# A VOLUME PRESERVING DIFFEOMORPHISM WITH ESSENTIAL COEXISTENCE OF ZERO AND NONZERO LYAPUNOV EXPONENTS

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Abstract. We show that there exists a $C^\infty$ volume preserving topologically transitive diffeomorphism of a compact smooth Riemannian manifold which is ergodic (indeed is Bernoulli) on an open and dense subset $\mathcal{G}$ of not full measure and has zero Lyapunov exponent on the complement of $\mathcal{G}$.

## 1. Introduction.

It is shown in [7, 11, 22, 23] that on any manifold $\mathcal{M}$ and for any sufficiently large $r$ one has what can be viewed as a discrete version of the classical KAM theory phenomenon in the volume preserving category – there are open sets of volume preserving $C^r$ diffeomorphisms of $\mathcal{M}$ all of which possess positive measure sets of codimension-1 invariant tori; on each such torus the diffeomorphism is $C^1$ conjugate to a Diophantine translation; all of the Lyapunov exponents are zero on the invariant tori. It is expected that the set of invariant tori is surrounded by "chaotic sea", i.e., outside this set the Lyapunov exponents are nonzero and the system has at most countably many ergodic components. It has since been an open problem to find out to what extend this picture is true.

A first step towards understanding this picture is to establish "essential" coexistence of completely chaotic and regular non-chaotic behavior for the class of volume preserving systems in the spirit of the results mentioned above. To this end in this paper we prove the following result.

**Main Theorem.** *Given $\alpha > 0$, there exists a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a $C^\infty$ diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ preserving the Riemannian volume $m$ such that*

(1) $\|P - \mathrm{Id}\|_{C^1} \le \alpha$ *and $P$ is homotopic to* Id*;*

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(2) *$P$ is ergodic on an open and dense subset $\mathcal{G} \subset \mathcal{M}$ and $m(\mathcal{G}) < m(\mathcal{M})$; in particular, $P$ is topologically transtive on $\mathcal{M}$; furthermore, $P|\mathcal{G}$ is a Bernoulli diffeomorphism;*
(3) *the Lyapunov exponents of $P$ are nonzero for almost every $x \in \mathcal{G}$;*
(4) *the complement $\mathcal{G}^c = \mathcal{M} \setminus \mathcal{G}$ has positive volume, $P|\mathcal{G}^c = \mathrm{Id}$ and the Lyapunov exponents of $P$ on $\mathcal{G}^c$ are all zero.*

In our example the set $\mathcal{G}^c$ is the direct product of a 3-dimensional smooth compact manifold and a Cantor set of positive volume in a two dimensional torus and thus has codimension two. By modifying our construction one can obtain a $C^\infty$ diffeomorphism $P$ of a compact smooth Riemannian manifold of dimension 4, which is close to the identity map and has nonzero Lyapunov exponents on an open and dense set $\mathcal{G}$ of positive but not full volume and zero exponents on its compliment. The latter is the direct product of a 3-dimensional smooth compact manifold and a circle and thus has codimension one and $P$ has countably many ergodic components (see [6]).

Coexistence of elliptic islands and "chaotic sea" is one of the most interesting phenomena in dynamical systems and very few results are known in this direction. Przytycki [19] and Liverani [16] studied a one-parameter family $f_a$, $-\varepsilon \le a \le \varepsilon$, of area preserving diffeomorphisms for which the map $f_0$ lies on the boundary of the set of Anosov diffeomorphisms. This example demonstrates a route from uniform hyperbolicity (corresponding to $-\varepsilon \le a < 0$) to non-uniform hyperbolicity (corresponding to $a = 0$) and then to coexistence of regular and chaotic behavior, i.e., the appearance of an elliptic island (for $0 < a \le \varepsilon$).

An example of a billiard dynamical system – the so-called "mushroom billiards" – with coexistence of "elliptic islands" and "chaotic sea" was constructed by Bunimovich in [3]. However, this case differs substantially from the smooth case due to the presence of singularities.

In [10], Fayad obtained a weaker version of our theorem: only *some but not all* Lyapunov exponents for $P$ are zero on $\mathcal{G}^c$. Ensuring that *all* Lyapunov exponents are zero is a substantially more difficult problem and we use a completely different techniques than in [10] to make it happen. The matter is that if all Lyapunov exponents in $\mathcal{G}^c$ are zero, then a typical trajectory that originates in $\mathcal{G}$ will spend long time in the vicinity of $\mathcal{G}^c$ where contraction and expansion rates are very small. This should be compensated by even longer periods of time that the trajectory should spend away from $\mathcal{G}^c$ thus gaining sufficient contraction and expansion and ensuring nonzero Lyapunov exponents.

Let us briefly outline our construction. It starts with a $C^\infty$ volume preserving diffeomorphism $T$ of a compact smooth 5-dimensional manifold $\mathcal{M}$. The map $T$ is close to and homotopic to the identity and indeed is the identity on an invariant compact subset of positive volume. On its compliment $\mathcal{G}$ the map $T$ is partially hyperbolic with one-dimensional stable, one-dimensional unstable subspaces and 3-dimensional center subspace along which $dT$ acts as an isometry and hence has zero Lyapunov exponents. These subspaces are integrable to three transverse one-dimensional stable, one-dimensional unstable and 3-dimensional central invariant foliations of $\mathcal{G}$. Since this set is open, partial hyperbolicity appears in its weaker *pointwise* form (see Section 2 for the definition of pointwise partial hyperbolicity).

Pointwise partially hyperbolic maps on compact manifolds were introduced in [5]. They have properties that are pretty much similar to those of uniformly partially hyperbolic systems: 1) stable and unstable subspaces are integrable to continuous stable and unstable foliations that are uniformly transverse to each other; 2) Lyapunov exponents along stable (unstable) subspaces are negative (positive); 3) any sufficiently small perturbation of a pointwise partially hyperbolic map is also pointwise partially hyperbolic. These properties fail to be true if one considers, as we do, pointwise partially hyperbolic maps on open subsets thus providing one of the major obstacles for our construction. To overcome this problem we only consider small perturbations of $T$ that are *gentle*, i.e., they coincide with $T$ outside a neighborhood of the Cantor set $\mathcal{G}^c$. For those perturbations the above three properties hold. However, the final map $P$ is *not* a gentle perturbation of $T$ and additional arguments are needed to establish these properties for $P$.

Our next step is to perturb $T$ gently to a $C^\infty$ volume preserving diffeomorphism $Q$, which is concentrated in an open set, which is "far away" from the Cantor set. We arrange this perturbation in such a way that the average Lyapunov exponents of $Q$ in the central direction are positive for points in $\mathcal{G}$ while the Lyapunov exponents on the compliment $\mathcal{G}^c$ of $\mathcal{G}$ are all zero. Our construction of the map $Q$ is built upon some ideas from [21, 8, 14, 2, 9] but requires substantial modifications and new arguments due to nonuniform hyperbolicity of the map $T$. Note that the restriction $Q|\mathcal{G}$ is not ergodic.

Finally, we perturb $Q$ to a $C^\infty$ volume preserving diffeomorphism $P$, which is pointwise partially hyperbolic on $\mathcal{G}$ and, similarly to the maps $T$ and $Q$, possesses three transversal continuous one-dimensional stable, one-dimensional unstable and 3-dimensional central invariant foliations. In doing so we first construct a sequence of small perturbations $P_n$ of $Q$ such that each $P_n$ coincides with $T$ outside some open invariant subset $\mathcal{U}_n \subset \mathcal{G}$ (hence, $P_n$ is a gentle perturbation of $T$) and has the accessibility property on $\mathcal{U}_n$. The sets $\mathcal{U}_n$ are nested and exhaust $\mathcal{G}$ and the sequence $P_n$ converges to the desired map $P$. In constructing the maps $P_n$ we use some techniques developed in [8, 14].

Although the map $P$ is *not* a gentle perturbation of $T$ (it coincides with $T$ on the Cantor set only) we shall prove that $P$ has the three properties described above. Furthermore, we show that $P$ has the accessibility property on $\mathcal{G}$ via its stable and unstable foliations and that the average Lyapunov exponents of $P|\mathcal{G}$ in the central direction remain positive and in fact, central Lyapunov exponents are positive on a subset of positive volume. We then show that $P|\mathcal{G}$ is ergodic and indeed, is a Bernoulli diffeomorphism. To achieve this we extend the argument in [4] to the case of maps that are pointwise partially hyperbolic on open sets. This implies that $P$ has four positive and one negative Lyapunov exponents on $\mathcal{G}$ while the Lyapunov exponents on the Cantor set $\mathcal{G}^c$ are all zero.

In Section 2 we provide some background information and introduce some basic notations. In Section 3 we describe our construction of the map $P$ and prove our result subject to two propositions. In the remaining sections we present the proofs of these propositions and other supporting statements.

## 2. Preliminaries

See [17, 1] for more details.

Let $f$ be a diffeomorphism of a compact smooth Riemannian manifold $\mathcal{M}$ and $\Lambda \subset \mathcal{M}$ an $f$-invariant compact subset. The map $f$ is said to be *uniformly partially hyperbolic on* $\Lambda$ if for every $x \in \Lambda$ the tangent space at $x$ admits an invariant splitting

$$T_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x) \tag{2.1}$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces. More precisely, there are numbers $0 < \lambda < \lambda' \le 1 \le \mu' < \mu$ such that for every $x \in \Lambda$,

$$\begin{aligned} \|dfv\| &\le \lambda \|v\|, && v \in E^s(x),\\ \lambda' \|v\| \le \|dfv\| &\le \mu' \|v\|, && v \in E^c(x),\\ \mu \|v\| \le \|dfv\|&, && v \in E^u(x). \end{aligned}$$

Given $x \in \Lambda$, one can construct a *strongly stable local manifold* $V^s(x) = V^s_f(x)$ and a *strongly unstable local manifold* $V^u(x) = V^u_f(x)$ at $x$. This local manifolds have uniform size, i.e., there are numbers $r > 0$ and $D > 0$ such that for every $x \in \Lambda$ there are smooth functions $\varphi^i : B^i(r) \to T_x\mathcal{M}$, $i = s$ or $u$ (here $B^i(r) \subset E^i(x)$ is the ball centered at zero of radius $r$) such that

$$\varphi(0) = 0, \quad d\varphi(0) = 0, \quad \max\{\|d\varphi(a)\| : a \in B^i(r)\} \le A,$$

and

$$V^i(x) = \exp_x\{(a, \varphi(a)) : a \in B^i(r)\}.$$

We define the *strongly stable* and *strongly unstable global manifolds* at $x$ by

$$W^u(x) = W^u_f(x) = \bigcup_{n\geq 0} f^n(V^u(f^{-n}(x))),$$
$$W^s(x) = W^s_f(x) = \bigcup_{n\geq 0} f^{-n}(V^s(f^n(x))).$$

We denote by $B(x,r)$ the ball centered at the point $x$ of radius $r$. Further, we adopt the following notation: for a smooth submanifold $V \subset \mathcal{M}$ and a point $x \in V$ we denote by $B_V(x,r)$ the ball in $V$ centered at $x$ of radius $r$ (with respect to the intrinsic Riemannian metric). We also set

$$B^s(x,r) = B^s_f(x,r) = B_{V^s(x)}(x,r),$$
$$B^u(x,r) = B^u_f(x,r) = B_{V^u(x)}(x,r).$$

In this paper we need a weaker property than uniform partial hyperbolicity. Let $\mathcal{S} \subset \mathcal{M}$ be an $f$-invariant open subset. We say that $f$ is *pointwise partially hyperbolic on* $\mathcal{S}$ if for every $x \in \mathcal{S}$ the tangent space at $x$ admits an invariant splitting (2.1) and there are continuous positive functions $\lambda(x) < \lambda'(x) \leq 1 \leq \mu'(x) < \mu(x)$, $x \in \mathcal{S}$ such that

$$\begin{aligned} \|dfv\| &\leq \lambda(x)\,\|v\|, && v \in E^s(x),\\ \lambda'(x)\,\|v\| \leq \|dfv\| &\leq \mu'(x)\,\|v\|, && v \in E^c(x),\\ \mu(x)\,\|v\| \leq \|dfv\|&, && v \in E^u(x). \end{aligned}$$

Given a subset $\mathcal{S}$ we call a partition $\mathcal{P}$ of $\mathcal{S}$ a $(\delta,q)$*-foliation with smooth leaves* or simply a *foliation with smooth leaves* if there exist continuous functions $\delta = \delta(x) > 0$, $q = q(x) > 0$, and an integer $k > 0$ such that for each $x \in \mathcal{S}$:

(1) there exists a smooth immersed $k$-dimensional manifold $W(x)$ containing $x$ for which $\mathcal{P}(x) = W(x)$ where $\mathcal{P}(x)$ is the element of the partition $\mathcal{P}$ containing $x$. The manifold $W(x)$ is called the *global leaf* of the foliation at $x$; the connected component of the intersection $W(x) \cap B(x,\delta(x))$ that contains $x$ is called the *local leaf* at $x$ and is denoted by $V(x)$;
(2) there exists a continuous map $\phi_x : B(x,q(x)) \to C^1(D,\mathcal{M})$ (where $D$ is the unit ball) such that $V(y)$ is the image of the map $\phi_x(y) : D \to \mathcal{M}$ for each $y \in B(x,q(x))$; the number $q(x)$ is called *the size* of $V(x)$.

We say that a foliation with smooth leaves is *absolutely continuous* if for almost every $x \in \mathcal{S}$ and almost every $y \in B(x,q(x))$ the conditional measure generated on $V(y)$ by volume $m$ (with respect to the partition of $B(x,q(x))$ by local leaves) is absolutely continuous with respect to the leaf volume $m_{V(y)}$ on $V(y)$.

The strongly stable and strongly unstable global manifolds of a uniformly partially hyperbolic diffeomorphism form two $(\delta, q)$-foliations of $\Lambda$ with smooth leaves where $\delta$ and $q$ are constants. These foliations are absolutely continuous and transversal at every point $z \in \Lambda$.

Let $W_1$ and $W_2$ be two continuous foliations with smooth leaves of a subset $\mathcal{S}$. Assume that these foliations are transversal at every point $z \in \mathcal{S}$. We say that these foliations have the *accessibility property* if any two points $z, z' \in \mathcal{S}$ are *accessible*; this means that

(1) there exists a collection of points $z_1, \dots, z_n \in \mathcal{S}$ such that $x = z_1$, $y = z_n$ and $z_k \in W_i(z_{k-1})$ for $i = 1$ or 2 and $k = 2, \dots, n$;
(2) the points $z_{k-1}$ and $z_k$ can be connected by a smooth curve $\gamma_k \subset W_i(z_{k-1})$ for $i = 1$ or 2 and $k = 2, \dots, n$.

The collection of such points $z_k$ and curves $\gamma_k$ is called the *leaf-wise path* connecting $x$ and $y$. In particular, if $W_1$ and $W_2$ are the stable and unstable foliations, then we say that $f$ has the *accessibility property* and the leaf-wise path is called the $(u, s)_f$*-path* or simply $(u, s)$*-path*.

It may not be true in general that a diffeomorphism, which is pointwise partially hyperbolic on an open set $\mathcal{S}$, has strongly stable and unstable local manifolds at every point in $\mathcal{S}$. However, this is the case for all pointwise partially hyperbolic diffeomorphisms that we construct and in fact their global strongly stable and unstable manifolds form two transversal foliations with smooth leaves.

A uniformly partially hyperbolic diffeomorphism $f$ is called *dynamically coherent* if the subbundles $E^{cu} = E^c \oplus E^u$, $E^c$, and $E^{cs} = E^c \oplus E^s$ are integrable to continuous foliations with smooth leaves $W^{cu}$, $W^c$ and $W^{cs}$, called respectively the *center-unstable*, *center* and *center-stable foliations.* Furthermore, the foliations $W^c$ and $W^u$ are subfoliations of $W^{cu}$, while $W^c$ and $W^s$ are subfoliations of $W^{cs}$.

The following result (see [12, 20]) shows that dynamical coherence is a robust property.

**Theorem 2.1.** *Suppose that $f$ is a partially hyperbolic diffeomorphism. If the center foliation $W^c$ is smooth, then $f$ is dynamically coherent. Moreover, any diffeomorphism that is close to $f$ in the $C^1$ topology is dynamically coherent.*

Since both subbundle $E^{cu}$ and $E^{cs}$ vary continuously with the map, so does $E^c$ and the corresponding center foliation $W^c$.

