# Ergodic Theory of Almost Hyperbolic Systems

Huyi Hu
Department of Mathematics, Pennsylvania State University,
University Park, PA 16802, USA,
E-mail:hu@math.psu.edu

(*In memory of Professor Liao Shantao*)

**Abstract**

Almost hyperbolic systems are smooth dynamical systems that are hyperbolic everywhere except on a finite set of points. The systems exhibit some new ergodic behavior, which is quite different from the one of uniformly hyperbolic systems, and which has not yet been found in any other nonuniformly hyperbolic systems. The systems may admit either SRB measures or infinite SRB measures. And even if they have SRB measures, correlation decay may change from exponential to power law. In this paper we introduce some results and describe some open problems in this area.

## 0. Introduction

A smooth dynamical system is almost hyperbolic if it is hyperbolic everywhere except on a finite set of points. In recent years, it is found that if hyperbolicity fails at even one point, the long term behavior of the system changes dramatically. Some almost Anosov systems admit infinite rather than finite Sinai-Ruelle-Bowen measures, which implies that orbits are concentrated on an area arbitrarily close to exceptional points. Some piecewise smooth expanding maps with indifferent fixed points have polynomial instead of exponential decay of correlations. These properties are not enjoyed by uniformly hyperbolic systems, and have not been found in any other nonuniformly hyperbolic systems so far.

In this paper we introduce some known results related to above phenomena, most of which were obtained in the 1990's. Though systems that fail hyperbolicity at a finite set has been studied for long time (see e.g [K], also see e.g [T] for psesdo-Anosov maps), this subject is relatively new and many important problems are open. We describe some of them. Since the systems still have some hyperbolic structure, they should be relatively easy to study, compared to other nonuniformly hyperbolic systems like Hénon maps, and we can expect more general results for these systems. On the other hand, to study almost hyperbolic systems, more delicate estimates are usually involved since hyperbolicity is very weak near the exceptional points.

We are interested in the following systems.

Almost Anosov diffeomorphisms

Almost hyperbolic invariant sets and attractors

Piecewise expanding maps with indifferent fixed points

Invariant sets of expanding maps with indifferent fixed points

There are many other interesting systems that could belong to this category, for example, almost hyperbolic systems with singularities, almost Anosov systems with holes, almost hyperbolic systems in Banach spaces, etc.

Here we give some precise definitions for above systems.

**Definition 0.1** *Let $f$ be a $C^r$, $r > 1$, diffeomorphism from a manifold $M$ to itself. An invariant closed subset $\Lambda \subset M$ is called an almost hyperbolic set, if there exist two continuous families of cones $\mathcal{C}^u_x$ and $\mathcal{C}^s_x$ on the tangent bundle $T_\Lambda M$ such that except at a finite set $S \subset \Lambda$,*

i) *(invariance)* $Df_x\mathcal{C}^u_x \subset \mathcal{C}^u_{fx}$, $Df_x\mathcal{C}^s_x \supset \mathcal{C}^s_{fx}$;

ii) *(hyperbolicity)* $|Df(v)| > |v|$ $\forall v \in \mathcal{C}^u_x$, $|Df(v)| < |v|$ $\forall v \in \mathcal{C}^s_x$.

*If $\Lambda = M$, then we call the system an almost Anosov diffeomorphism.*

One may also assume that $S$ is a submanifold of $M$. However, we are not going to discuss the case here.

Without loss generality, we assume that $S$ is an $f$-invariant set. We assume further that $S$ consists of fixed points. In this paper we always assume that $S$ consists of only one single fixed point $p$ except when otherwise stated.

By the definition, it may happen that for some $v \in \mathcal{C}^u_p$ or $v \in \mathcal{C}^s_p$,

$$|Df_p(v)| = |v|.$$

So $f$ is not uniformly hyperbolic in general. On the other hand, by continuity, we can see that $f$ is uniformly hyperbolic away from $S$.

**Definition 0.2** *A fixed point of $f$ is indifferent if $Df_p$ has an eigenvalue $\lambda$ with $|\lambda| = 1$.*

For a piecewise smooth expanding map $f$ from $D = [0,1]^m$ to itself with an indifferent fixed point $p$, we always assume the following.

1. There exists a finite partition of $D$ into $\{D_j\}$ such that for each $j$, $f|_{\mathrm{int}\, D_j} : \mathrm{int}\, D_j \to \mathrm{int}\, D$ is a $C^r$, $r > 1$, diffeomorphism.

2. $f$ is expanding except at $p$, that is, if $x \in D \backslash \{p\}$, then $|Df_x v| > |v|$ $\forall v \in \mathbb{R}^m$

The first condition implies that $D = \cup D_j$ is a Markov partition. This requirement could be relaxed just like in the study of piecewise smooth uniformly expanding maps.

