cdots \circ P_1 \mathbf{1}_{\tilde{J}} \geq$$

$$P^l_{n_p+m_{p-1}+n_{p-1}+n_{p-2}+m_{p-2}\cdots n_1+m_1+1} \circ P^{n_p}_{m_{p-1}+n_{p-1}+n_{p-2}+m_{p-2}\cdots n_1+m_1+1}$$

$$\circ P^{m_p-1}_{n_{p-1}+n_{p-2}+m_{p-2}\cdots n_1+m_1+1} \circ \cdots \circ P^{n_2}_{n_1+m_1+1} \circ P^{m_1}_{n_1+1} \circ P^{n_1}_1 \mathbf{1}_{\tilde{J}} \geq$$

$$|J| \frac{c_0}{3} \exp\{-c_5 - c_2 |\overline{T}^{n_p+\cdots+m_1+n_1} J| - \cdots - c_5 |\overline{T}^{m_1+n_1} J| - c_2 - |\overline{T}^{n_1} J|\} \geq$$

$$|J| \frac{c_0}{3} \exp\{-(c_5 + c_2)(1 + r^{n_p} + r^{n_p+n_{p-1}} + \cdots + r^{n_p+n_{p-1}+\cdots+n_2})\} \geq$$

$$|J| \frac{c_0}{3} \exp\{\frac{-(c_5 + c_2) r}{1 - r}\} := \gamma |J|.$$

Since the first derivatives of all the $T_\beta$ is strictly increasing on the circle, the supremum over all possible values of $t = n_1 + m_1 + ... + n_p + l$ associated to intervals $J$ of size $2\varepsilon$, will be attained when case $III$ will happen at the beginning with $J$ located around 0, and in this case we should consider one third of the length of such an interval (see above), which means we should consider the iterates of the interval $(0, 2\varepsilon/3)$. This implies $a_{d+t} \leq 2\varepsilon/3$ which in turn provides the value for $n_\varepsilon = n_\varepsilon = [\frac{3c_\alpha}{2\varepsilon^\alpha}]$.

**Acknowledgement**. RA and SV were supported by the ANR- Project *Perturbations* and by the PICS ( Projet International de Coopération Scientifique), *Propriétés statistiques des sysèmes dynamiques detérministes et aléatoires*, with the University of Houston, n. PICS05968. SV thanks the University of Houston for supporting his visits during the preparation of this work and the Newton Institute in Cambridge where this paper was completed during the Workshop *Mathematics for Planet Earth*. MN was supported by NSF grant DMS 1101315. AT was partially supported by the Simons Foundation grant 239583.

# References

[1] M. Stenlund, Non-stationary compositions of Anosov diffeomorphisms, *Nonlinearity* **24** (2011) 2991-3018.

[2] W. Ott, M. Stenlund and L.-S. Young, Memory loss for time-dependent dynamical systems, *Math. Res. Lett.*, **16** (2009), 463-475.

[3] C. Liverani, B. Saussol and S. Vaienti, A probabilistic approach to intermittency, *Ergodic theory and dynamical systems*, **19** (1999), 671-685.

[4] C. Gupta, W. Ott, and A. Török, Memory loss for time-dependent piecewise expanding systems in higher dimension, *Mathematical Research Letters*, **20** (2013), 155-175.

[5] M. Stenlund, L-S. Young and H. Zhang, Dispersing billiards with moving scatterers, *Comm. Math. Phys.*, to appear (2013).

[6] H. Hu, Decay of correlations for piecewise smooth maps with indifferent fixed points, *Ergodic Theory and Dynamical Systems*, **24** (2004), 495–524.

[7] J.-P. Conze and A. Raugi, Limit theorems for sequential expanding dynamical systems on $[0, 1]$, *Ergodic theory and related fields*, **89121**, Contemp. Math., 430, Amer. Math. Soc., Providence, RI, 2007.

[8] W. Bahsoun, Ch. Bose and Y. Duan, Decay of correlation for random intermittent maps, arXiv:1305.6588.

[9] R. Aimino and J. Rousseau, Concentration inequalities for sequential dynamical systems of the unit interval, in preparation.

[10] S. Gouezel, Central limit theorem and stable laws for intermittent maps *Probab. Theory Relat. Fields*, **128**(2004) 82-122.

[11] W. Shen and S. Van Strien, On stochastic stability of expanding circle maps with neutral fixed points, *Dynamical Systems, An International Journal*, **28** (2013).

[12] O. Sarig, Subexponential decay of correlations, *Invent. Math.*, **150** (2002), 629-653.