We denote by

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \|df^n v\|$$

the *Lyapunov exponent* of a nonzero vector $v$ at $x \in \mathcal{M}$ and by $\lambda_i(x) = \lambda_i(x, f)$, $i = 1, \dots, \dim \mathcal{M}$, the values of the Lyapunov exponents at $x$. Note that the functions $\lambda_i(x, f)$ are invariant. We assume that these values are ordered so that

$$\lambda_1(x, f) \geq \cdots \geq \lambda_{\dim \mathcal{M}}(x, f).$$

We also denote by

$$L_k(f) := \int_{\mathcal{M}} \sum_{i=1}^{k} \lambda_i(x, f)\, dm(z), \tag{2.2}$$

where $m$ is the Riemannian volume. We call this number the *$k$-th average Lyapunov exponent* of $f$.

Consider a volume preserving $C^2$ diffeomorphisms $f$ of a compact smooth manifold $\mathcal{M}$ that is pointwise partially hyperbolic on an open set $\mathcal{S}$. We say that $f$ has *positive central exponents* if there is an invariant set $\mathcal{A} \subset \mathcal{S}$ of positive volume such that for every $x \in \mathcal{A}$ and every $v \in E^c(x)$ the Lyapunov exponent $\lambda(x, v) > 0$. The following result plays an important role in the proof of our Main Theorem.

**Theorem 2.2.** *Assume that the following conditions hold:*

1. *$f$ has strongly stable and unstable $(\delta, q)$-foliations $W^s$ and $W^u$ where $\delta = \delta(x)$ and $q = q(x)$ are continuous functions on $\mathcal{S}$;*
2. *the foliations $W^s$ and $W^u$ are absolutely continuous; more precisely, any two points $z_1, z_2 \in \mathcal{S}$ can be connected in $\mathcal{S}$ via a $W^s$ and $W^u$ foliations;*
3. *$f$ has the accessibility property via the foliations $W^s$ and $W^u$;*
4. *$f$ has positive central exponents.*

*Then $f$ has positive central exponents at almost every point $x \in \mathcal{S}$, $f|\mathcal{S}$ is ergodic and indeed, is a Bernoulli diffeomorphism.*

*Proof.* In the case when $f$ is uniformly partially hyperbolic on the whole manifold $\mathcal{M}$, has positive central exponents and the accessibility property this theorem was proved in [4]. We shall show how to extend the argument presented there to our case.

Note that $f$ is a $C^2$ volume preserving diffeomorphism, with nonzero Lyapunov exponents on a set $\mathcal{A}$ of positive volume. Hence, it has at most countably many ergodic components of positive volume in $\mathcal{A}$. Each such component contains the set

$$A(x) = \bigcup_{y \in V^+(x)} V^s(y),$$

where $x$ is a density point of $\mathcal{A}$ and $V^+(x)$ is a center-unstable local manifold at $x$. Since the strongly stable foliation $W^s$ is continuous, the set $A(x)$ is open in $\mathcal{A}$ and hence the set $\mathcal{A}$ itself is open (mod 0). We shall show that the accessibility property of $f$ in $\mathcal{S}$ and absolute continuity of strongly stable and unstable foliations imply that the trajectory of almost every point in $\mathcal{S}$ is dense. Clearly, this yields that $\mathcal{A} = \mathcal{S}$ (mod 0) and that $f|\mathcal{S}$ is ergodic and indeed, is a Bernoulli diffeomorphism.

To this end, it suffices to show that if $U$ is an open set then the orbit of almost every point enters $U$. To see this let us call a point *good* if it has a neighborhood in which the orbit of almost every point enters $U$. We wish to show that an arbitrary point $p$ is good. Since $f$ is accessible, there is a $(u, s)$-path $[z_0, \dots, z_k]$ with $z_0 \in U$ and $z_k = p$. We shall show by induction on $j$ that each point $z_j$ is good. This is obvious for $j = 0$. Now suppose that $z_j$ is good. Then $z_j$ has a neighborhood $N$ such that $\mathrm{Orb}(x) \cap U \neq \emptyset$ for almost every $x \in N$. Let $B$ be the subset of $N$ consisting of points with this property that are also both forward and backward recurrent. It follows from the Poincaré recurrence theorem that $B$ has full measure in $N$. If $x \in B$, any point $y \in W^s(x) \cup W^u(x)$ has the property that $\mathrm{Orb}(y) \cap U \neq \emptyset$. The absolute continuity of the foliations $W^s$ and $W^u$ means that the set

$$\bigcup_{x \in B} W^s(x) \cup W^u(x)$$

has full measure in the set

$$\bigcup_{x \in N} W^s(x) \cup W^u(x).$$

The latter is a neighborhood of $z_{j+1}$. Hence $z_{j+1}$ is good. □

## 3. Construction of the map $P$: Proof of Main Theorem

We describe a construction of the map $P$ splitting it into several steps.

3.1. **Step 1: A Special Flow $T^t$.** Let $A$ be an Anosov automorphism of the torus $X = \mathbb{T}^2$. We denote by $\eta_A$ the constant expanding rate of $A$ along the unstable direction.

Consider the special flow $T^t$ over $A$ with a constant roof function. The flow acts on the the manifold

$$\mathcal{N} = \{(x, t) : x \in X,\ t \in [0, 1]\}/\sim,$$

where "$\sim$" is the identification $(x, 1) = (Ax, 0)$. We may choose the metric on $\mathcal{N}$ in such a way that the expansion rate of $T^t$ along the one-dimensional unstable direction is $t\eta_A$ at every point $(x, t) \in \mathcal{N}$. For each $t \neq 0$ the map

$T^t$ is uniformly partially hyperbolic with one-dimensional stable $E^s_{T^t}$, one-dimensional unstable $E^u_{T^t}$ and one-dimensional center $E^c_{T^t}$ subbundles (the latter is the direction of the flow). These subbundles are integrable to smooth stable $W^s_{T^t}$, unstable $W^u_{T^t}$ and center $W^c_{T^t}$ foliations of $\mathcal{N}$.

3.2. **Step 2: The Original Map $T$.** Set $Y = \mathbb{T}^2$ and $\mathcal{M} = \mathcal{N} \times Y$. We endow $\mathcal{M}$ with the product metric and denote by $m$ its Riemannian volume. We also denote the fiber

$$\mathcal{N}_y = \mathcal{N} \times \{y\}. \tag{3.1}$$

For our construction we choose:

($A1$) a Cantor set $C \subset Y$ of positive measure whose complement $G = Y \setminus C$ is an open connected subset;
($A2$) an open square $G_0$ such that $\overline{G}_0 \subset G$;
($A3$) a $C^\infty$ function $\kappa : Y \to \mathbb{R}$ satisfying: (1) $\kappa(y) = 0$ if $y \in C$ and $\kappa(y) > 0$ if $y \in G$; (2) $|\operatorname{grad} \kappa| < 1/4$, and (3) $\kappa(y) = \kappa_0$ for $y \in U_1$, where $\kappa_0$ is a constant and $U_1$ is a neighborhood of $G_0$ whose choice is specified in Subsection 5.1.

The set $\mathcal{G}$ in the Main Theorem is given by $\mathcal{G} = \mathcal{N} \times G$ and is open, dense and of positive but not full measure. We let $\mathcal{G}^c$ be the complement of $\mathcal{G}$.

We define a map $T : \mathcal{M} \to \mathcal{M}$ by

$$T((x, t), y) = (T^{\kappa(y)}(x, t), y),$$

where $(x, t) \in \mathcal{N}$ and $y \in Y$. The proof of the following proposition is immediate.

**Proposition 3.1.** *The map $T$ is a $C^\infty$ volume preserving diffeomorphism of $\mathcal{M}$ with the following properties:*

(1) *given $\delta_T > 0$, one can chose the function $\kappa$ such that $\|T - \operatorname{Id}\|_{C^1} \le \delta_T$; moreover, $T$ is homotopic to* Id*;*
(2) *$T$ preserves the fibers $\mathcal{N}_y$;*
(3) *$T$ is uniformly partially hyperbolic on any invariant subset $\mathcal{N} \times A$ where $A \subset G$ is compact; moreover, $T$ is dynamically coherent with the cenral foliation $W^c_T = W^c_{T^t} \times Y$;*
(4) *$T$ is pointwise partially hyperbolic on $\mathcal{G}$ with one-dimensional stable $E^s_T(z)$, one-dimensional unstable $E^u_T(z)$ and 3-dimensional center $E^c_T(z)$ subspaces; the subspaces $E^s_T(z)$ and $E^u_T(z)$ are integrable to strongly stable and unstable foliations $W^s_T(z)$ and $W^u_T(z)$ with smooth leaves; these foliations are uniformly transversal and their local leaves have uniform size; in addition, these foliations are absolutely continuous;*

(5) $T|\mathcal{G}^c = \mathrm{Id}$ *and* $dT_z = \mathrm{Id}$ *for any* $z \in \mathcal{G}^c$*; in particular, the Lyapunov exponents of* $T|\mathcal{G}^c$ *are all zero;*

(6) *for every* $z \in \mathcal{G}$ *the Lyapunov exponents of* $T$ *are as follows:*

$$\lambda_1(z,T) = \lambda^u(z,T) > 0 = \lambda_2(z,T) = \lambda_3(z,T) = \lambda_4(z,T) \\ > \lambda_5(z,T) = \lambda^s(z,T),$$

*where* $\lambda^u(z,T)$ *and* $\lambda^s(z,T)$ *correspond to the directions* $E^u_{T^t}$ *and* $E^s_{T^t}$ *respectively and* $\lambda_2(z,T)$*,* $\lambda_3(z,T)$ *and* $\lambda_4(z,T)$ *correspond to the direction of the flow and the* $Y$*-direction respectively. Moreover,* $|\lambda^u(z,T)| = |\lambda^s(z,T)|$*. Consequently,*

$$L_1(T) = L_2(T) = L_3(T) = L_4(T) > 0 \text{ and } L_5(T) = 0,$$

*where each* $i$*-th average Lyapunov exponents* $L_i(\cdot)$ *is given by* (2.2)*.*

We say that a diffeomorphism $F$ is a *gentle perturbation* of $T$ if $F$ is $C^1$ close to $T$ and the following conditions hold:

(1) $F(\mathcal{G}) = \mathcal{G}$ and $F$ is pointwise partially hyperbolic in $\mathcal{G}$;
(2) the one-dimensional strongly stable and unstable subbundles for $F$ are integrable to one-dimensional strongly stable and unstable foliations with smooth leaves on $\mathcal{G}$; the 3-dimensional central subbundle of $F$ is integrable to a central foliation;
(3) $F|\mathcal{G}^c = \mathrm{Id}$.

Further, if $F^\natural$ is a gentle perturbation of $T$ that is sufficiently $C^1$ close to $F$, then we say that $F^\natural$ is a gentle perturbation of $F$ as well.

Let $F$ be a diffeomorphism of $\mathcal{M}$ that is $C^1$ close to $T$. Assume that there is an open set $\mathcal{U}$ such that $\overline{\mathcal{U}} \subset \mathcal{G}$ and $F|\mathcal{U} = T|\mathcal{U}$; in particular, $\mathcal{U}$ is invariant under $F$. Then $F$ is a gentle perturbation of $T$ and in fact, $F|\mathcal{U}$ is uniformly partially hyperbolic.

3.3. **Step 3: The Perturbation $Q$.** We perturb the map $T$ to a map $Q$ such that it has one negative and four positive average Lyapunov exponents but is not necessarily ergodic. We then perturb $Q$ to a map $P$ which is ergodic on $G$ and has all the desired properties.

Given $z \in \mathcal{M}$, we choose a local coordinate system $(s,u,t,a,b)$ such that

$$F^s(z) := \partial/\partial s = E^s_T(z),\ F^u(z) := \partial/\partial u = E^u_T(z),\ F^t(z) := \partial/\partial t \tag{3.2}$$

are the unstable, stable and central (flow) directions of $T$ respectively, and

$$F^b(z) := \partial/\partial b, \quad F^a(z) := \partial/\partial a \tag{3.3}$$

are tangent to $Y$. We shall assume that in these coordinates the square $G_0$ has the form

$$G_0 = B_{F^a}(a_0, \alpha_0) \times B_{F^b}(b_0, \alpha_0) \tag{3.4}$$

for some $(a_0, b_0) \in Y$ and $\alpha_0 > 0$.

The following statement describes some properties of the map $Q$; its proof is given in Section 4.

**Proposition 3.2.** *Given $\delta_Q > 0$, one can construct a $C^\infty$ volume preserving diffeomorphism $Q : \mathcal{M} \to \mathcal{M}$ which satisfies:*

1. $\|Q - T\|_{C^1} \le \delta_Q$ *and $Q$ is homotopic to* Id*;*
2. $Q = T$ *on the set $\mathcal{N} \times (Y \setminus G_0)$; in particular, $Q$ preserves $\mathcal{N}_y$-fibers if $y \notin G_0$ and is a gentle perturbation of $T$;*
3. *$Q$ satisfies Statements (3)–(5) of Proposition 3.1;*
4. *for every $z \in \mathcal{G}$ we have*

$$E_Q^{utab}(z) = E_T^{utab}(z), \quad \det(dQ|E_Q^{utab}(z)) = \det(dT|E_T^{utab}(z)).$$

5. $L_1(Q) < L_2(Q) < L_3(Q) < L_4(Q) = L_4(T)$ *and* $L_5(Q) = 0$ *where $L_i(\cdot)$ is given by* (2.2)*.*

3.4. **Step 4: The Final Perturbation $P$.** Our next step is to perturb the map $Q$ to a map $P$ that is pointwise partially hyperbolic on the open set $\mathcal{G}$, and hence possesses two transversal stable and unstable foliations $W_P^s$ and $W_P^u$ of $\mathcal{G}$. Furthermore, we shall ensure that $P$ has two transversal strongly stable and unstable foliations $W_P^s$ and $W_P^u$ of $\mathcal{G}$ and satisfies the accessibility property on this set via these foliations. We shall also show that $P$ can be constructed in such a way that the Lyapunov exponents of $P$ on $\mathcal{G}^c$ are all zero and that $\int_{\mathcal{M}} \lambda_i(z, P)\, dm > 0$ for $i = 1, 2, 3, 4$.

In order to construct the map $P$ we choose two sequences of open subsets $U_n, \widetilde{U}_n \subset G$, $n = 1, 2, \dots$ such that

(*A4*) $G_0 \subset \widetilde{U}_1$;
(*A5*) $\widetilde{U}_n \subset \overline{\widetilde{U}}_n \subset U_n \subset \overline{U}_n \subset \widetilde{U}_{n+1} \subset G$ and $\bigcup_{n\ge 1} U_n = G$;
(*A6*) $\widetilde{U}_n$ and $U_n$ are connected sets for any $n \ge 1$.

We set

$$\mathcal{U}_n = \mathcal{N} \times U_n, \quad \widetilde{\mathcal{U}}_n = \mathcal{N} \times \widetilde{U}_n. \tag{3.5}$$

We will construct a sequence of diffeomorphisms $\{P_n\}$, whose limit is the desired map $P$. The following statement is proven in Section 5.

**Proposition 3.3.** *Given a number $\delta_P > 0$, one can find two sequences of positive numbers $\{\delta_n\}$ and $\{\theta_n\}$ with $\delta_n \le \delta_P/2^n$ and $\delta_n \le d(C, U_n)^2$ as well as* *a sequence of $C^\infty$ volume preserving diffeomorphisms $P_n : \mathcal{M} \to \mathcal{M}$ such that for $n \ge 1$*

(1) $||P_n - P_{n-1}||_{C^n} < \delta_n$ *and $P_n$ is homotopic to* Id*;*
(2) $P_n(\mathcal{U}_n) = \mathcal{U}_n$*,* $P_n = T$ *on* $\mathcal{M}\setminus\mathcal{U}_n$*, and* $P_n = P_{n-1}$ *on* $\mathcal{U}_{n-2}$*; in particular, $P_n$ is a gentle perturbation of $T$;*
(3) *$P_n$ satisfies Statements (3)–(5) of Proposition 3.1;*
(4) *for every $z \in \mathcal{M}$ we have*

$$E^{utab}_{P_n}(z) = E^{utab}_Q(z), \quad \det(dP_n|E^{utab}_{P_n}(z)) = \det(dQ|E^{utab}_Q(z));$$

(5) *for all $z \in \mathcal{U}_j$, $j = 1, \dots, n$ and $i = u, s, c$,*

$$\angle(E^i_{P_{n-1}}(z), E^i_{P_n}(z)) \le \theta_j/2^{n-j};$$

(6) *if the number $\delta_Q > 0$ (see Proposition 3.2) is sufficiently small, then each map $P_n$ is stably accessible in the following sense: let $P^\natural$ be a $C^2$ volume preserving diffeomorphism of $\mathcal{M}$ that is a gentle perturbation of $T$; assume that for all $z \in \mathcal{U}_n$ and $i = u, s, c$*

$$\angle(E^i_{P^\natural}(z), E^i_{P_n}(z)) \le \theta_n;$$

*then any two points $z_1, z_2 \in \widetilde{\mathcal{U}}_n$ are accessible via a $(u, s)_{P^\natural}$-path in $\mathcal{G}$; in particular, $P_n$ has the accessibility property on $\widetilde{\mathcal{U}}_n$.*

Statement (1) and (2) of this proposition implies that the limit $P = \lim_{n\to\infty} P_n$ exists. We shall show that the map $P$ has all the desired properties.