It is generally believed that an almost hyperbolic system is topologically conjugate to a uniformly hyperbolic system. In particular, this should be easy to prove for almost Anosov systems on a torus by using some results in [AH].

# 1 Invariant Measures

In this section we discuss existence and finiteness of various of interesting invariant measures.

## 1.1 SRB measures and and Infinite SRB Measures

For an Anosov system $f : M \to M$, a result of Sinai (see e.g. [S]) says that $f$ admits a unique invariant Borel probability measure $\mu$ with the property that $\mu$ has absolutely continuous conditional measures on unstable manifolds. This is the invariant measure that is observed physically, for if $\phi : M \to \mathbb{R}$ is a continuous function, then for Lebesgue almost every point $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) \to \int \phi d\mu.$$

These results have been extended to Axiom A attractors by Ruelle, Bowen, etc. (See e.g. [B].) This measure is called an Sinai-Ruelle-Bowen measure or an SRB measure. Due to the work of Oseledec, Pesin, Ledrappier and Young for nonuniformly hyperbolic sets, the notion of SRB measures was extended to a more general setting. (See e.g. [LS] for related references.) Now there are some results about existence of SRB measure outside Axiom A systems such as Hénon maps and hyperbolic systems with singularities. (See [BY2], [P], [JN], etc.)

**Definition 1.1** *An $f$-invariant probability measure $\mu$ is called an SRB measure for $f : M \to M$ if*

i) *$(f, \mu)$ has positive Lyapunov exponents almost everywhere;*

ii) *$\mu$ has absolutely continuous conditional measures on unstable manifolds.*

For the precise meaning of the second condition above, we refer [LS].

**Definition 1.2** *An $f$-invariant measure $\mu$ is called an infinite SRB measure for $f : M \to M$ if $\mu M = \infty$, and for any $R \subset M$ with $\mu(R) < \infty$ and $R = \overline{\operatorname{int} R}$,*

i) *$(f_R, \mu_R)$ has positive Lyapunov exponents almost everywhere, and*

ii) *$\mu_R$ has absolutely continuous conditional measures on unstable manifolds of $f$.*

*where $f_R$ is the first return map of $f$ on $R$, and $\mu_R$ is the conditional measure of $\mu$ on $R$.*

Consider a topologically transitive almost Anosov system on an compact surface. Suppose $f : M^2 \to M^2$ is hyperbolic everywhere except at a fixed point $p$. Assume that $Df_p$ has eigenvalues $\overline{\lambda} \geq \underline{\lambda} > 0$.

**Theorem 1.1 ([HY])** *If $\overline{\lambda} = 1$, $\underline{\lambda} < 1$, then $f$ has an infinite SRB measure.*

Other the other hand, it is easy to obtain the following.

**Fact 1.2** *If $\overline{\lambda} > 1$, $\underline{\lambda} = 1$, then $f$ has an SRB measure.*

**Definition 1.3** *An almost Anosov diffeomorphism $f : M \to M$ is said to be nondegenerate, if there are constants $C^+, C^- > 0$ such that for $x$ near $p$,*

$$|Df_x v| \geq \big(1 + C^+ d(x,p)^2\big)|v| \qquad \forall v \in \mathcal{C}^u_x,$$

$$|Df_x v| \leq \big(1 - C^- d(x,p)^2\big)|v| \qquad \forall v \in \mathcal{C}^s_x.$$

Since $Df_x \to \mathrm{id}$ as $x \to p$ if $\overline{\lambda} = 1 = \underline{\lambda}$, expansion and contraction are weak as $x$ near $p$. The nondegeneracy conditions assure that they are not too weak.

**Theorem 1.3 ([H1])** *If $\overline{\lambda} = 1 = \underline{\lambda}$ and $f$ is nondegenerate, then $f$ has either an SRB measure or an infinite SRB measure.*

Note that if $\overline{\lambda} > 1 > \underline{\lambda}$, then $f$ is uniformly hyperbolic on $M$ and therefore $f : M \to M$ is an Anosov diffeomorphism and has an SRB measure.

**Theorem 1.4** *Every nondegenerate almost Anosov diffeomorphism on a surface has either an SRB measure or an infinite SRB measure.*

Since the stable foliation is absolutely continuous, we can get the following.

**Corollary 1.5** *Let $\phi : M \to \mathbb{R}$ be a continuous function.*

*If $f$ admits an SRB measure $\mu$, then for Leb-a.e. $x \in M$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \int \phi d\mu;$$

*If $f$ admits an infinite SRB measure $\mu$, then for Leb-a.e. $x \in M$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \phi(p).$$

The second part of the corollary is true because the measure of any neighborhood of $p$ is infinite, while the measure of its complement is finite.

From the corollary we can see that if $f$ admits an infinite SRB measure, then the statistic behavior of the system changes dramatically. In a such system, for Lebesgue almost every initial condition, orbit spends one hundred percent of its time arbitrarily near $p$. Hence from statistic point of view, the systems is deterministic. However, almost every orbit is dense in $M$, and the system is still "chaotic" from the topological point view. Another feature of the system is that the forward Lyapunov exponents of $f$ are zero at Lebesgue almost every point, even though $f$ is hyperbolic everywhere except at $p$.

Both cases in Theorem 1.3 do occur. Here we give sufficient conditions for each case. Suppose that $f$ satisfies the conditions in Theorem 1.3. Assume further that $W^u_\epsilon(p)$ and $W^s_\epsilon(p)$, the local unstable and stable manifolds at $p$, are $C^4$ curves. Since $f$ satisfies the nondegeneracy conditions, we can choose a suitable coordinate system such that $f$ can be written as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x\big(ax^2 + rxy + by^2 + O((x^2+y^2)^{3/2})\,\big) \\ y - y\big(cx^2 + sxy + dy^2 + O((x^2+y^2)^{3/2})\,\big) \end{pmatrix}.$$

**Theorem 1.6 ([H1])** *With the above notations,*

- *$f$ has an SRB measure if $b > 2d$, $r = 0 = s$, $ad > bc$;*
- *$f$ has an infinite SRB measure if $2b < d$, $rs \neq 0$.*

Note that $p$ is a saddle. When a point near the stable manifold $W^s_\epsilon(p)$ approaches $p$, the $y$-coordinate is much bigger than the $x$-coordinate of the point, and therefore the terms $by^2$ and $dy^2$ dominate the expanding and contracting rates respectively. The above theorem implies that $f$ has an SRB measure if expansion is stronger, and has an infinite SRB measure if contraction is stronger. The same phenomena can also be seen from Theorem 1.1 and Fact 1.2.

**Question 1.7** *Does every almost Anosov system in a higher dimensional space have either SRB measure or an infinite SRB measure?*

**Question 1.8** *Is a similar statement true for almost hyperbolic attractors?*

**Question 1.9** *Can the nondegeneracy conditions in Theorem 1.4 be relaxed or eliminated?*

The same results should hold if $Df_p - \mathrm{id}$ is of higher order, instead of the second order as in Theorem 1.3. However, no one has tried the problem yet.

**Question 1.10** *If $f$ has two indifferent fixed points $p$ and $q$ and both cause infinite SRB measures, what is the orbit distribution?*

By Theorem 1.14, it seems that in general the time average doesn't exist for Lebesgue almost every initial point.

## 1.2 Absolutely Continuous Invariant Measures

A folklore theorem says that if $f$ is a piecewise smooth uniformly expanding map on the unit interval, then $f$ has an absolutely continuous invariant measure. The results has been extended to more general setting, including piecewise smooth uniformly expanding maps in higher dimensional spaces. (See e.g. [KS], [W], [Yu], etc.)

Absolutely continuous invariant measures have also been found in other type of dynamical systems, for example, the quadratic maps $f_a(x) = ax(1-x)$, $x \in [0,1]$, (see [J], also see [BY1]).

For piecewise smooth expanding maps with indifferent fixed points on the interval, it is known in the early 80's that absolutely continuous invariant measures exists, which may be finite or $\sigma$-finite. (See e.g. [Pi].)

**Theorem 1.11** *Let $f : I \to I$ be a piecewise smooth expanding map with an indifferent fixed point $0$. Assume that*

$$f(x) = x + x^{1+\alpha} + o(x^{1+\alpha}), \qquad x \to 0.$$

*Then $f$ has an absolutely continuous invariant measure $\mu$, and $\mu$ is finite if $0 < \alpha < 1$; $\mu$ is $\sigma$-finite if $\alpha \geq 1$.*

**Question 1.12** *Does the statement remain true if $f$ maps $I^m$ to itself?*

However, if bounded distortion estimates are easy to obtain, then existence of absolutely continuous invariant measures follows from standard arguments. Here is an example.