3.5. **Step 5: Proof of the Main Theorem.** By Proposition 3.3 (1), we have for any $k \ge 1$ and any $n > k$,

$$||P_n - P_{n-1}||_{C^k} \le ||P_n - P_{n-1}||_{C^n} < \delta_P/2^n.$$

It follows that $P_n$ converges to $P$ in the $C^k$ topology. Since $k$ is arbitrary, $P$ is a $C^\infty$ diffeomorphism. Clearly, $P$ preserves volume and $\|P - \mathrm{Id}\| \le \delta$ if $\delta_T$, $\delta_Q$ and $\delta_P$ are small enough. In addition, since $P = P_{n+1}$ on $\mathcal{U}_n$, by Proposition 3.3 (1), $P$ is homotopic to Id on $\mathcal{U}_n$ for any $n$. The first statement of the Main Theorem follows.

By Proposition 3.3, each diffeomorphism $P_n$ is pointwise partially hyperbolic on $\mathcal{U}$ and uniformly partially hyperbolic on $\overline{\mathcal{U}}_n$. By Theorem **??** in the Appendix, if the sequence $\delta_n$ decreases sufficiently fast, the limit diffeomorphism $P$ is pointwise partially hyperbolic on $\mathcal{U}$.

We now claim that the one-dimensional strongly stable $E^s_P$ and unstable $E^u_P$ subbundles are integrable to invariant strongly stable $W^s_P$ and unstable $W^u_P$ foliations with smooth leaves, which are transversal and absolutely continuous. Recall that the "start-up" map $T$ has strongly stable and unstable local manifolds $V_T^s(z)$ and $V_T^u(z)$ respectively at each $z \in \mathcal{U}$. Moreover, these local manifolds are of uniform size, say larger than a certain number $4r > 0$.

By Proposition 3.3(3), $P_n|\mathcal{U}_n^c = T|\mathcal{U}_n^c$, and thus $V_{P_n}^\omega(z) = V_T^\omega(z)$ for all $z \in \mathcal{U}\backslash\mathcal{U}_n$, $\omega = s, u$. On the other hand, each $P_n$ is a perturbation of $P_{n-1}$ on the compact set $\overline{\mathcal{U}}_n$ on which both $P_n$ and $P_{n-1}$ are uniformly partially hyperbolic if $\delta_n$ is sufficiently small. Furthermore, if $r_n$ is the size of $V_{P_n}^\omega(z)$ for $z \in \mathcal{U}_n$, one can arrange that $r_n/r_{n-1} \geq 2^{-1/2^n}$, and thus by induction we obtain that the size of local manifolds for $P_n|\mathcal{U}_n$ is bigger than $r$. Therefore, given $z \in \mathcal{U}$, we obtain that the size of $V_{P_n}^\omega(z)$ has a lower bound $r > 0$, which is independent of $z$ and $n$.

We can describe the local stable manifold at a point $z \in \mathcal{U}$ in the following way

$$V_{P_n}^s(z) = \exp_z\{(v, \psi_{P_n}^s(v)) : \ v \in B_T^s(0, r)\},$$

where $\psi_{P_n}^s : B_T^s(0, r) \to E_T^{cu}(z)$ is a $C^1$ map satisfying $\psi_{P_n}^s(0) = 0$ and $d\psi_{P_n}^s(0) = 0$. The $C^1$-norm of each $\psi_{P_n}^s$ is small provided $\delta_n$ are sufficiently small. We may assume that the sequence of maps $\psi_{P_n}^s$ converges in the $C^1$ topology to a map $\psi_P^s$, so that the local stable manifold through $z$ for $P$ is given by

$$V_P^s(z) = \exp_z\{(v, \psi_P^s(v)) : \ v \in B_T^s(0, r)\}.$$

Clearly, $T_z V_P^s(z) = E_P^s(z)$ and hence $V_P^s(z)$ is a strongly stable manifold of size at least $r$. Ina similar fashion we can obtain strongly unstable local manifolds for $P$. Since $P$ is nonuniformly partially hyperbolic on $\mathcal{U}$, by Theorem 8.6.1 in[1], we obtain that its strongly stable and unstable foliations are absolutely continuous.

We shall now show that the Lyapunov exponent $\lambda_P^s(z)$ in the direction $E_P^s(z)$ is negative at almost every point $z \in \mathcal{G}$. Indeed, let $Z \subset \mathcal{G}$ be the set of points at which $\lambda_P^s(z) = 0$. If $m(Z) > 0$ then

$$\begin{aligned} 0 = \int_Z \lambda_P^s(z)\, dm &= \int_Z \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \lambda_P(P^i(z)) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_Z \sum_{i=0}^{n-1} \log \lambda_P(P^i(z))\, dm(z) \\ &= \int_Z \log \lambda_P(z)\, dm(z) < 0 \end{aligned}$$

(recall that $\lambda_P(z)$ is the contraction coefficient along $E_P^s(z)$). This contradiction proves our claim. Similarly, one can prove that the Lyapunov exponent $\lambda_P^u(z)$ in the direction $E_P^u(z)$ is positive at almost every point $z \in \mathcal{G}$.

Our next step is to show that the map $P$ has the accessibility property on $\mathcal{G}$ via its invariant foliations $W^s_P$ and $W^u_P$. Indeed, by Proposition 3.3 (6), for any $n > k$ and any $z \in \mathcal{U}_k$, $i = s, u, c$,

$$\angle(E^i_{P_n}(z), E^i_{P_k}(z)) \le \theta_k\big(1 - \frac{1}{2^{n-k}}\big) < \theta_k.$$

Taking the limit as $n \to \infty$ yields for $i = s, u, c$ and any $z \in \mathcal{U}_k$,

$$\angle(E^i_P(z), E^i_{P_k}(z)) \le \theta_k. \tag{3.6}$$

Hence, by Proposition 3.3 (6), the map $P$ has the accessibility property on $\widetilde{\mathcal{U}}_k$. Since $k$ is arbitrary, we obtain that the map $P$ has the accessibility property on $\mathcal{G}$.

To prove that the map $P$ has nonzero central Lyapunov exponents almost everywhere we let $c = L_4(Q) - L_3(Q) > 0$. By semicontinuity of $L_i$ with respect to the map, we may take $\delta_P$ in Proposition 3.3 so small that $L_3(P) < L_3(Q) + c/2$. Note that by Proposition 3.3 (4), for all $n \ge 1$,

$$\begin{aligned} L_4(P_n) &= \int_{\mathcal{G}} \log|\det(dP_n|E^{utab}_{P_n}(z))|\,dm \\ &= \int_{\mathcal{G}} \log|\det(dQ|E^{utab}_{Q}(z))|\,dm = L_4(Q). \end{aligned}$$

Since $P_n$ converges to $P$ in the $C^1$ topology, by Proposition 3.3 (4), we have that $L_4(P_n) \to L_4(P)$ as $n \to \infty$ and hence $L_4(P) = L_4(Q)$. It follows that $L_4(P) - L_3(P) \ge c/2 > 0$. Therefore,

$$\int_{\mathcal{G}} \lambda_4(z, P)dm(z) \ge c/2 > 0.$$

It follows that there is a subset $\mathcal{A} \subset \mathcal{G}$ of positive volume such that $\lambda_4(z) > 0$ for every $z \in \mathcal{A}$. Hence, $\lambda_2(z) \ge \lambda_3(z) \ge \lambda_4(z) > 0$. Thus the map $P$ has positive central exponents at every point in a set of positive volume. Since $P$ is volume preserving, the total sum of the Lyapunov exponents is zero at every point. Therefore, $\lambda_5(z, P) < 0$ at every point in $\mathcal{A}$. Since $P$ has the accessibility property and its strongly stable and unstable foliations are absolutely continuous, by Theorem 2.2, we obtain that $P$ has positive central exponents at almost every point in $\mathcal{G}$, $P|\mathcal{G}$ is ergodic and indeed, is a Bernoulli diffeomorphism.

It follows from Proposition 3.3 (3) and the fact that $\delta_n \le d(C, U_n)^2$, that $P = \mathrm{Id}$ on the set $\mathcal{N} \times C$ and that $dP_z = \mathrm{Id}$ for all $z \in \mathcal{N} \times C$. In other words, all Lyapunov exponents at every point in the set $\mathcal{N} \times C$ are zero. Since this set has positive volume this completes the proof of the Main Theorem.

## 4. Construction of the map $Q$: Proof of Proposition 3.2

We use an approach which is similar to the one in [14] and obtain $Q$ as a result of three consecutive perturbations. First, we perturb the map $T$ to a diffeomorphism $S$ via a gentle perturbation $h_S$ so that $S = h_S \circ T$ preserves the fibers $\mathcal{N}_y$, $y \in G$ and has two positive average Lyapunov exponents in the $E_T^{ut}$ subbundle, i.e, $L_1(S) < L_2(S)$ (see Lemma 4.1). Next, we perturb $S$ to a diffeomorphism $R$ via a gentle perturbation $h_R$ so that $R = h_R \circ S$ has three positive average Lyapunov exponents, i.e., $L_1(R) < L_2(R) < L_3(R)$ (see Lemma 4.2). Finally, we obtain the desired map $Q$ as a perturbation of $R$ via a gentle perturbation $h_Q$ so that $Q = h_Q \circ R$ satisfies

$$L_1(Q) < L_2(Q) < L_3(Q) < L_4(Q)$$

(see Lemma 4.6), or equivalently, $\int_{\mathcal{M}} \lambda_4(z, Q)\, dm(z) > 0$.

Given $\delta > 0$ and $k = S, R, Q$, the perturbations $h_k$ are concentrated on pairwise disjoint small open subsets $\Omega_k \subset \mathcal{G}_0$ such that $\|h_k - \mathrm{Id}\,\|_{C^1} \le \delta$ and $h_k = \mathrm{Id}$ outside $\Omega_k$. It follows that $Q = T$ outside $\Omega_S \bigcup \Omega_R \bigcup \Omega_Q$.

To effect our construction we choose periodic points $q$, $p^t$, $p^a$ and $p^b$ of the Anosov automorphism $A$, which are close to each other and whose orbits are pairwise disjoint. Let $V_A^s(q)$, $V_A^u(q)$, $V_A^s(p^i)$ and $V_A^u(p^i)$, $i = t, a, b$ be stable and unstable local manifolds at these periodic points. We may assume that each intersection $V_A^u(q) \cap V_A^s(p^i)$ and $V_A^u(p^i) \cap V_A^s(q)$ consists of exactly one point, which we denote by $[q, p^i]$ and $[p^i, q]$ respectively. Consider the closed quadrilateral path with the collection of points $q$, $[q, p^i]$, $p^i$, $[p^i, q]$ and $q$, and let

$$\gamma(q) = V_A^u(q) \cup V_A^s(q), \quad \gamma(p^i) = V_A^u(p^i) \cup V_A^s(p^i).$$

Given positive numbers $\nu$ and $\sigma$ whose choice will be specified later (see (4.4)), we set for $i = t, a, b$,

$$\begin{aligned} &\Omega^i(\nu) = \Big( \bigcup_{t \in [0, \tau(p^i)]} B_{\mathcal{N}}\big(T^t(p^i, 0), \nu\big) \Big) \times G, \\ &\hat{\Omega}^i(\sigma) = \Big( \bigcup_{(x,t) \in (\gamma(q) \times [0, \tau(q)]) \cup (\gamma(p^i) \times [0, \tau(p^i)])} B_{\mathcal{N}}\big((x,t), \sigma\big) \Big) \times G, \\ &\Omega(\nu, \sigma) = \Big( \bigcup_{i=t,a,b} \Omega^i(\nu) \Big) \cup \Big( \bigcup_{i=t,a,b} \hat{\Omega}^i(\sigma) \Big), \end{aligned} \tag{4.1}$$

where $\tau(q)$ and $\tau(p^i)$ are the periods of $q$ and $p^i$ and $B_{\mathcal{N}}((x,t)), r)$ is the ball in $\mathcal{N}$ of radius $r$ centered at the point $(x, t)$. Finally, we set

$$\Omega_0(\nu, \sigma) = \Omega(\nu, \sigma) \cap \mathcal{G}_0 \tag{4.2}$$

(recall that $G_0$ is defined in (A2) and is in the form of (3.4)).

Given $\delta_Q > 0$, choose the number $\theta > 0$ according to Sublemma 4.5 below and an integer $k_0 > 0$ such that

$$\pi/2k_0 < \theta. \tag{4.3}$$

Now choose positive numbers $\nu$ and $\sigma$ to ensure that the measure of the set $\Omega_0(\nu, \sigma)$ is so small that

$$20k_0 m(\Omega_0(\nu, \sigma)) < 1. \tag{4.4}$$

### 4.1. Construction of the map $S$.

We obtain the map $S$ as a small perturbation of the map $T$ via a perturbation $h_S$, which is a small rotation in the $E_T^{ut}$ subbundle at every point of a small subset of $\mathcal{G}_0 = \mathcal{N} \times G_0$. This approach is an elaboration of the approach developed in [21, 8] for some uniformly partially hyperbolic systems.

To this end we observe that by the construction of the map $T$ for every $z \in \mathcal{G}_0$ the expansion rate in the $E_T^u$-direction at $z$, $|dT|E_T^u|$, is a constant. We denote this constant by $\eta$. Choose a $C^\infty$ function $\psi = \psi(r) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(1) $\psi(r) = \psi_0 > 0$ if $r \in [0, 0.9]$;
(2) $\psi(r) > 0$ if $r \in [0, 1)$ and $\psi(r) = 0$ if $r \geq 1$;
(3) $\|\psi\|_{C^1} \leq 1$.

Given $N_0 \geq 20k_0$, choose a point $(x_0, t_0) \in \mathcal{N}$ and a number $\epsilon_1 > 0$ such that

$$B_{\mathcal{N}}((x_0, t_0), 2\epsilon_1) \cap \operatorname{Proj}_{\mathcal{N}}(\Omega_0) = \emptyset,$$
$$f^{-k\kappa_0}(B_{\mathcal{N}}((x_0, t_0), 2\epsilon_1)) \cap B_{\mathcal{N}}((x_0, t_0), 2\epsilon_1) = \emptyset, \quad k = 1, \ldots, N_0,$$

where $\operatorname{Proj}_{\mathcal{N}}$ is the projection onto $\mathcal{N}$, i.e., $\operatorname{Proj}_{\mathcal{N}}(x, t, y) = (x, t)$ and $\kappa_0$ is defined by (A3) (see Subsection 3.2). Set

$$\Omega_S = B_{\mathcal{N}}((x_0, t_0), \epsilon_1) \times G_0.$$

Our choice of $\epsilon_1$ guarantees that $\Omega_S \cap \Omega_0 = \emptyset$ and for $k = 1, \ldots, N_0$,

$$T^{-k}(\Omega_S) \cap \Omega_S = \emptyset. \tag{4.5}$$

To define the desired map $h_S$ we switch from the coordinate system $(s, u, t, a, b)$ (see (3.2) and (3.3)) in $\Omega_S$ to the cylindrical coordinate system $(r, \theta, s, a, b)$ originated at $z_0 = (x_0, t_0, a_0, b_0)$, where $u = r\cos\theta$ and $t = r\sin\theta$.

Given $\tau > 0$, define the map $h_S = h_{S,\tau}$ on $\Omega_S$ as a small rotation in the $(u, t)$-subspace. More precisely, we set

$$h_S(r, \theta, s, a, b) = \Big(r,\ \theta + \tau\alpha_0^2\epsilon_1^2\psi\big(\frac{r}{\epsilon_1}\big)\psi\big(\frac{|s|}{\epsilon_1}\big)\psi\big(\frac{|b|}{\alpha_0}\big)\psi\big(\frac{|a|}{\alpha_0}\big),\ s, a, b\Big) \tag{4.6}$$

(here $\alpha_0$ is defined in (3.4)). We extend the map $h_S = h_{S,\tau}$ to the whole manifold $\mathcal{M}$ by letting it to be the identity outside of $\Omega_S$. It is easy to see that $h_S$ is a $C^\infty$ volume preserving diffeomorphism satisfying:

(1) $\|h_{S,\tau} - \mathrm{Id}\,\|_{C^1} \to 0$ as $\tau \to 0$;
(2) $dh_S$ preserves $E_T^{ut}$ bundle;
(3) $\det(dh_S|E_T^{ut}(z)) = 1$ for any $z \in \mathcal{M}$.

We define the map $S = S_\tau = T \circ h_{S,\tau}$ and we set

$$\alpha_1 = 0.9\alpha_0, \quad G_1 = B_{F^a}(a_0, \alpha_1) \times B_{F^b}(b_0, \alpha_1). \tag{4.7}$$

The following statement describes some properties of the map $S$.

**Lemma 4.1.** *Given $\delta_Q > 0$, there exist $\tau > 0$ such that the map $S = S_\tau$ is a $C^\infty$ diffeomorphism with the following properties:*

(1) *$\|S - T\|_{C^1} \le \delta_Q$ and $S$ is homotopic to* Id*;*
(2) *$S = T$ on the sets $\mathcal{N} \times (Y \setminus G_0)$ and $\Omega_0$; in particular, $S$ is a gentle perturbation of $T$;*
(3) *$S$ satisfies Statements (3)–(5) of Proposition 3.1;*
(4) *for every $z \in \mathcal{M}$,*
$$E_S^{ut}(z) = E_T^{ut}(z), \quad \det(dS|E_S^{ut}(z)) = \det(dT|E_T^{ut}(z));$$
(5) *for any $y_1, y_2 \in G_1$,*
$$\mathrm{Proj}_{\mathcal{N}}(S(x,t,y_1)) = \mathrm{Proj}_{\mathcal{N}}(S(x,t,y_2));$$
(6) *$L_1(S) < L_1(T)$ and hence,*
$$L_1(S) < L_2(S) = L_3(S) = L_4(S) = L_4(T), \;\; L_5(S) = 0;$$
(7) *there exist a number $\lambda_S$ and a set $\Pi_S = \mathrm{Proj}_{\mathcal{N}}(\Pi_S) \times G_1$ such that*
$$m(\Pi_S) \ge 20k_0 m(\Pi_S \cap \Omega_S) > 0,$$
*and for any $z \in \Pi_S$ the map $S$ has two positive Lyapunov exponents $\lambda_1(z,S) > \lambda_2(z,S) \ge \lambda_S$ along the $E_S^{ut} = E_T^{ut}$ subbundle.*

*Proof.* Statements (1)–(5) follow easily from the construction of the map $h_S$. In particular, $S$ is dynamically coherent in view of Theorem 2.1. It remains to prove Statements (6) and (7).

We prove that there exists $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0]$,

$$L_1(S_\tau|\mathcal{G}_0) < L_1(T|\mathcal{G}_0). \tag{4.8}$$

Since on the complement of $\mathcal{G}_0$ we have $S = T$, this implies that $L_1(S) < L_1(T)$.

We outline the proof of (4.8) referring the reader to the proof of Proposition 5.1 in [8] for details. Since $E^u_{S_\tau}(z)$ is one-dimensional, it is easy to see that

$$L_1(S_\tau|\mathcal{G}_0) = \int_{\mathcal{G}_0} \lambda_1(z, S_\tau)\, dm(z) = \int_{\mathcal{G}_0} \log|dS_\tau(z)|E^u_{S_\tau}(z)|\, dm(z).$$

Since the perturbation $h_S = h_{S,\tau}$ preserves the $E^{ut}_T$ subbundle, we can write

$$dh_{S,\tau}|E^{ut}_T(z) = \begin{pmatrix} A(\tau,z) & B(\tau,z) \\ C(\tau,z) & D(\tau,z) \end{pmatrix},$$

where

$$\begin{aligned}
A &= A(\tau,z) = 1 - \tau r\tilde{\rho}_r \sin\theta\cos\theta - \frac{\tau^2\tilde{\rho}^2}{2} - \tau^2 r\tilde{\rho}\tilde{\rho}_r\cos^2\theta + O(\tau^3),\\
B &= B(\tau,z) = -\tau\tilde{\rho} - \tau r\tilde{\rho}_r\sin^2\theta - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin\theta\cos\theta + O(\tau^3),\\
C &= C(\tau,z) = \tau\tilde{\rho} + \tau r\tilde{\rho}_r\cos^2\theta - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin\theta\cos\theta + O(\tau^3),\\
D &= D(\tau,z) = 1 + \tau r\tilde{\rho}_r \sin\theta\cos\theta - \frac{\tau^2\tilde{\rho}^2}{2} - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin^2\theta + O(\tau^3),
\end{aligned}$$

and

$$\tilde{\rho}(r,s,a,b) = \alpha_0^2\epsilon_1^2\psi(\frac{r}{\epsilon_1})\psi(\frac{|s|}{\epsilon_1})\psi(\frac{|b|}{\alpha_0})\psi(\frac{|a|}{\alpha_0}).$$

Recall that the expanding rate $\eta = \eta_A$ of $dT$ along $E^u_z(T)$ is constant for all $z \in \mathcal{G}_0$. By the choice of the coordinate systems, we can write

$$dT|E^{ut}_T(z) = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $dS_\tau = dT \circ dh_{S,\tau}$, we have

$$dS_\tau(z)|E^{ut}_{S_\tau}(z) = \begin{pmatrix} \eta A(\tau,z) & \eta B(\tau,z) \\ C(\tau,z) & D(\tau,z) \end{pmatrix}.$$

Denote by $e_\tau(z)$ the unique number such that the vector $v_\tau(z) = (1, e_\tau(z))^* \in E^u_{S_\tau}(z)$, where $^*$ denote the transpose of the vector. Repeating the arguments in the proof of Lemma B.7 in [8], one can show that

$$L_1(S_\tau|\mathcal{G}_0) = \int_{\mathcal{G}_0} \log\eta\, dm(z) - \int_{\mathcal{G}_0} \log[D(\tau,z) - \eta B(\tau,z)e_\tau(S_{\tau 0}(z))]\, dm(z).$$

Now we compute the first and second derivatives of $L_1$ with respect to $\tau$. To apply the arguments in [8], we use the Fubini theorem

$$\int_{\mathcal{G}_0} \cdot\ dm(z) = \int_{G_0}\int_{\mathcal{N}_y} \cdot\ dm^{\mathcal{N}}_y(x,t)dm^Y(y),$$

where the fiber $\mathcal{N}_y$ is given by (3.1) and $m_y^{\mathcal{N}}$, $m^Y$ denote the Lebesgue measures on $\mathcal{N}_y$ and on $Y$ respectively. Hence, applying the same arguments in the proof of Lemma B.8 in [8] one can show that

$$\frac{dL_1(S_\tau|\mathcal{G}_0)}{d\tau}\Big|_{\tau=0} = \int_{\mathcal{G}_0} D'_\tau dm(z) = 0$$

and

$$\frac{d^2L_1(S_\tau|\mathcal{G}_0)}{d\tau^2}\Big|_{\tau=0} = \int_{\mathcal{G}_0} \left[(D'_\tau)^2 - D''_{\tau\tau} + 2\eta B'_\tau \frac{\partial e_\tau(z)}{\partial \tau}(S_\tau(z))\right]_{\tau=0} dm(z).$$

Similar to Lemma B.9 in [8], this integral can be wrriten as the sum

$$\int_{\mathcal{G}_0} \left[D'_\tau(0,w)^2 - D''_{\tau\tau}(0,w) + 2B'_\tau(0,w)C'_\tau(0,w)\right] dm(z)$$
$$+ \int_{\mathcal{G}_0} \sum_{i=1}^{\infty} \frac{1}{\eta^i} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))\, dm(z).$$

The first term is bounded above by

$$-(1-\epsilon_1)\int_{\mathcal{G}_0} \tilde\rho^2 dm(z) - \frac{1}{8}\int_{\mathcal{G}_0} r^2\tilde\rho_r^2 dm(z).$$

To estimate the second term, we notice that for any $i \geq 1$, $\epsilon_1 > 0$ and $y \in G_0$,

$$\int_{\Omega_y} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))\, dm_y^Y(x,t) \leq \frac{1}{4}\int_{\Omega_y} \left(\tilde\rho^2 + r^2\tilde\rho_r^2\right) dm_y^Y(x,t),$$

where $\Omega_y = \Omega \cap \mathcal{N}_y$. This allow us to choose $N_0 > 0$ large enough such that for all $y \in G_0$,

$$\sum_{i=N_0}^{\infty} \frac{1}{\eta^i}\int_{\Omega_y} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))\, dm_y^Y(x,t) \leq \frac{1}{10}\int_{\Omega_y} \left(\tilde\rho^2 + r^2\tilde\rho_r^2\right) dm_y^Y(x,t).$$

By (4.5), for $k = 1,\dots,N_0$ we have

$$\int_{\Omega_y} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))\, dm_y^Y(x,t) = 0.$$

We conclude that

$$\frac{d^2L_1(S_\tau|\mathcal{G}_0)}{d\tau^2}\Big|_{\tau=0} \leq -\Big(\frac{9}{10} - \epsilon_1\Big)\int_{\mathcal{G}_0} \tilde\rho^2 dm(z) - \frac{1}{40}\int_{\mathcal{G}_0} r^2\tilde\rho_r^2 dm(z) < 0.$$

It follows that there exists $\tau_0 > 0$ such that (4.8) hold for any $\tau \in (0,\tau_0]$. Therefore, $L_1(S_\tau) < L_1(T)$.

Note that for any $y \in Y$ the fibers $\mathcal{N}_y$ are $S_\tau$-invariant and that the subbundles $E_T^{utab}$, $E_T^{uta}$ and $E_T^{ut}$ are preserved by the perturbation $h_S$. Furthermore, since $\det(dh_{S,\tau}|E_T^{ut}(z)) = 1$, we have for $i = ut, uta, utab$,

$$\det(dS_\tau|E_T^i) = \det(dT|E_T^i).$$

Hence, the three smallest Lyapunov exponents remain unchanged and so does the sum of the two largest ones. This implies that $L_i(S_\tau) = L_i(T)$ for $i = 3, 4, 5$ and hence,

$$L_1(S_\tau) < L_2(S_\tau) = L_3(S_\tau) = L_4(S_\tau) = L_4(T)$$

and $L_5(S_\tau) = 0$. This proves Statement (6) of the lemma.

To prove Statement (7) we first notice that for any $y \in G_1$, the arguments similar to the above ones yield

$$\frac{dL_1(S_\tau|\mathcal{N}_y)}{d\tau}\Big|_{\tau=0} = 0, \qquad \frac{d^2L_1(S_\tau|\mathcal{N}_y)}{d\tau^2}\Big|_{\tau=0} < 0.$$

It follows that if $\tau_0 > 0$ is small enough, then $L_1(S_\tau|\mathcal{N}_y) < L_1(T|\mathcal{N}_y)$ for any $\tau \in (0, \tau_0]$. Let us fix such a $\tau$. There is a subset of $\mathcal{N}_y$ on which $S_\tau$ has two positive Lyapunov exponents $\lambda_1(z, S_\tau) > \lambda_2(z, S_\tau) > 0$. Given $\lambda_S > 0$, consider the level set $\Pi_S(y) = \{z \in \mathcal{N}_y : \lambda_2(z, S_\tau) \geq \lambda_S\}$. If $\lambda_S$ is sufficiently small this set has positive Lebesgue measure. Set $\Pi_S = \Pi_S(y) \times G_1$, where the set $G_1$ is defined by (4.7). Clearly, $\Pi_S$ is invariant under $S_\tau$. Since $N_0 \geq 20k_0$, we obtain by (4.4) that $20k_0 m(\Pi_S \cap \Omega_S) \leq m(\Pi_S)$. Furthermore, by Statement (5) and definition of $\Pi_S$, for any $z \in \Pi_S$ we have that $\lambda_2(z, S_\tau) \geq \lambda_S$ and the lemma follows. □

4.2. **Construction of the map $R$.**

We shall obtain the map $R$ as a small perturbation of the map $S$ by a diffeomorphism $h_S$, i.e., $R = h_R \circ S$. We use some ideas from [2, 9] and construct $h_R$ as a composition of rotations in the $F^{ta}$-subspace along pieces of orbits so that the total rotation is $\pi/2$. This allows us to interchange the $F^t$- and $F^a$-directions making the Lyapunov exponents along these directions to be close to each other.

Let us briefly outline the construction. It starts with a choice of the Rokhlin-Halmos tower for $S$ within an invariant set $\Gamma'$ of positive measure where at every point the map $S$ has two positive Lyapunov exponents along the $E_T^{ut}$-subspace. The tower of height $7K + k_0$ consists of disjoint subsets called floors, where $K > 0$ is a given number and $k_0$ is given by(4.3). We then consider a subtower $\Gamma \subset \Gamma'$ of height $2K + k_0$. The number $K$ should be sufficiently large to ensure that the $k_0$ floors in the middle of $\Gamma$ are disjoint from $\Omega_S$ and $\Omega_0$ and consist of "good" points $z$ in the sense that every vector $v \in E_T^{ut}$-subspace expands by about $e^{i\lambda}$ times under $dS^i$ and contracts by about $e^{-i\lambda}$ under $dS^{-i}$

for any $i \geq K/2$. We then approximate these $k_0$ floors by finitely many sets of a special type – in our global coordinate system these sets are cylinders. We obtain the perturbation $h_R$ as a composition of finitely many maps where each of these maps rotates the core of the corresponding cylinder by the angle $\pi/2k_0$ in the $F^{ta}$-subspace at each level so that the total rotation is $\pi/2$.

Now consider a "good" orbit, which starts at a point $z$ on the bottom of the subtower $\Gamma$, and a vector $v \in E^{uta}(z)$. If $v$ is close to the $E^{ut}$-subspace, then the length of the $ut$-component of $dR^K v = dS^K v$ becomes at least about $e^{K\lambda}$ times longer than the length of $v$. Since $dS$ does not contract vectors in the $E^{uta}$-subspace very much during the remaining $k_0 + K$ steps, the length of the $ut$-component stays about the same. If $v$ is close to the $E^a_T$-subspace, the length of the $a$-component of $v$ does not change under the map $dR^K = dS^K$. During the next $k_0$ iterations the vector $dR^K v$ is rotated by $\pi/2$ degree into the $E^t$-subspace. During the next $K$ iterations the length of the vector becomes at least about $e^{K\lambda}$ times longer. It follows that every vector in $E^{uta}(z)$ expands by about $e^{K\lambda}$ times under $dR^{2K+k_0}$. Thus we obtain a set on which $R$ has three positive Lyapunov exponents.