**Fact 1.13** *Let $f : I^m \to I^m$ be a piecewise smooth expanding map with an indifferent fixed point* $0$. *Assume that*

$$f(x) = x\big(1 + |x|^\alpha + o(|x|^\alpha)\big), \qquad x \to 0.$$

*Then $f$ has an absolutely continuous invariant measure $\mu$, and $\mu$ is finite if $0 < \alpha < m$; $\mu$ is $\sigma$-finite if $\alpha \geq m$.*

If a system has an absolutely continuous invariant measure $\mu$, then a statement similar in Corollary 1.5 can be made. That is, for Lebesgue almost every point, the time average is equal to $\int \phi d\mu$ if $\mu$ is finite, and is equal to $\phi(0)$ if $\mu$ is $\sigma$-finite.

In one dimensional systems, if $f$ has two indifferent fixed points that cause $\sigma$-finite measure, then the following happens.

**Theorem 1.14 ([I])** *Suppose $f$ is a piecewise smooth expanding maps on the interval with indifferent fixed points $p$ and $q$ such that*

$$\begin{aligned} |f(x) - x| &= a|x-p|^\alpha + o(|x-p|^\alpha), \qquad & x \to p, \\ |f(x) - x| &= b|x-q|^\beta + o(|x-q|^\beta), \qquad & x \to q. \end{aligned}$$

*Then for any small neighborhoods $U$ and $V$ of $p$ and $q$ respectively, Leb-a.e. $x$,*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n 1_U\big(f^i(x)\big)}{\sum_{i=0}^n 1_V\big(f^i(x)\big)} = \begin{cases} 0 & \text{if } \beta > \alpha \geq 2 \\ c > 0 & \text{if } \beta = \alpha = 2 \\ \text{doesn't exist} & \text{otherwise.} \end{cases}$$

## 1.3 Measures of Full Hausdorff Dimensions

A measure $\nu$ on a bounded subset $\Lambda$ in $\mathbb{R}^n$ called an measure of full Hausdorff dimension if $HD(\nu) = HD(\Lambda)$, where $HD(\Lambda)$ is the Hausdorff dimension of $\Lambda$, and

$$HD(\nu) = \inf\big\{HD(\Lambda_0) : \Lambda_0 \subset \Lambda,\ \nu(\Lambda \backslash \Lambda_0) = 0\big\}.$$

It is conjectured that if $\Lambda$ is an invariant set of an smooth expanding map, then $f$ has an invariant probability measure $\mu$ such that $HD(\nu) = HD(\Lambda)$. However, for an almost expanding set on the interval, the measure of full Hausdorff dimension may not be finite.

**Fact 1.15** *Let $0 \in \Lambda \subset I$ be an invariant set of a piecewise smooth expanding map $f$ with indifferent fixed point* $0$. *Assume that*

$$f(x) = x + x^{1+\alpha} + o(x^{1+\alpha}), \qquad x \to 0.$$

*Then $f$ has an invariant measure $\mu$ of full Hausdorff dimension, and $\mu$ is finite if $\dfrac{1}{\alpha} + 1 > \dfrac{2}{\delta}$; $\mu$ is $\sigma$-finite if $\dfrac{1}{\alpha} + 1 < \dfrac{2}{\delta}$, where $\delta = HD(\Lambda)$.*

### 1.4 Equilibrium States

An invariant measure $\mu$ is an equilibrium state for $\phi$, if

$$P(\phi) = h_\mu(f) + \int \phi d\mu,$$

where $P$ is the topological pressure of $\phi$.

All measures we discussed in above subsections are equilibrium states for some Hölder functions. For example, the measure $\mu$ in Fact 1.15 is an equilibrium state of the function $\phi(x) = -\log |f'(x)|^\delta$.

The equilibrium states are not unique because the point mass $\delta_p$ is also an equilibrium state, though the functions $\phi$ satisfy the Hölder conditions.

**Question 1.16** *Under which conditions is an equilibrium state $\mu$ for a Hölder function $\phi$ finite?*

## 2 Convergence Rate

We discuss decay rates of correlations and convergence rates of functions under Perron-Frobenius operators. All the results are for expanding maps. However, some methods can be adapted for diffeomorphisms.

### 2.1 Decay of Correlations

**Definition 2.1** *The covariance of two functions $g_1, g_2 : M \to \mathbb{R}$ is defined by*

$$\operatorname{Cov}(g_1, g_2) = \int g_1 g_2 d\mu - \int g_1 d\mu \int g_2 d\mu.$$

If a system is mixing, then by taking $g_1 = \chi_A$, $g_2 = \chi_B$ for measurable sets $A$ and $B$, we have

$$\operatorname{Cov}(g_1 \circ f^n, g_2) = \mu(f^{-n}A \cap B) - \mu A \cdot \mu B \to 0.$$

If a system is chaotic enough, then the speed of convergence could be exponentially fast.