To effect this construction let $\lambda = \lambda_S$ and $\Pi = \Pi_S$ be as in Statement (7) of Lemma 4.1. Given $K > 0$, let

$$\begin{aligned} \Lambda' = \Lambda'(K) = \{ & z \in \Pi : \log \|dS^k(z,v)\| - k\lambda \geq -0.1k\lambda, \\ & \log \|dS^{-k}(z,v)\| + k\lambda \leq 0.1k\lambda \\ & \text{for all } v \in E^{ut}_S(z),\ \|v\| = 1 \ \text{ and all } |k| \geq 0.5K\}, \end{aligned} \tag{4.9}$$

and let also

$$\Lambda = \Lambda(K) = \bigcap_{i=0}^{k_0-1} S^{-i}(\Lambda'(K)), \tag{4.10}$$

where $k_0 > 0$ is given by (4.3). Note that $m(\Lambda'(K)) \to m(\Pi)$ as $K \to \infty$ and hence, $m(\Lambda(K)) \to m(\Pi)$ as $K \to \infty$. Therefore, given a number $\delta_Q > 0$, we can choose $K$ so large that

$$K\lambda \geq \max\{5k_0\lambda,\ 10\log 2,\ -10k_0 \log(1-\delta_Q)\}, \tag{4.11}$$

$$\lambda m(\Pi) + 40\log(1-\delta_Q) m(\Pi \setminus \Lambda) > 0, \tag{4.12}$$

$$20 m(\Pi \setminus \Lambda) \leq m(\Pi). \tag{4.13}$$

Note that if $z \in \Lambda(K)$ then for $n \geq 0.5K$ and $v \in E^{ut}_S(z)$,

$$\|dS^n(z,v)\| \geq e^{0.9n\lambda}\|v\|.$$

Set

$$\Lambda^* = \Lambda \setminus \bigcup_{i=0}^{k_0-1} S^{-i}(\Omega_0 \cup \Omega_S) \tag{4.14}$$

(recall that $\Omega_0$ and $\Omega_S$ are given by (4.2) and (4.4) respectively). By Lemma 4.1 (7),

$$m(\Omega_S \cap \Pi) \le m(\Pi)/20k_0. \tag{4.15}$$

Furthermore, by choosing the numbers $\nu$ and $\sigma$ in (4.1) appropriately, we may assume that

$$m(\Omega_0 \cap \Pi) \le m(\Pi)/20k_0. \tag{4.16}$$

Combining (4.13), (4.14), (4.16) and (4.15), we find that

$$m(\Lambda^*) \ge ((1-0.05) - 0.05 - 0.05)m(\Pi) \ge 0.8m(\Pi).$$

By the Rokhlin-Halmos Lemma (see [15]), given $K > 0$, one can choose a measurable set $\Gamma' \subset \Pi$ such that $S^i(\Gamma') \cap \Gamma' = \emptyset$ for any $-K \le i \le 6K + k_0 - 1$, $i \ne 0$ and

$$m\Big(\bigcup_{i=-K}^{6K+k_0-1} S^i(\Gamma')\Big) \ge 0.9m(\Pi). \tag{4.17}$$

Set

$$\Gamma_0 = \{S^j(z) : \ z \in \Gamma',\, 0 \le j \le 5K-1,\, S^j(z) \in \Lambda^*,\, S^i(z) \notin \Lambda^* \text{ for } i < j\}.$$

In other words, $\Gamma_0$ is the set of first entries to $\Lambda^*$ of trajectories $\{S^i(z)\}_{i=0}^{5K-1}$ with $z \in \Gamma'$. By Lemma 4.1 (5), both sets $\Lambda$ and $\Pi$ are of the form

$$\Lambda = \mathrm{Proj}_{\mathcal{N}}(\Lambda) \times G_1, \quad \Pi = \mathrm{Proj}_{\mathcal{N}}(\Pi) \times G_1$$

and hence so is the set $\Gamma_0$, i.e., $\Gamma_0 = \mathrm{Proj}_{\mathcal{N}}(\Gamma_0) \times G_1$. Let

$$\Gamma_i = S^i(\Gamma_0), \quad \Gamma = \bigcup_{i=-K}^{K+k_0-1} \Gamma_i. \tag{4.18}$$

Clearly, the sets $\{\Gamma_i\}$ are pairwise disjoint for $i = -K, \dots, K + k_0 - 1$. We approximate the set $\Gamma_0$ by finitely many disjoint sets $\Sigma_{0j}$ of the form

$$\Sigma_{0j} = B_{F^u}(u_j, r'_j) \times B_{F^s}(s_j, r''_j) \times B_{F^{ta}}((t_j, a_j), r_j) \times B_{F^b}(b_0, \alpha_1),$$

where

$$z_j = (u_j, s_j, t_j, a_j, b_j) \in \mathcal{M}, \quad r'_j \ge r_j,\, r''_j \ge r_j \eta^{k_0},\, j = 1, \dots, J.$$

For $i = -K, \dots, K + k_0 - 1$, let

$$\Sigma_{ij} = S^i(\Sigma_{0j}), \qquad \Delta_i = \bigcup_{j=1}^{J} \Sigma_{ij}.$$

We can choose the sets $\Sigma_{0j}$ in such a way that

$$\Sigma_{ij} \cap \Sigma_{kl} = \emptyset$$

for $(i, j) \neq (k, l)$, $-K \leq i, k \leq K + k_0$, $1 \leq j, l \leq J$ and that

$$\Sigma_{ij} \cap (\Omega_0 \cup \Omega_S) = \emptyset$$

for $0 \leq i \leq k_0 - 1$, $0 \leq j \leq J$. It follows that for $i = 1, \dots, k_0$, the set $\Delta_i$ is an approximation of $\Gamma_i$ and $\Gamma_i \cap (\Omega_0 \cup \Omega_S) = \emptyset$. We may assume that for each $i = 0, \dots, k_0$,

$$m(\Gamma_i \triangle \Delta_i) \leq 0.05 \max\{m(\Gamma_i), m(\Delta_i)\}. \tag{4.19}$$

Note that each set $\Sigma_{ij}$ is a cylinder in the form described in Sublemma 4.5 below. Applying this sublemma with $\Delta = \Sigma_{ij}$, we obtain a map $\rho_{ij}$ and a subcylinder $\Sigma'_{ij} \subset \Sigma_{ij}$ such that $\|\rho_{ij} - \mathrm{Id}\| \leq \delta_Q$ and

$$m(\Sigma'_{ij})/m(\Sigma_{ij}) \geq 3/4. \tag{4.20}$$

Furthermore, restricted to $\Sigma'_{ij}$, the map $\rho_{ij}$ is the rotation by the angle $\pi/2k_0$ along the $F^t \times F^a$-subspace and is the identity outside $\Sigma_{ij}$. In fact, by the construction of the sets $\Sigma'_{ij}$ (see Sublemma 4.5 below), we can assume that $S(\Sigma'_{ij}) = \Sigma'_{i+1,j}$ for $i = 0, \dots, k_0 - 1$. Let

$$\Omega_R = \bigcup_{i=0}^{k_0-1} \Delta_i, \quad \Delta'_i = \bigcup_{j=1}^{J} \Sigma'_{ij}. \tag{4.21}$$

Hence, by (4.20) and by definition of $\Delta_i$ and $\Delta'_i$, we have

$$m(\Delta'_i)/m(\Delta_i) \geq 3/4. \tag{4.22}$$

Then define $h_R = \rho_{ij}$ on $\Sigma_{ij}$, and $h_R = \mathrm{Id}$ otherwise. Clearly, $h_R$ is a $C^\infty$ volume preserving diffeomorphism. Moreover, $dh_R$ preserves $E_T^{uta}$ bundle and $\det(dh_R | E_T^{uta}(z)) = 1$ for any $z \in \mathcal{M}$. We define the map $R = h_R \circ S$. Some of the properties of $R$ are described in the following lemma.

**Lemma 4.2.** *The following statements hold:*

1. $\|R - T\|_{C^1} \leq \delta_Q$ *and* $R$ *is homotopic to* Id*;*
2. $R = T$ *on the sets* $\mathcal{N} \times (Y \setminus G_0)$ *and* $\Omega_0$*; in particular,* $R$ *is a gentle perturbation of* $T$*;*
3. $R$ *satisfies Statements (3)–(5) of Proposition 3.1;*

(4) *for any* $(a,b) \in G_0$, *the set* $\mathcal{N} \times I_b$, *where* $I_b = \{(a',b) : a' \in B_{F^a}(a_0,\alpha_0)\}$, *is* $R$*-invariant and for* $y \notin G_0$ *the set* $\mathcal{N}_y$ *is* $R$*-invariant;*

(5) *for every* $z \in \mathcal{M}$,

$$E_R^{uta}(z) = E_S^{uta}(z) = E_T^{uta}(z),$$
$$\det(dR|E_R^{uta}(z)) = \det(dS|E_S^{uta}(z)) = \det(dT|E_T^{uta}(z));$$

(6) *for* $\alpha_2 = 0.9\alpha_1$, $y' = (a,b'), y_2 = (a,b'') \in B_{F^a}(a_0,\alpha_1) \times B_{F^b}(b_0,\alpha_2)$ *we have* $\mathrm{Proj}_{\mathcal{N}\times B_{F^a}(a_0,\alpha_1)} R(x,t,a,b') = \mathrm{Proj}_{\mathcal{N}\times B_{F^a}(a_0,\alpha_1)} R(x,t,a,b'')$, *where* $\mathrm{Proj}_{\mathcal{N}\times B_{F^a}(a_0,\alpha_1)}$ *is the projection onto the set* $\mathcal{N} \times B_{F^a}(a_0,\alpha_1)$ *given by* $\mathrm{Proj}_{\mathcal{N}\times B_{F^a}(a_0,\alpha_1)}(x,t,a,b) = (x,t,a)$;

(7) $L_2(R) < L_3(R)$ *and hence,*

$$L_1(R) < L_2(R) < L_3(R) = L_4(R) = L_4(T), \quad L_5(R) = 0;$$

(8) *there exist a number* $\lambda_R > 0$ *and a subset* $\Pi_R = (\mathrm{Proj}_{\mathcal{N}\times B_{F^a}(a_0,\alpha_1)}\Pi_R) \times B_{F^b}(b_0,\alpha_2)$ *of positive measure such that* $m(\Pi_R) \ge 20k_0 m(\Pi_R \cap \Omega_i)$ *for* $i = R, S$, *and at any* $z \in \Pi_R$, $R$ *has three positive Lyapunov exponents* $\lambda_1(z,R)$, $\lambda_2(z,R)$, $\lambda_3(z,R) \ge \lambda_R$ *along the* $E_R^{uta} = E_T^{uta}$ *subbundle.*

*Proof.* Statements (1)–(6) follows immediately from the construction of $h_R$. In particular, the fact that $\alpha_2 = 0.9\alpha_1$ follows from Statement (4) of Sublemma 4.5.

Now we prove Statements (7) and (8).

Set $\Delta_0^* = \Delta_0' \cap \Lambda$, where $\Delta_0'$ is given by (4.21), and $\Lambda$ is given by (4.10) (we shall see later that $\Delta_0^*$ is not empty and indeed has positive measure). Then set

$$U_1 = R^{-K}\Delta_0^*, \qquad U_2 = \Delta_0 \setminus \Delta_0^*,$$
$$U_3 = R^{k_0}((\Delta_0 \cap \Lambda) \setminus \Delta_0^*), \quad U_4 = R^{k_0}(\Delta_0 \setminus \Lambda).$$

Let $U = U_1 \cup U_2 \cup U_3 \cup U_4$ and $\bar{R} = R^\beta : U \to U$ be the first return map where $\beta = \beta(z)$ is the first return time of the point $z \in U$ to $U$ under $R$. By Poincarẽs Recurrence Theorem, the map $\bar{R}$ is defined for almost every $z \in U$. In the proof below, for any $z \in U$, we shall assume that $v \in E_R^{uta}(z) = E_S^{uta}(z)$.

Let $\wedge^k(E_S^{uta}(z))$ denote the exterior power of $E_S^{uta}(z)$ and

$$\wedge^k(dR|E_S^{uta}(z)) : \wedge^k(E_S^{uta}(z)) \to \wedge^k(E_S^{uta}(R(z)))$$

be the exterior power of $dR|E_S^{uta}(z)$. It is easy to see that if there exists $c \ge 1$ such that $\|dRv\| \ge c\|v\|$ for any $v \in E_S^{uta}(z)$, then

$$\frac{\|\wedge^3(dR|E_S^{uta}(z))\|}{\|\wedge^2(dR|E_S^{uta}(z))\|} \ge c. \tag{4.23}$$

First we consider the case when $z \in U_1$. Then $\beta(z) \geq 2K + k_0$. By Sublemma 4.3 below and (4.11),

$$\log \|d\bar{R}_z v\| \geq 0.9K\lambda - 0.5\log 2 + \log \|v\| \geq 0.85K\lambda + \log \|v\|.$$

Hence,

$$\log \|\wedge^3(dR|E_S^{uta}(z))\| - \log \|\wedge^2(dR|E_S^{uta}(z))\| \geq 0.85K\lambda.$$

Note that by definition, $\Gamma_0 \subset \Lambda^*$. Since $\Sigma_{ij} \cap (\Omega_0 \cup \Omega_S) = \emptyset$ for $0 \leq i \leq k_0 - 1$ and $0 \leq j \leq J$, we have that $\Delta_0' \cap \Lambda = \Delta_0' \cap \Lambda^* \supset \Delta_0 \cap \Gamma_0$. Hence, by (4.22) and (4.19),

$$\begin{aligned} m(U_1) =& m(\Delta_0^*) = m(\Delta_0' \cap \Lambda) \geq m(\Delta_0' \cap \Gamma_0) = m(\Delta_0') - m(\Delta_0' \setminus \Gamma_0) \\ \geq& m(\Delta_0') - m(\Delta_0 \setminus \Gamma_0) \geq \frac{3}{4}m(\Delta_0) - 0.05m(\Delta_0) = 0.7m(\Delta_0). \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{U_1} \left(\log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\| - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z))\|\right) dm(z) \\ \geq& 0.85K\lambda \cdot 0.7m(\Delta_0). \end{aligned} \tag{4.24}$$

Now we consider the case when $z \in U_2$. Note that $\|dR - dT\|_{C^1} \leq \delta_Q$ and$E_R^{uta}(z) = E_R^{uta}(z)$ for all $z$. Then $\bar{R}|U_2 = R^{k_0}|U_2$ and

$$\log \|d\bar{R}_z v\| \geq k_0 \log(1 - \delta_Q) + \log \|v\|.$$

In addition, by definition of $\Delta_0^*$ and (4.22),

$$m(U_2) = m(\Delta_0 \setminus \Delta_0^*) \leq m(\Delta_0 \setminus \Delta_0') \leq \frac{1}{4}m(\Delta_0).$$

We conclude that

$$\begin{aligned} &\int_{U_2} (\log \|\wedge^3(d\bar{R}|E_S^{uta}(z)) - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z))\|\big) dm(z) \\ \geq& k_0 \log(1 - \delta_Q) \cdot 0.25m(\Delta_0). \end{aligned} \tag{4.25}$$

If $z \in U_3$, then $z \in R^{k_0}(\Lambda) \subset \Lambda'$, where $\Lambda'$ is defined in (4.9), and $\beta(z) > K$. Hence, $R^k(z) = S^k(z)$ for $0 \leq k \leq \beta(z)$ and

$$d\bar{R}|E_S^{ut}(z) = dS^{\beta(z)}|E_S^{ut}(z).$$

Therefore if $v \in E_S^{ut}(z)$, then $\|d\bar{R}_z v\| \geq 0.9K\lambda\|v\|$, and if $v \in E_S^a(z)$, then $\|d\bar{R}v\| = \|dS^{\beta(z)}v\| = \|v\|$. It follows that $\|d\bar{R}_z v\| \geq \|v\|$ for any $v \in E_S^{uta}(z)$. Hence, by (4.23) with $c = 1$, we have

$$\int_{U_3} \left(\log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\| - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z)\|\right) dm(z) \geq 0. \tag{4.26}$$

Finally, let us consider the case $z \in U_4$. Let $\beta'(z)$ be the smallest positive integer such that $R^{\beta'(z)}(z) \in \Lambda$ for some $0 \le \beta'(z) \le \beta(z)$ and let $\beta'(z) = \beta(z)$ if there is no such integer. Denote by

$$U_4' = U_4 \cap \{z : \beta(z) - \beta'(z) \ge 0.5K\}, \ U_4'' = U_4 \cap \{z : \beta(z) - \beta'(z) < 0.5K\}.$$

Since $\beta(z) \ge K$ for $z \in U_4''$, we have $\beta(z) \le 2\beta'(z)$. Note that by (4.9), if $n \ge 0.5K$ then $\|dS_z^n v\| \ge \|v\|$ for any $z \in \Lambda$ and $v \in E_S^{uta}(z)$. Also note that $R = S$ on $\Pi \setminus \Omega_R$. If $z \in U_4'$ then

$$\|d\bar{R}_z v\| = \|dR_z^{\beta(z)} v\| = \|dS_{R^{\beta'(z)}(z)}^{\beta(z)-\beta'(z)}(dR_z^{\beta'(z)} v)\| \ge \|dR_z^{\beta'(z)} v)\|.$$

Hence, by Statement (6) of the lemma,

$$\log \|d\bar{R}_z v\| \ge \log \|R_z^{\beta'(z)}(v)\| \ge \beta'(z) \log(1-\delta_Q) + \log \|v\|.$$