**Definition 2.2** *We say that $(f, \mu)$ has exponential decay of correlations for functions in $\mathcal{F}$, if there exists $0 < \tau < 1$ such that for all $g_1, g_2 \in \mathcal{F}$, $\exists C = C(g_1, g_2) > 0$ s.t.*

$$|\operatorname{Cov}(g_1 \circ f^n, g_2)| \leq C\tau^n \qquad \forall n \geq 1.$$

If $f$ is an uniformly hyperbolic diffeomorphism and $\mu$ is an SRB measure of $f$, then $(f, \mu)$ has exponential decay of correlations ([R]). The same is true for uniformly expanding maps and their absolutely continuous invariant measures ([HF]). It has been found that some other systems have exponentially decay of correlations. These systems include certain unimodal maps on the intervals, Hénon maps, and hyperbolic systems with singularities like dispersing billiards. (See e.g [L], [Y1], etc.)

On the other hand, it is natural to ask if there are systems with slower decay rates. Expanding maps with indifferent fixed points are the only known examples that have polynomial decay of correlations.

Since decay rates not only depend on the systems, but also on the class of functions. For simplicity here we only consider Lipschitz functions.

**Theorem 2.1 ([Y2], [H2])** *Let $f$ be a piecewise smooth expanding map on the interval with an indifferent fixed point $0$. Assume that for some $0 < \alpha < 1$,*

$$f(x) = x + x^{1+\alpha} + o(x^{1+\alpha}), \qquad x \to 0.$$

*Then for all Lipschitz functions $g_1$,$g_2$, there exists $C = C(g_1, g_2)$ s.t. $\forall n \geq 1$,*

$$\left|\mathrm{Cov}(g_1 \circ f^n, g_2)\right| \leq \frac{C}{n^{\beta-1}}, \quad \text{where} \quad \beta = \frac{1}{\alpha}.$$

The power law of upper bound for decay of correlations was first proved in 1993 by Mori ([M]) and by Lambert, Siboni and Vaienti ([LaSV]) independently for piecewise linear maps with an indifferent fixed point (Takahashi model). They obtained the same estimates. A slightly different model was also studied by Fisher and Lopes in 1997 ([FL]). For maps similar to that in the above theorem, Liverani, Saussol and Vaienti (LiSV]), and Pollicott and Yuri ([PY]) also obtained $C/(\log n)^{\beta} n^{\beta-1}$ and $C/n^{\beta_- -1}$ as upper bounds for $\left|\mathrm{Cov}(g_1 \circ f^n, g_2)\right|$ respectively, where $\beta_-$ means $\beta - \epsilon$ for any $\epsilon > 0$.

The result below means that the upper bounds in Theorem 2.1 are also sharp.

**Theorem 2.2 ([H2])** *Let $f$ be the same as in Theorem2.1. Then there exist smooth functions $g_1$, $g_2$, and a constant $C > 0$ such that $\forall n \geq 1$,*

$$\left|\mathrm{Cov}(g_1 \circ f^n, g_2)\right| \geq \frac{C}{n^{\beta-1}}, \quad \text{where} \quad \beta = \frac{1}{\alpha}.$$

There are also other systems with polynomial decay of correlations.

If the maps introduced in Fact 1.13 have absolutely continuous invariant probability measures, then ([H3])

$$\left|\mathrm{Cov}(g_1 \circ f^n, g_2)\right| = O(n^{m\beta-1}).$$

If the maps introduced in Fact 1.15 admit invariant probability measures of full Hausdorff dimensions, then ([H3])

$$\big|\mathrm{Cov}(g_1 \circ f^n, g_2)\big| = O(n^{\delta(\beta+1)-2}).$$

The inhomogeneous Diophantine approximation transformation defined by

$$f(x,y) = \Big(\frac{1}{x} - \big[\frac{1-y}{x}\big] + \big[-\frac{y}{x}\big], -\big[-\frac{y}{x}\big] - \frac{y}{x}\Big),$$

where $0 \le y \le 1$, $-1 \le x \le -y+1$, and $[x]$ is the integer part of $x$. This map has two indifferent periodic points $(1,0)$ and $(-1,1)$ with period 2, and admits an absolutely continuous measure with density $1/(2\log 2)(1-x^2)$. The rate of decay of correlations satisfies ([PY])

$$\big|\mathrm{Cov}(g_1 \circ f^n, g_2)\big| = O(n^{-1+\epsilon}) \qquad \forall \epsilon > 0.$$

This is very close to the rate for the maps in Fact 1.13 with $\alpha = 1$ and $m = 2$.

It is nature to ask the following questions.

**Question 2.3** *Does every almost hyperbolic system that admits an SRB measure have decay rate of correlations strictly slower than exponential?*

**Conjecture 2.4** *A nondegenerate almost Anosov diffeomorphism on the surface that admits an SRB measure has polynomial decay of correlations, and the degree of the polynomial is related to the coefficients of the third-order terms.*

Now we know that smooth dynamical systems with SRB measures may have exponential or polynomial decay of correlations.

**Question 2.5** *Do there exist smooth or piecewise smooth dynamical systems with SRB measures whose decay rates of correlations are between exponential and polynomial, or slower than polynomial?*

It seems by the work of Young ([Y2]) that such systems exist.

## 2.2 Convergence to Density Functions

We consider a piecewise smooth expanding map $f$ on the interval with an indifferent fixed point 0, though what we discuss here can be extended to a more general setting.

Suppose $f(x) = x + x^{1+\alpha} + o(x^{1+\alpha})$, as $x \to 0$, and $\beta = \alpha^{-1}$. Let

$$\phi(x) = -\log|f'(x)|.$$

and

$$\mathcal{L}_\phi g(x) = \sum_{y \in f^{-1}x} e^{\phi(y)} g(y).$$

$\mathcal{L}_\phi$ has a fixed point $h$, i.e. $\mathcal{L}_\phi h = h$, which is the density function of the absolute continuous invariant measure $\mu$. It is well known that if $\mu$ is a probability measure, then for any continuous function $g$ (see e.g. [W]),

$$\lim_{n\to\infty} \mathcal{L}_\phi^n g = \mu(g)h. \tag{$*$}$$

Put $\psi(x) = \phi(x) + \log h(x) - \log h\big(f(x)\big)$. Then define a new operator

$$\mathcal{L}_\psi g(x) = \sum_{y \in f^{-1}x} e^{\psi(y)} g(y) = \frac{1}{h(x)}\big(\mathcal{L}_\phi(hg)\big)(x).$$

It is easy to see by $\mathcal{L}_\phi h = h$ that $\lim\limits_{n\to\infty} \mathcal{L}_\psi^n g = \mu(g)$ for any continuous function $g$. Since on a closed subset $E \subset (0,1]$, $h$ is bounded away from zero and infinity, $\mathcal{L}_\psi^n g$ and $\mathcal{L}_\phi^n g$ have the same convergence rate on $E$.

**Theorem 2.6 ([H2])** *For any Lipschitz function $g$, there is a constant $C > 0$ such that for all $n > 0$,*

$$\int \big|\mathcal{L}_\psi^n g - \mu(g)\big| d\mu \le \frac{C}{n^{\beta-1}}.$$

*Also for any closed subset $E \subset (0,1]$, there is $C_1 > 0$ such that for all $n > 0$,*

$$\big|\mathcal{L}_\psi^n g - \mu(g)\big| \le \frac{C_1}{n^{\beta-1}} \qquad \forall x \in E.$$

The first part of the results is finer than that in Theorem 2.1 because

$$\big|\mathrm{Cov}(g_1 \circ f^n, g_2)\big| = \Big|\int \big(\mathcal{L}_\psi^n g_2 - \mu(g_2)\big) g_1 d\mu\Big| \le |g_1| \int \big|\mathcal{L}_\psi^n g_2 - \mu(g_2)\big| d\mu.$$

If $\mu$ is a $\sigma$-finite measure, then we cannot expect $(*)$ for any function $g$ with $|\int g dx| < \infty$, because $\int \mathcal{L}_\phi g dx = \int g dx$ and $\int h dx = \infty$. However, if we multiply the left side by suitable factors, we may still get a density function $h$ as a limit.

**Theorem 2.7 ([CF])** *Suppose $f$ is a piecewise $C^3$ expanding map on $[0,1]$ with an indifferent fixed point $0$ such that $f\big([0,1/2)\big) = [0,1)$ and $f\big((1/2,1]\big) = [0,1)$ or $(0,1]$. Then for any real function $g$ bounded away from $0$ and $\infty$,*

$$\lim_{n\to\infty} \frac{A_n}{n} \sum_{i=0}^{n-1} \mathcal{L}_\phi^i g = h$$

*uniformly on any compact subset $E \subset (0,1]$ for some $h$ with $\mathcal{L}_\phi h = h$, where $A_n$ is of order $\log n$.*

Note that a typical piecewise $C^3$ map $f$ has the form $f(x) = x+ax^2+O(x^3)$.

For $0 < \alpha < 1$, restricted to a compact subset $E \subset (0,1]$, the difference between $\mathcal{L}_\psi^n g$ and a constant, which can be measured by the Hilbert metric, is of order $n^{-\beta}$. (See [H3] for more details.) Based on this observation, we make the following conjecture.