If $z \in U_4''$ then

$$\log \|d\bar{R}_z v\| \ge \beta(z) \log(1-\delta_Q) + \log \|v\| \ge 2\beta'(z) \log(1-\delta_Q) + \log \|v\|.$$

It follows that

$$\begin{aligned} &\int_{U_4} \big(\log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\| - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z)\|\big)\, dm(z) \\ \ge& 2\log(1-\delta_Q) \int_{U_4} \beta'(z)\, dm(z). \end{aligned}$$

Furthermore, if $z \in U_4$, then $z, R(z), \dots, R^{\beta'(z)-1}(z) \in \Pi \setminus \Lambda$. Hence, we obtain $\int_{U_4} \beta'(z)\, dm(z) \le m(\Pi \setminus \Lambda)$ and therefore

$$\begin{aligned} &\int_{U_4} \big(\log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\| - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z)\|\big)\, dm(z) \\ \ge& 2\log(1-\delta_Q) m(\Pi \setminus \Lambda). \end{aligned} \tag{4.27}$$

Note that the sets $R^K(U_1)$, $R^{-k_0}(U_3)$ and $R^{-k_0}(U_4)$ form a partition of $\Delta_0$ and hence, by (4.24)–(4.27), we have

$$\begin{aligned} &\int_{U} \big(\log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\| - \log \|\wedge^2(d\bar{R}|E_S^{uta}(z))\|\big)\, dm(z) \\ \ge& 0.595\lambda K m(\Delta_0) + 0.25 k_0 \log(1-\delta_Q) m(\Delta_0) + 2\log(1-\delta_Q) m(\Pi \setminus \Lambda). \end{aligned} \tag{4.28}$$

Using (4.11), and then Sublemma 4.4 and (4.12), we conclude that the right hand side of (4.28) is greater than

$$\begin{aligned} &0.57\lambda K m(\Delta_0) + 2\log(1-\delta_Q) m(\Pi \setminus \Lambda) \\ \ge& 0.0627\lambda m(\Pi) - 0.05\lambda m(\Pi) \ge 0.0127\lambda m(\Pi) > 0. \end{aligned}$$

Hence,

$$\int_U \log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\|\, dm(z) > \int_U \log \|\wedge^2(d\bar{R}|E_S^{uta}(z))\|\, dm(z).$$

Denote $\Pi' = \cup_{i=-\infty}^{\infty} R^i(U)$. Clearly we have

$$\int_U \log \|\wedge^3(d\bar{R}|E_S^{uta}(z))\|\, dm(z) = \int_{\Pi'} \log \|\wedge^3(dR|E_S^{uta}(z))\|\, dm(z)$$
$$= \int_{\Pi'} (\lambda_1(z,R) + \lambda_2(z,R) + \lambda_3(z,R))\, dm(z)$$

and

$$\int_U \log \|\wedge^2(d\bar{R}|E_S^{uta}(z))\|\, dm(z) = \int_{\Pi'} \log \|\wedge^2(dR|E_S^{uta}(z))\|\, dm(z)$$
$$= \int_{\Pi'} (\lambda_1(z,R) + \lambda_2(z,R))\, dm(z).$$

It follows that $L_3(R|\Pi') > L_2(R|\Pi')$, where $L_i$ is defined by (2.2). Since $R = S$ outside $\Pi'$, we obtain that $L_3(R) > L_2(R)$. Furthermore, there is an $R$-invariant subset of $\Pi'$ on which $R$ has three positive Lyapunov exponents. Note that the subbundles $E_T^{utab}$ and $E_T^{uta}$ are preserved by $dS$ and $dR$ and that $\det(dS|E_T^i) = \det(dT|E_T^i)$ for $i = uta, utab$. Hence, the two smallest Lyapunov exponents remain unchanged, and so does the sum of the three largest ones. This implies that $L_i(S_\tau) = L_i(T)$ for $i = 4, 5$ and hence,

$$L_1(R) < L_2(R) < L_3(R) = L_4(R) = L_4(S) = L_4(T) \text{ and } L_5(R) = 0.$$

Statement (7) of the lemma follows.

To prove Statement (8) observe that the above argument applies to the sets

$$\tilde{U} = U \bigcap \mathcal{N} \times B_{F^a}(a_0, \alpha_1) \times B_{F^b}(b_0, \alpha_2)$$

and

$$\tilde{\Pi}' = \Pi' \bigcap \mathcal{N} \times B_{F^a}(a_0, \alpha_1) \times B_{F^b}(b_0, \alpha_2).$$

Denote by $\lambda_1(z,R)$, $\lambda_2(z,R)$ and $\lambda_3(z,R)$ the positive Lyapunov exponents of $z \in \Pi'$. Given $\lambda_R > 0$, consider the level set

$$\Pi_R = \{z \in \tilde{\Pi}' : \lambda_1(z,R), \lambda_2(z,R), \lambda_3(z,R) \geq \lambda_R\}.$$

If $\lambda_R$ is sufficiently small, this set has positive Lebesgue measure. Note that by (4.11), we have $K \geq 5k_0$. Furthermore, by definition of sets $\Gamma'$, $\Gamma_0$ and $\Omega_R$, we have that every piece of an orbit visiting all set $S^i(\Gamma')$ with $-K \leq$

$i \le 6K + k_0 - 1$ consecutively meets $\Omega_R$ exactly $k_0$ times. Moreover, $\Omega_R$ is contained in the union of these $S^i(\Gamma')$. Since $R$ preserves volume, we have that

$$m(\Pi_R) \ge (7K + k_0)m(\Pi_R \cap \Omega_R) > 20k_0 m(\Pi_R \cap \Omega_R).$$

Since $N_0 > 20k_0$, we obtain by (4.5), that $m(\Pi_R) \ge 20k_0 m(\Pi_R \cap \Omega_S)$. This completes the proof of the lemma. □

### 4.3. Sublemmas.

We shall prove now the technical sublemmas used in the previous subsection.

**Sublemma 4.3.** *Let $z \in R^{-K}(\Delta_0^*)$. Then for any $v \in E_S^{uta}(z)$,*

$$\|d\bar{R}_z(v)\| \ge \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.$$

*Proof.* Note that $h_R = \mathrm{Id}$ on $\cup_{i=-K}^{-1}\Gamma_i$ and hence, $R^K(z) = S^K(z)$. Since $dh_S$ preserves the subbundle $E^{ut}(S)$, we have $E_S^{ut}(z) = E_T^{ut}(z)$. Write $v = v^{ut} + v^a$, where $v^{ut} \in E_T^{ut}(z)$ and $v^a \in E_T^a(z)$.

We first consider the case $\|v^a\| \le \frac{\sqrt{2}}{2}\|v\|$. Note that $\|v^{ut}\| \ge \frac{\sqrt{2}}{2}\|v\|$. Since $dS^K v^{ut} \in E_S^{ut}(S^K(z))$ and $S^K(z) \in \Lambda$, using (4.9) and (4.10), we find that

$$\|v^{ut}\| = \|dS^{-K}(dS^K v^{ut})\| \le \|dS^K v^{ut}\|e^{-0.9K\lambda}.$$

Hence,

$$\|dR^K v\| = \|dS^K v\| \ge \|dS^K v^{ut}\| \ge \|v^{ut}\|e^{0.9K\lambda} \ge \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.$$

Note that at $R^K(z), \dots, R^{K+k_0-1}(z)$ the map $dh_R$ is a rotation and that $dS|E_S^{uta}(R^i(z)) = dT|E_T^{uta}(R^i(z))$ is non-contracting for $i = K, \dots, K + k_0 - 1$. Therefore, $dR^{k_0}|E_S^{uta}(R^K(z))$ is non-contracting. Further, since

$$\{R^i(z)\}_{i=K+k_0}^{\beta} \cap \Omega_R = \emptyset$$

and $R^{K+k_0}(z) \in \Lambda'$, we have that the map

$$dR^{\beta-(K+k_0)}|E_S^{ut}(R^{K+k_0}(z)) = dS^{\beta-(K+k_0)}|E_S^{ut}(R^{K+k_0}(z))$$

is expanding and the map

$$dR^{\beta-(K+k_0)}|E_S^{uta}(R^{K+k_0}(z))$$

is non-contracting. It follows that

$$\begin{aligned}\|d\bar{R}v\| =& \|dR^{\beta-(K+k_0)}_{R^{K+k_0}(z)}(dR_z^{K+k_0}v)\| = \|dS^{\beta-(K+k_0)}_{R^{K+k_0}(z)}(dR_z^{K+k_0}v)\| \\ \ge& \|dR_z^{K+k_0}v\| \ge \|dR_z^K v\| = \|dS_z^K v\| \ge \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.\end{aligned}$$

We now consider the case $\|v^a\| \geq \frac{\sqrt{2}}{2}\|v\|$. Note that $dS^K v^a \in E^a_S(S^K(z))$. By construction of $h_R$, we see that $dR^{k_0}_{S^K(z)}$ rotates the vector in $E^{ta}_S(S^K(z)) = E^{ta}_T(S^K(z))$ by $\pi/2$. It means that

$$dR^{K+k_0}v^a = dR^{k_0}(dS^K v^a) \in E^{ut}_S(R^{K+k_0}(z)).$$

Using the fact that $R^{K+k_0}(z) \in \Lambda$ we obtain

$$\begin{aligned}\|d\bar{R}v^a\| =&\|dR^{\beta-(K+k_0)}_{R^{K+k_0}(z)}(R^{K+k_0}(z)v^a\| \geq \|dR^K(dR^{K+k_0}v^a)\| \\ \geq&\|dR^{K+k_0}v^a\|e^{0.9K\lambda} \geq \|v^a\|e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.\end{aligned}$$

This implies the desired result. □

**Sublemma 4.4.** *$m(\Gamma_0) \geq 0.12K^{-1}m(\Pi)$ and hence, $m(\Delta_0) \geq 0.11K^{-1}m(\Pi)$.*

*Proof.* Denote by

$$\hat{\Gamma}' = \bigcup_{i=0}^{5K-1} S^i(\Gamma'), \quad \overline{\Gamma}' = \bigcup_{i=-K}^{6K+k_0-1} S^i(\Gamma')$$

(recall that $\Gamma'$ is given by the Rokhlin-Halmos Lemma in Subsection 4.2). Since $K \geq 5k_0$, we have that

$$\frac{m(\hat{\Gamma}')}{m(\overline{\Gamma}')} = \frac{5K}{7K+k_0} \geq \frac{5K}{7K+0.2K} \geq \frac{50}{72}.$$

By (4.17),

$$m(\hat{\Gamma}') \geq (50/72)\cdot 0.9m(\Pi) = 0.625m(\Pi).$$

For $z \in \Gamma'$ denote by $O(z) = \{Q^i(z) : i = 0,\dots,5K+k_0-1\}$ the piece of the orbit from 0 to $5K-1$ that start at $z$. Let

$$\hat{\Gamma}'_1 = \{O(z) : z \in \Gamma',\, O(z)\cap\Lambda \neq \emptyset\},\ \hat{\Gamma}'_2 = \{O(z) : z \in \Gamma',\, O(z)\cap\Lambda = \emptyset\}.$$

Clearly $\{\hat{\Gamma}'_1, \hat{\Gamma}'_2\}$ forms a partition of $\hat{\Gamma}'$ and $\hat{\Gamma}'_2 \subset \Pi\setminus\Lambda$. Therefore by (4.13),

$$\begin{aligned}m(\hat{\Gamma}'_1) =&m(\hat{\Gamma}') - m(\hat{\Gamma}'_2) \geq m(\hat{\Gamma}') - m(\Pi\setminus\Lambda) \\ \geq&0.625m(\Pi) - 0.025m(\Pi) = 0.6m(\Pi).\end{aligned}$$

Note that $\Gamma_0$ consists of exactly one point from each orbit $O(z)$ in $\hat{\Gamma}_1$. It follows that

$$m(\Gamma_0) \geq \frac{m(\hat{\Gamma}_1)}{5K} \geq \frac{0.6m(\Pi)}{5K} \geq 0.12K^{-1}m(\Pi).$$

By (4.19),

$$m(\Delta_0) \geq m(\Gamma_0) - m(\Gamma_0\setminus\Delta_0) \geq 0.95m(\Gamma_0) \geq 0.11K^{-1}m(\Pi).$$

This is the desired result. □

**Sublemma 4.5.** *For any $\delta > 0$, there is $\theta_0 > 0$ such that for any $\theta \in [0, \theta_0]$, any positive numbers $s, s', s'', s'''$ satisfying $s', s'', s''' \geq s$ and any cylinder $\Delta \subset \mathbb{R}^5$ of the form*

$$\Delta = \Delta_{s,s',s'',s'''} = B_1(z_1, s') \times B_2(z_2, s'') \times B_{34}((z_3, z_4), s) \times B_5(z_5, s''')$$

*there exists a set $\Delta' \subset \Delta$ of the form*

$$\Delta' = \Delta'_{s_0,s'_0,s''_0,s'''_0} = B_1(z_1, s'_0) \times B_2(z_2, s''_0) \times B_{34}((z_3, z_4), s_0) \times B_5(z_5, s'''_0)$$

*and a $C^\infty$ map $\rho : \mathbb{R}^5 \to \mathbb{R}^5$ with the following properties:*

(1) *$\rho = r_\theta$ on $\Delta'$ where $r_\theta$ is the rotation*

$$r_\theta(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, z_3 \cos\theta - z_4 \sin\theta, z_3 \sin\theta + z_4 \cos\theta, z_5);$$

(2) *$\rho = \mathrm{Id}$ outside $\Delta$;*
(3) *$m(\Delta')/m(\Delta) \geq 3/4$;*
(4) *$s_0/s, s'_0/s', s''_0/s'', s'''_0/s''' > 9/10$.*
(5) *$\|\rho - \mathrm{Id}\|_{C^1} \leq \delta$;*

*Proof.* Due to the particular form of our cylinders there is a number $\kappa \in (0, 1/10)$ such that for any $r > 0$ and $r', r'', r''' > r$ we have that

$$\frac{m(\Delta_{r(1-\kappa),r'(1-\kappa),r''(1-\kappa),r'''(1-\kappa)})}{m(\Delta_{rr'r''r'''})} \geq 3/4.$$

Consider a family of $C^\infty$ functions $\zeta_r = \zeta_r(s) : \mathbb{R}^+ \to \mathbb{R}^+$, for $r \geq 1$ such that

(a) $\zeta_1(s) = 1$ if $s \in [0, 1-\kappa]$ and $\zeta_1(s) = 0$ if $s \geq 1$;
(b) $\zeta_r(s) = 1$ if $s \in [0, r-1)$ and $\zeta_r(s) = \zeta_1(s - r + 1)$ if $s \geq r - 1$.

Define the map $\rho$ by $\rho(z) = r_{\theta(\tau,s,s',s'',s''')}(z)$, where

$$\theta(\tau, s, s', s'', s''') = \tau\zeta_{s'/s}(z_1/s')\zeta_{s''/s}(z_2/s'')\zeta_1\Big(\frac{\sqrt{z_3^2 + z_4^2}}{s}\Big)\zeta_{s'''/s}(z_5/s'''),$$

and $r_{(s,s',s'',s''')}$ is given in Condition (1) of the sublemma. By construction, $\rho$ satisfies Statements 1 and 2. Statement 3 and 4 follows from the choice of the number $\kappa$ and the definition of $\zeta_1$ and $\zeta_r$. To obtain Statement 5, we first note that if $\tau = 0$ then $\rho = \mathrm{Id}$ and that the $C^1$ norm of $\rho$ changes smoothly with $\tau$. It is also easy to check that the $C^1$ norm of the rotation is independent of the choice of the size $s$ if $s' = s'' = s''' = s$, and the $C^1$ norm does not increase if we increase $s'$, $s''$ and $s'''$. □

4.4. **Construction of the map $Q$.**

We shall obtain the map $Q$ as a small perturbation of the map $R$ by a diffeomorphism $h_Q$, i.e., $Q = h_Q \circ R$. The construction of $h_Q$ is similar to the construction of the map $h_R$: it is a composition of rotations in the $F^{ba}$-subspace along pieces of orbits so that the total rotation is $\pi/2$.