**Conjecture 2.8** *Let $f$ be a piecewise smooth expanding map on $[0,1]$ with an indifferent fixed point $0$ such that $f(x) = x + x^{1+\alpha} + o(x^{1+\alpha})$ as $x \to 0$, where $\alpha > 0$. Then for any continuous function $g$, there exist $a_n = O(n^{-\beta})$ such that*

$$\lim_{n\to\infty} A_n \mathcal{L}_\phi^n g = h$$

*uniformly on any compact subset $E \subset (0,1]$ for some $h$ with $\mathcal{L}_\phi h = h$, where $A_n = \prod_{i=0}^{n-1}(1+a_i)$. Further, the convergence rate is of order $n^{-\beta}$.*

Since $\{A_n\}$ is convergent if $\alpha \in (0,1)$, the conjecture is consistent with $(*)$.

# 3 Other Ergodic Properties

It is also interesting to know whether almost hyperbolic systems display any new phenomena respect to other ergodic properties. Unfortunately, there are few known results other than invariant measures and decay of correlations.

## 3.1 Stochastic Stability

Let $f : M \to M$ be a dynamical system preserving an invariant measure $\mu_0$. A random perturbation $\mathcal{F}$ of $f$ can be regarded as a Markov chain with the state space $M$ and transition probabilities $P(\cdot|x)$. A invariant measure $\mu$ for $\mathcal{F}$ is the measure such that for any Borel subset $E$ of $M$,

$$\mu(E) = \int P(E|x)d\mu.$$

The system $(f, \mu_0)$ is *stochastically stable* if for any 1-parameter family of random perturbations $\{\mathcal{F}_\epsilon : \epsilon > 0\}$ converging to $f$, (i.e. $P_\epsilon(\cdot|x) \to \delta_{f(x)}$ weakly for all $x$,) the invariant measures $\mu_\epsilon$ of $\mathcal{F}_\epsilon$ converge to $\mu_0$ as $\epsilon \to 0$.

For Axiom A attractors and the SRB measures, stochastic stability was first proved by Kifer (see e.g. [Ki]). Absolutely continuous conditional measures are stochastically stable for some piecewise expanding maps, logistic maps and unimodal maps. (See [Ke], [KK], [BY1], [BV], etc.)

There are no results about stochastic stability for almost hyperbolic systems so far. It is nature to ask the following.

**Question 3.1** *Are almost hyperbolic systems stochastically stable under reasonable random perturbations?*

Even though the systems contain points at which hyperbolicity fails, it seems that the answer would be positive in many cases. On the other hand, it would be very interesting to find some systems which are stochastically unstable under certain reasonable random perturbations. Almost hyperbolic systems might be good candidates.

## 3.2 Poisson Distributions for Returning Time

This is a new research subject developed in the 1990s. Consider a measure preserving transformation $f$ from a probability space $(X, \mu)$ to itself. Poincaré's recurrence theorem says that for any measurable set $E$, almost every point in $E$ returns to the set infinite many times. Later, Kac's theorem states that if $\mu$ is an ergodic measure, then the expectation of the return time to $E$ of points starting from $E$ is just $\mu(E)^{-1}$. It is easy to see that if a system is chaotic enough, then the measure of the set of points whose returning time $\tau_E(x)$ is greater than $t$ is of order $e^{-at}$ for some $a > 0$.

Recently, it is found that in some systems the $k$th returning time $\tau^k$ is related to a Poisson law. That is, for $\mu$-a.e. $z$, if one take a decreasing sequence of neighborhoods $U_\epsilon$ of $z$, then $\tau^k_{U_\epsilon}$ satisfies

$$\mu_{U_\epsilon}\big\{x \in U_\epsilon \big| \tau^k_{U_\epsilon}(x) \le t < \tau^{k+1}_{U_\epsilon}(x)\big\} \to \frac{t^k}{k!}e^{-t} \qquad \text{as} \qquad \epsilon \to 0.$$

These systems includes Axiom A systems, transitive Markov chains, expanding maps of the interval, rational maps. These properties were also proved for piecewise smooth expanding maps on the interval with an indifferent fixed point by Hirata, Saussol and Vaienti. (See [HSV] for related references.)

# References

[AH] N. Aoki and K. Hiraide, *Topological theory of Dynamical Systems. Recent advances. North-Holland Mathematical Library, 52.* (North-Holland Publishing Co., Amsterdam, 1994).

[B] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms Lecture Notes in Math. Vol. 470* (Springer-Verlag, Berlin-New York, 1975).

[BV] V. Baladi and M. Viana, Strong stochastic stability and rate of mixing for unimodal maps, *Ann. Scient. Éc. Norm. Sup.* **29** (1996), 483–517.

[BY1] M. Benedicks and L.-S. Young, Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps, *Ergod. Th. & Dynam. Sys.* **12** (1992), 13–37.

[BY2] M. Benedicks and L.-S. Young, Sinai-Bowen-Ruelle measure for certain Hénon maps, *Invent. Math.* **112** (1993), 541–576.

[C] N. Chernov, Statistical properties of piecewise smooth hyperbolic systems in high dimensions, *Preprint.*

[CF] P. Collet and P. Ferrero, Some limit ratio theorem related to a real endomorphism in case of neutral fixed point, *Ann. Inst. Henri Poicaré* **52** (1990), 283–301.

[FL] A. Fisher and A. Lopes, Polynomial decay of correlations and the central limit theorem for the equilibrium states of a non-Hölder potential, *Preprint.*

[HSV] M. Hirata, B.Saussol and S. Vaienti, Statistics of returning times: A general fromwork and new applications, *Preprint.*

[HK] F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations, *Math. Z.* **180** (1982), 119-140.

[H1] H. Huyi, Conditions for the Existence of SBR Measures for "Almost Anosov" Diffeomorphisms, *Trans. Amer. Math. Soc.* (to appear).

[H2] H. Huyi, Decay of Correlations for Piecewise Smooth Maps with Indifferent Fixed Points, *Preprint.*

[H3] H. Huyi, Equilibrium states with Polynomial Decay of Correlations, *Preprint.*

[HY] H. Huyi and L.-S. Young, Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov", *Ergod. Th. & Dynam. Sys.* **15** (1995), 67–76.

[I] T. Inoue, Ratio ergodic theorems for maps with indifferent fixed points, *Ergod. Th. & Dynam. Sys.* **17** (1997), 625–642.

[J] M. Jakobson, Absolute continuous invariant measures for one-parameter families of one-dimensional maps, *Commun. Math. Phys.* **81** (1981), 39–88.

[JN] M. Jakobson and S. Newhouse, On the structure of non-hyperbolic attractors, in *Dynamical systems and chaos, Vol. 1*, ed. N. Aoki, etc. (World Sci. Publishing, River Edge, 1972), pp 103–111.

[KK] A. Katok, Bernoulli diffeomorphisms on surfaces, *Ann. of Math. (2)* **110** (1979), 529–547.

Random perturbations of transformations of an interval, *J. Analyse Math.* **47** (1986), 193–237.

[Ke] G. Keller, Stochastic stability in some chaotic dynamical systems, *Monatsh. Math.* **94** (1982), 313–333.

[Ki] Yu. Kifer, General random perturbations of hyperbolic and expanding transformations, *J. Analyse Math.* **47** (1986), 111–150.

[KS] K. Krzyzewski and W. Szlenk, On invariant measure for expanding differentiable mappings, *Studia Math.* **33** (1969), 83–92.

[LS] F. Ledrappier and J.-M. Strelcyn, Estimation from below in Pesin's entropy formula, *Ergod. Th. & Dynam. Sys.* **2** (1982 ), 203–219.

[LaSV] A. Lambert, S. Siboni and S. Vaienti, Statistical properties of a nonuniformly hyperbolic map of the interval, *J. Statist. Phys.* **72** (1993), 1305–1330.

[Li] C. Liverani, Decay of correlations, *Ann. Math.* **142** (1995 ), 239-301.

[LiSV] C. Liverani, B.Saussol and S. Vaienti, A probabilistic approach to intermittency, *Preprint.*

[M] M. Mori, On the intermittency of a piecewise linear map, *Tokyo J. Math.* **16** (1986), 411–428.

[P] Ya. B. Pesin, Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties, *Ergod. Th. & Dynam. Sys.* **12** (1992), 123-151.

[Pi] G. Pianigiani, First return map and invariant measures in case of neutral fixed point, *Isreal J. Math.* **35** (1980), 32–48.

[Po] M. Pollicott and M. Yuri, Statistical Properties of maps with indifferent periodic points, *Preprint.*

[R] D. Ruelle, A measure associated with axiom-A attractors, *Amer. J. Math.* **98** (1976), 619–654.

[S] Ya. G. Sinai, Gibbs measure in ergodic theory, *Russian Math. Surveys* **27** (1972), 21–69.

[T] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc. (N.S.)* **19** (1988), 417–431.

[W] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances, *Trans. Amer. Math. Soc.* **236** (1978), pp. 121–153.

[Y1] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. of Math.* **147** (1998), 585–650.

[Y2] L.-S. Young, Recurrence time and rate of mixing, *Preprint.*

[Yu] M. Yuri, Invariant measures for certain multi-dimensional maps, *Nonlinearity* **7** (1994), 1093–1124.