Let $\lambda = \lambda_R$ and $\Pi = \Pi_R$ be as in Lemma 4.2 (8). Note that for any $z \in \Pi$ the map $R$ has three positive Lyapunov exponents $\lambda_1(z, R)$, $\lambda_2(z, R)$, and $\lambda_3(z, R) \geq \lambda$ along the $E_R^{uta} = E_T^{uta}$ subbundle. Consider the set

$$\begin{aligned}\Lambda' = \Lambda'(K) = \{z \in \Pi : & \log \|dR_z^k v\| - k\lambda \geq -0.1k\lambda, \\ & \log \|dR_z^{-k} v\| + k\lambda \leq 0.1k\lambda, \\ & \text{for all } v \in E^{uta}(z, R), \|v\| = 1, \text{ and all } |k| \geq 0.5K\}\end{aligned}$$

and define the set $\Lambda$ and the number $K > 0$ similar to (4.10)-(4.13). Set

$$\Lambda^* = \Lambda \setminus \bigcup_{i=0}^{k_0-1} R^{-i}(\Omega_0 \cup \Omega_S \cup \Omega_R).$$

Similar to (4.16), we may assume

$$m(\Omega_0 \cap \Pi) \leq m(\Pi)/20k_0.$$

Hence, by the choice of $K$, and Lemma 4.2 (4), we have

$$m(\Lambda^*) \geq ((1 - 0.05) - 0.05 - 0.05 - 0.05)m(\Pi) = 0.8m(\Pi).$$

We then construct the set $\Gamma'$, $\Gamma_0$ in a way similar to the previous subsection and set $\Gamma_i = R^i(\Gamma)$ for $-K \leq i \leq K + k_0 - 1$. Finally, we approximate $\Gamma_0$ by the sets of the form

$$\Sigma_{0j} = B_{F^u}(u, t_j') \times B_{F^s}(s, t_j'') \times B_{F^t}(t, t_j'') \times B_{F^{ab}}((a_j, b_j), r_j),$$

where $r_j', r_j''' \geq r_j$, $r_j'' \geq r_j \eta^{k_0}$ and set for $i = -K, \cdots, K + k_0 - 1$,

$$\Sigma_{ij} = R^i(\Sigma_{0j}), \quad \Delta_i = \bigcup_{j=1}^{J} \Sigma_{ij}.$$

Define $\Omega_Q = \bigcup_{i=0}^{k_0-1} \Delta_i$. Applying Sublemma 4.5 to each set $\Sigma_{ij}$ we obtain a map $\rho_{ij}$ and then set $h_Q = \rho_{ij}$ on each $\Sigma_{ij}$ and $h_Q = \mathrm{Id}$ otherwise. Finally, define $Q = h_Q \circ R$.

**Lemma 4.6.** *The map $Q$ satisfies all the properties stated in Proposition 3.2. In particular, $L_1(Q) < L_2(Q) < L_3(Q) < L_4(Q) = L_4(T)$.*

*Proof.* Statements (1)–(4) of Proposition 3.2 follow from the construction of the map $Q$. The proof that $L_3(Q) < L_4(Q)$ is the same as in the proof of Lemma 4.2.

Note that the subbundle $E_T^{utab}$ is preserved by both $Q$ and $T$ and that both maps $T$ and $Q$ are volume preserving. Hence the smallest Lyapunov exponents remains unchanged, and so does the sum of the four largest ones. It follows that $Q$ has four positive Lyapunov exponents along the $E_R^{utab} = E_T^{utab}$ subbundle on a set of positive measure. □

## 5. Construction of the maps $P_n$: Proof of Proposition 3.3

Recall that the map $Q$ is pointwise partially hyperbolic with one-dimensional stable, one-dimensional unstable and 3-dimensional central subbundles. The stable and unstable subbundles are integrable to (one-dimensional) transversal stable and unstable foliations. The central subbundle corresponds to the flow direction and two directions, $F^a$ and $F^b$, in the $Y$-space and is integrable to a smooth central foliation. However $Q$ does not have the accessibility property: for $(a, b) \notin G_0$ the accessibility class of every point $z = (u, s, t, a, b)$ is the 2-torus $(X, t, a, b)$.

For each $n$, we construct the map $P_n$ to be a sufficiently small gentle perturbation of $Q$ such that $P_n$ has the accessibility property on an invariant open set $\mathcal{U}_n$, and is stably accessible on an open set $\widetilde{\mathcal{U}}_n$ (see (3.5)). These sets are nested and exhaust the set $\mathcal{G}$, and the sequence of maps $P_n$ converges to a map $P$ that is accessible on $\mathcal{G}$. In our construction we use methods similar to those in [8] and [14], and we obtain each $P_n$ as a result of three gentle perturbations $h^t$, $h^a$ and $h^b$ that ensure accessibility in the flow direction and two directions in $Y$ respectively.

### 5.1. Construction of sets $U_n$.

In our construction we will heavily exploit the fact that the 2-torus $Y$ has a global coordinate system. This will enable us to define the sets $U_n$ in an explicit and specific way, which will serve our goal. At this point we regard the 2-torus $Y$ as the square $[0, 8] \times [0, 8]$ whose opposite sides are identified. For each $n \geq 1$, consider the partition of $Y$ into squares

$$\widehat{Z}_{ij}^{(n)} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \qquad i, j = 0, 1, \ldots, 2^{n+3} - 1.$$

Without loss of generality we shall assume that the square $G_0$, constructed in Subsection 3.2, is contained in some $\widehat{Z}_{i_0 j_0}^{(1)}$ so that

$$d(G_0, \partial \widehat{Z}_{i_0 j_0}^{(1)}) \geq 1/2^4 \text{ and } d(C, \widehat{Z}_{i_0 j_0}^{(1)}) > 2$$

(here $C$ is the Cantor set constructed in (A1), see Subsection 3.2).

Consider the open squares

$$Z_{ij}^{(n)} = \Big(\frac{i}{2^n} - \frac{1}{2^{n+2}}, \frac{i+1}{2^n} + \frac{1}{2^{n+2}}\Big) \times \Big(\frac{j}{2^n} - \frac{1}{2^{n+2}}, \frac{j+1}{2^n} - \frac{1}{2^{n+2}}\Big),$$
$$\widetilde{Z}_{ij}^{(n)} = \Big(\frac{i}{2^n} - \frac{1}{2^{n+5}}, \frac{i+1}{2^n} + \frac{1}{2^{n+5}}\Big) \times \Big(\frac{j}{2^n} - \frac{1}{2^{n+5}}, \frac{j+1}{2^n} - \frac{1}{2^{n+5}}\Big).$$

Clearly, these squares have the same center as $\widehat{Z}_{ij}^{(n)}$ and $\widehat{Z}_{ij}^{(n)} \subset \widetilde{Z}_{ij}^{(n)} \subset Z_{ij}^{(n)}$.

For $n \geq 1$ consider the set

$$Y_n = \{y \in Y : d(y, C) \geq 1/2^{n-2}\}.$$

Since $G_0 \subset Y_1$, we let $Y_n'$ be the connected component of $Y_n$ that contains $G_0$. Finally, consider the sets

$$\widehat{U}_1 = \widehat{Z}_{i_0 j_0}^{(1)}, \quad U_1 = Z_{i_0 j_0}^{(1)} \text{ and } \widetilde{U}_1 = \widetilde{Z}_{i_0 j_0}^{(1)}$$

and for $n > 1$,

$$\widehat{U}_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} \widehat{Z}_{ij}^{(n)}, \quad U_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} Z_{ij}^{(n)}, \quad \widetilde{U}_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} \widetilde{Z}_{ij}^{(n)}.$$

It is clear that the sets $U_n$ and $\widetilde{U}_n$ satisfy Conditions (A4)–(A6) in Subsection 3.4.

Let $\widehat{\mathcal{Z}}_n = \{\widehat{Z}_{ij}^{(n)} : \widehat{Z}_{ij}^{(n)} \subset \widehat{U}_n \setminus \widehat{U}_{n-1}\}$ and $\mathcal{Z}_n = \{Z_{ij}^{(n)} : \widehat{Z}_{ij}^{(n)} \in \widehat{\mathcal{Z}}_n\}$. Relabeling elements of $\mathcal{Z}_n$ we shall denote them by $Z_1^{(n)}, \dots, Z_{k_n}^{(n)}$, and we shall use the notations $\widehat{Z}_\ell^{(n)}$ and $\widetilde{Z}_\ell^{(n)}$ for the corresponding squares contained in $Z_\ell^{(n)}$. Thus we have (see Figure 1)

$$U_n = U_{n-1} \cup \Big( \bigcup_{Z_{ij}^{(n)} \in \mathcal{Z}_n} Z_{ij}^{(n)}\Big) = U_{n-1} \cup \Big(\bigcup_{\ell=1}^{k_n} Z_\ell^{(n)}\Big).$$

Clearly, $\widehat{Z}_\ell^{(n)} \cap \widehat{Z}_j^{(m)} = \emptyset$ if $(n, \ell) \neq (m, j)$ and hence, the collection of sets $\{\widehat{Z}_\ell^{(n)} : n = 1, 2, \dots, \ell = 1, \dots, k_n\}$ forms a countable partition of $G$ up to a set of measure 0 while the collection of sets $\{Z_\ell^{(n)} : n = 1, 2, \dots, \ell = 1, \dots, k_n\}$ forms a cover of $G$ of multiplicity at most 4.

Note that the requirement $d(G_0, \partial \widehat{Z}_1^{(1)}) \geq 1/2^4$ yields that $G_0 \cap Z_\ell^{(n)} = \emptyset$ for any $n > 1$ and $\ell = 1, \dots, k_n$.

**Lemma 5.1.** *There is a labeling of the squares $\{Z_\ell^{(n)}\}$ by integers from 1 to 8 such that for any $y \in G$, the labels of the squares $Z_\ell^{(n)}$ containing $y$ are all different. In particular, $Z_1^{(1)}$ can be labelled by 1.*

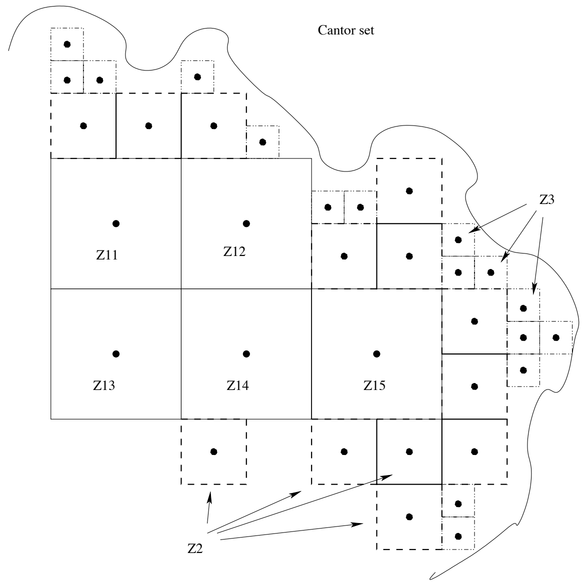


FIGURE 1. Sets $U_n$ and $U_{n+1}$

*Proof.* For each odd number $n > 0$, we use 1, 2, 3, 4 to label the squares $\{Z_{ij}^{(n)}\} \in \mathcal{Z}_n$ in such a way that $Z_{ij}^{(n)}$ and $Z_{kl}^{(n)}$ have the same label if $i \equiv k (\mathrm{mod}\, 2)$ and $j \equiv l (\mathrm{mod}\, 2)$. An alternative way of describing this process is that we first label the 4 squares $\widehat{Z}_{ij}^{(n)}$ inside of some $\widehat{Z}_{kl}^{(n-1)}$ by the numbers 1 to 4, and then translate the square $\widehat{Z}_{kl}^{(n-1)}$ to all other such squares. We then let $Z_{ij}^{(n)}$ have the same labeling as $\widehat{Z}_{ij}^{(n)}$. Clearly, for any $y \in G$, the label of the squares $Z_{ij}^{(n)}$ with $Z_{ij}^{(n)} \ni y$ are all different. Hence, we obtain a labeling on $\mathcal{Z}_n$ by restriction.

For even $n > 0$, we use numbers 5 to 8 to label the squares $\{Z_{ij}^{(n)}\}$ in a similar way. Since any squares $Z_{ij}^{(n)} \in \mathcal{Z}_n$ and $Z_{kl}^{(n+2)} \in \mathcal{Z}_{n+2}$ are disjoint, we obtain the desired labeling. $\square$

5.2. **Construction of maps $P_n$.**

Let $q_j$, $j = 1, \dots, 8$ be eight distinct periodic points of the Anosov automorphism $A$. There is $\epsilon_0 > 0$ such that $B_X(A^l q_j, \epsilon_0) \cap B_X(A^l q_{j'}, \epsilon_0) = \emptyset$ whenever $j \neq j'$ and $l = -1, 0, 1$. For each $q_j$ we choose three distinct periodic points $p_j^t, p_j^a, p_j^b \in B_X(q_j, \epsilon_0/3)$ for $A$. We shall assume that $q_1 = q$ and $p_1^i = p^i$ for $i = t, a, b$ where $q$ and $p^i$ are chosen as in the beginning of Section 3.4.

Denote by $[q_j, p_j^i] = V^u(q_j) \cap V^s(p_j^i)$, $i = t, a, b$ (where $V^s$ and $V^u$ are the stable and unstable local manifolds respectively). For $i = a, b, t$ and $j =$

$1, \dots, 8$ consider the closed quadrilateral $(u,s)_A$-path $\gamma_j^i$ with the collection of points $q_j$, $[q_j, p_j^i]$, $p_j^i$, $[p_j^i, q_j]$, and $q_j$. In the case $n = 1$, we take $\gamma_1^i$, $i = a, b, t$ as introduced in the beginning of Section 4.

Recall that $\eta = \eta_A$ is the expanding rate of $A$ along its unstable direction. Clearly, $\eta^{-1}$ is the contracting rate along the stable direction of $A$. Recall also that $\kappa$ is the function in (A2) such that $\kappa = \kappa_0$ on $U_1$ and $|\operatorname{grad} \kappa| < 1/4$. We have that the expanding rate of $T|\mathcal{N}_y$ along $W_T^u$ is $\eta\kappa(y)$ (here $\mathcal{N}_y$ is given by (3.1)).

For $n \geq 1$ let us choose a rectangle $Z_\ell^{(n)} \in \mathcal{Z}_n$ and assume that it is labelled by a number $j$. Consider the case $n > 1$ and let

$$\eta_- = \eta_-(n, \ell) = \min\{\eta\kappa(y) : y \in Z_\ell^{(n)}\}$$

and

$$(5.1) \qquad \alpha_u^i = \alpha_u^i(j) = d(p_j^i, [p_j^i, q_j]), \qquad \alpha_s^i = \alpha_s^i(j) = d(p_j^i, [q_j, p_j^i])$$
$$\breve{\alpha}_u^i = \breve{\alpha}_u(n, \ell) = \alpha_u^i(j)/\eta_-(n, \ell), \quad \breve{\alpha}_s^i = \breve{\alpha}_s(n, \ell) = \alpha_s^i(j)/\eta_-(n, \ell),$$

where we write $\breve{\alpha}_s(n, \ell)$ instead of $\breve{\alpha}_s(j, n, \ell)$ since $j$ is determined by $n$ and $\ell$ (see Figure 2).

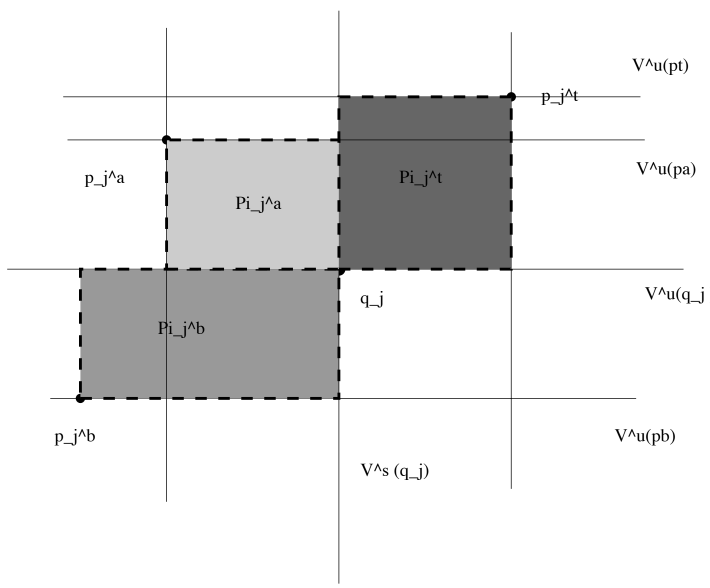


FIGURE 2. Quadrilaterals

Next for $i = t, a, b$ and $j = 1, \dots, 8$ we set

$$\Pi_j^i = B_{F^u}(p_j^i, \alpha_u^i) \times B_{F^s}(p_j^i, \alpha_s^i), \quad \breve{\Pi}_j^i = B_{F^u}(p_j^i, \breve{\alpha}_u^i) \times B_{F^s}(p_j^i, \breve{\alpha}_s^i).$$

We shall assume that the points $p_j^i$ are chosen in such a way that all three rectangles $\Pi_j^i$, $i = t, a, b$, are pairwise disjoint. Hence, all the 24 rectangles $\Pi_j^i$, $i = t, a, b$, $j = 1, \ldots, 8$ are pairwise disjoint.

Finally, we let

$$\epsilon_t = \epsilon_t(n, \ell) = \min\{\kappa(y)/2 : y \in Z_\ell^{(n)}\}, \quad \breve{\epsilon}_t = \breve{\epsilon}_t(n, j) = 5\epsilon_t(n, j)/6. \tag{5.2}$$

In the case $n = 1$, we have $Z_1^{(1)} = U_1$. Choose $l_u^i$ and $l_s^i$ such that

$$A^{-l_u^i}([p_1^i, q_1]) \in B_X(p_1^i, \nu/2), \quad A^{l_s^i}([q_1, p_1^i]) \in B_X(p_1^i, \nu/2),$$

where $\nu > 0$ is given by (4.1). Then we set

$$\alpha_u^i = \alpha_u^i(1) = d(p_j^i, A^{-l_u^i}([p_j^t, q_j])), \quad \alpha_s^i = \alpha_s^i(1) = d(p_j^i, A^{l_s^i}([q_j, p_j^t]))$$

with other quantities and sets to be defined in a similar way.

To effect our construction of the maps $P_n$, in addition to the squares $\widehat{Z}_{ij}^{(n)}$, $\widetilde{Z}_{ij}^{(n)}$ and $Z_{ij}^{(n)}$ constructed in the previous subsection, we need to consider for $n \geq 1$ the following squares:

$$\breve{Z}_{ij}^{(n)} = \Big(\frac{i}{2^n} - \frac{1}{2^{n+3}}, \frac{i+1}{2^n} + \frac{1}{2^{n+3}}\Big) \times \Big(\frac{j}{2^n} - \frac{1}{2^{n+3}}, \frac{j+1}{2^n} - \frac{1}{2^{n+3}}\Big);$$

$$\bar{Z}_{ij}^{(n)} = \Big(\frac{i}{2^n} - \frac{1}{2^{n+4}}, \frac{i+1}{2^n} + \frac{1}{2^{n+4}}\Big) \times \Big(\frac{j}{2^n} - \frac{1}{2^{n+4}}, \frac{j+1}{2^n} - \frac{1}{2^{n+4}}\Big)$$

as well as the following intervals:

$$\breve{I}_n = \breve{J}_n = \left(-\frac{5}{2^{n+3}}, \frac{5}{2^{n+3}}\right), \ \bar{I}_n = \bar{J}_n = (-9/2^{n+4}, 9/2^{n+4}),$$

$$\widehat{I}_n = \left(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right), \ \widetilde{I}_n = \left(-\frac{17}{2^{n+5}}, \frac{17}{2^{n+5}}\right), \ I_n = \left(-\frac{3}{2^{n+2}}, \frac{3}{2^{n+2}}\right)$$

and

$$\breve{K} = \left(-\frac{1}{8}, 1 + \frac{1}{8}\right), \ \bar{K} = (-1/16, 1 + 1/16),$$

$$\widehat{K} = (0, 1)\,, \ \widetilde{K} = (-1/32, 1 + 1/32)\,, \ K = (-1/4, 1 + 1/4)\,.$$

We have that

$$\widehat{Z}_{ij}^{(n)} \subset \widetilde{Z}_{ij}^{(n)} \subset \bar{Z}_{ij}^{(n)} \subset \breve{Z}_{ij}^{(n)} \subset Z_{ij}^{(n)}$$

with similar relations for $I_n$ and $J_n$.

Fix $n \geq 1$ and choose $C^\infty$ functions $\phi^i$ and $\psi^i$ on $\mathbb{R}$ for $i = a, b, t$ satisfying:

- $\phi^i(r) = \text{const.}$ for $r \in (-\breve{\alpha}_u^i, \breve{\alpha}_u^i)$ and $\psi^i(r) = \text{const}$ for $r \in (-\breve{\alpha}_s^i, \breve{\alpha}_s^i)$;
- $\phi^i(r) = 0$ for $|r| \geq \alpha_u^i$, and $\psi^i(r) = 0$ for $|r| \geq \alpha_s^i$;
- $\int_0^{\pm\alpha_u^i} \phi^i(\tau)d\tau = 0$, and $\psi^i(x) > 0$ for any $|x| < \alpha_s^i$;
- $\|\phi^i(\cdot)\|_{C^n} < 1$ and $\|\psi^i(\cdot)\|_{C^n} < 1$.

Further, choose $C^\infty$ functions $\xi_t$ and $\xi_Y$ supported on $K$ and $I_n$ respectively such that:

- $\xi_t(r) = \text{const.}$ for $r \in \breve{K}$, and $\xi_Y(r) = \text{const.}$ for $r \in \breve{I}_n$;
- $\xi_t(r) > 0$ for $r \in K$ and $\xi_Y(r) > 0$ for $r \in I_n$;
- $\xi_t(r) = 0$ for $r \notin K$ and $\xi_Y(r) = 0$ for $r \notin I_n$;
- $\|\xi_t\|_{C^n}, \|\xi_Y\|_{C^n} < 1$.

Finally, choose $C^\infty$ functions $\zeta_t$ and $\zeta_Y$ supported on $(-\epsilon_t, \epsilon_t)$ and $I_n$ respectively such that:

- $\zeta_t(r) = \text{const.}$ for $r \in (-\breve{\epsilon}_t, \breve{\epsilon}_t)$ and $\zeta_Y(r) = \text{const.}$ for $r \in \breve{I}_n$;
- $\zeta_t(r) > 0$ for $r \in (\epsilon_t, \epsilon_t)$ and $\zeta_Y(r) > 0$ for $r \in I_n$;
- $\zeta_t(r) = 0$ for $r \notin (\epsilon_t, \epsilon_t)$ and $\zeta_Y(r) = 0$ for $r \notin I_n$;
- $\|\zeta\|_{C^n} < 1$.

Let $(a_0, b_0) = (a_0(n, \ell), b_0(n, \ell))$ be the center of the square $Z_\ell^{(n)}$.

In this section we shall use the coordinate system $z = (u, s, t, a, b) = (x, t, a, b)$ introduced in (3.2) and (3.3) with the origin at $(p_j^a, 1/2, a_0, b_0)$. In this coordinate system the interval $K$ is in the symmetric form $(-3/4, 3/4)$. Define

$$\Omega^a = \Omega^a_{n,\ell} = \{z = (x, r, \hat{a}, \hat{b}) :\ x \in \Pi_j^a,\ |r| \le \epsilon_t,\ (\hat{a}, \hat{b}) \in Z_\ell^{(n)}\}$$

(recall that $j$ labels the square $Z_\ell^{(n)}$) and for each $\beta > 0$ a vector field $X^a = X^a_{\beta,n,\ell}$ by

$$X^a(z) = \beta\zeta_Y(\hat{b})\zeta_t(r)\psi^a(s)\Big(-\xi_Y'(\hat{a})\int_0^u \phi^a(\tau)d\tau,\ 0,\ 0,\ \xi_Y(\hat{a})\phi^a(u),\ 0\Big), \tag{5.3}$$

(here $\xi_Y'$ denotes the derivative of $\xi_Y$). The choice of $\epsilon_t$ guarantees that $T(\Omega^a) \cap \Omega^a = \emptyset$. It is clear that $X^a$ is constant on the set

$$\breve{\Omega}^a = \{z = (x, r, \hat{a}, \hat{b}) :\ x \in \breve{\Pi}_j^a,\ |r| \le \breve{\epsilon}_t,\ (\hat{a}, \hat{b}) \in \breve{Z}^{(n)}\}.$$

We define the map $h^a_{n,\ell} = h^a_{\beta,n,\ell}$ on $\Omega^a$ to be the time-1 map of the flow generated by $X^a$, and we set $h^a_{n,\ell} = \text{Id}$ on the complement of $\Omega^a$. It is easy to see that the vector field $X^a$ is divergence free, the differential $dh^a_{n,\ell}$ preserves $E_T^{ua}$, and $\det(dh^a_{n,\ell}|E_T^{ua}(z)) = 1$.

Then we use the same coordinate system as above but with the origin at $(p_j^b, 1/2, a_0, b_0)$. Define

$$\Omega^b = \Omega^b_{n,\ell} = \{z = (x, r, \hat{a}, \hat{b}) :\ x \in \Pi_j^b,\ |r| \le \epsilon_t,\ (\hat{a}, \hat{b}) \in Z_\ell^{(n)}\}$$

and for each $\beta > 0$ a vector field $X^b = X^b_{\beta,n,\ell}$ by

$$X^b(z) = \beta\zeta_Y(\hat{a})\zeta_t(r)\psi^b(s)\Big(-\xi_Y'(\hat{b})\int_0^u \phi^b(\tau)d\tau,\ 0,\ 0,\ 0,\ \xi_Y(\hat{b})\phi^b(u)\Big). \tag{5.4}$$

Let $h^b_{n,\ell} = h^b_{\beta,n,\ell}$ on $\Omega^b$ be the time-1 map of the flow generated by $X^b$ and let $h^b_{n,\ell} = \mathrm{Id}$ on the complement of $\Omega^b$. It is clear that $X^b$ is divergence free, $dh^b_{n,\ell}$ preserves $E^{ub}_T$, and $\det(dh^b_{n,\ell}|E^{ub}_T(z)) = 1$.

Now we use the coordinate system but with the origin at $(p^t_j, 1/2, a_0, b_0)$. Define

$$\Omega^t = \Omega^t_{n,\ell} = \{z = (x, r, y) : \ x \in \Pi^t_j,\ r \in K,\ y \in Z^{(n)}_\ell\}$$

and for each $\beta > 0$ a vector field $X^t = X^t_{\beta,n,\ell}$ by

$$X^t(z) = \beta\zeta_Y(a)\zeta_Y(b)\psi^t(s)\Big(-\xi'_t(r)\int_0^u \phi^t(\tau)d\tau,\ 0,\ \xi_t(r)\phi^t(u),\ 0,\ 0\Big). \tag{5.5}$$

We define the map $h^t_{n,\ell} = h^t_{\beta,n,\ell}$ on $\Omega^t$ to be the time-1 map of the flow generated by $X^t$, and we set $h^t_{n,\ell} = \mathrm{Id}$ on the complement of $\Omega^t$. Obviously, $X^t$ is divergence free, $dh^t_{n,\ell}$ preserves $E^{ut}_T$ and $\det(dh^t_{n,\ell}|E^{ut}_T(z)) = 1$.

Our construction guarantees that all $\{Q^i_{n,\ell}\}$ are pairwise disjoint. For $n = 1, 2, \dots$ define $h_n = h_{\beta,n}$ by

$$h_{\beta,n} = h^b_{\beta,n,k_n} \circ h^a_{\beta,n,k_n} \circ h^t_{\beta,n,k_n} \circ \cdots \circ h^b_{\beta,n,1} \circ h^a_{\beta,n,1} \circ h^t_{\beta,n,1}.$$

Then we let $P_1 = h_{\beta_1,1,1} \circ Q$ and define $P_n$ inductively by setting $P_n = h_{\beta_n,n} \circ P_{n-1}$ for some suitable choice of $\{\beta_n\}$ which will be determined inductively later.

### 5.3. **Properties of maps $P_n$: Proof of Proposition 3.3.**

Statements (2) and (4) of Proposition 3.3 and the fact that the map $P_n$ is homotopic to the identity follow directly from the construction.

Note that the unperturbed map $T$ is uniformly partially hyperbolic on each set $\mathcal{U}_n$ with smooth 3-dimensional central foliation and is dynamically coherent. Note also that for each $n > 0$, by choosing $\beta_n$ in (5.3)-(5.5) sufficiently small, we can ensure that $\|h_n - \mathrm{Id}\|_{C^n}$ is arbitrarily small. Hence, we can choose a positive sequence $\{\delta'_n\}$ such that $\delta'_n \le \delta'_1/2^{n-1}$ and if $h_n$ and $P_n$ satisfy

$$\|P_n - P_{n-1}\|_{C^n} \le \delta'_n \text{ and } \|h_n - \mathrm{Id}\|_{C^n} \le \delta'_n, \tag{5.6}$$

then Statement (3) of the proposition holds. In particular, $P_n$ is pointwise partially hyperbolic on an open set $\mathcal{G}$; it is uniformly partially hyperbolic on $\mathcal{U}_n$ with 3-dimensional central foliation and is dynamically coherent. It remains to show how to choose sequences of positive numbers $\delta_n$ and $\theta_n$ such that $P_n$ also satisfies Statements (5) and (6) of the proposition.

We denote by $W^c_{P_n}(z)$ the center manifold of $P_n$ at the point $z \in \mathcal{M}$. Suppose a square $Z^{(n)}_\ell$ is labelled by a number $j$. Let $q_j$ be the periodic point chosen as in the previous subsection and $z_0 = z_0(n,\ell) = (q_j, 1/2, a_0(n,\ell), b_0(n,\ell))$. We denote by $W^c_{P_n}(z_0, K, Z^{(n)}_\ell)$ the connected component of $W^c_{P_n}(z_0) \cap (X \times K \times$

$Z_\ell^{(n)}$) that contains $z_0$. We shall also use similar notations $W^c_{P_n}(z_0, \breve{K}, \breve{Z}_\ell^{(n)})$, etc. Note that for all $\ell$ and $n$,

$$W^c_{P_n}(z_0, K, Z_\ell^{(n)}) = W^c_Q(z_0, K, Z_\ell^{(n)}) = W^c_T(z_0, K, Z_\ell^{(n)}).$$

Recall that $\gamma^i_j$ is the quadrilateral $(u,s)_A$-path with the collection of points $q_j$, $[q_j, p^i_j]$, $p^i_j$, $[p^i_j, q_j]$, and $q_j$ (for $i = a, b, t$ and $j = 1, \dots, 8$) introduced in the beginning of Subsection 5.2. In particular, $\gamma^i_1 = \gamma^i$ is given in the beginning of Section 4.

For any $n \geq 1$, $\ell = 1, \dots, k_n$, and $j = 1, \dots, 8$ such that the label of $Z^n_\ell$ is $j$, we consider a quadrilateral $(u,s)_{P_n}$-path $\widehat{\gamma}^a_j$ with the initial point $z_1$ such that $\mathrm{Proj}_X \widehat{\gamma}^a_j = \gamma^a_j$. More precisely, $\widehat{\gamma}^a_j = \{z_1, \dots, z_5\}$ where

$$\begin{aligned} z_2 &= V^u_{P_n}(z_1) \cap V^{sc}_{P_n}(p^a_j, 1/2, a_0, b_0), \\ z_3 &= V^s_{P_n}(z_2) \cap V^{uc}_{P_n}(p^a_j, 1/2, a_0, b_0), \\ z_4 &= V^u_{P_n}(z_3) \cap V^{sc}_{P_n}(z_1), \\ z_5 &= V^s_{P_n}(z_4) \cap V^{uc}_{P_n}(z_1). \end{aligned} \tag{5.7}$$

This path defines a map $\Theta = \Theta^a = \Theta^a_{n,\ell,P_n}$, given by $\Theta(z_1) = z_5$. Note that $z_4 \in V^{sc}_{P_n}(z_1)$ and $z_5 \in V^s_{P_n}(z_4)$. Hence, $z_5 \in V^{sc}_{P_n}(z_1)$. Since also $z_5 \in V^{uc}_{P_n}(z_1)$, we obtain that $z_5 \in V^c_{P_n}(z_1)$. This implies that $\Theta$ maps $W^c_{P_n}(z_0, K, Z^n_\ell)$ into itself.

We contract the $(u,s)_{P_n}$-path $\widehat{\gamma}^a_j$ to a line segment. Namely, let $\sigma : [0,1] \to V^u_{P_n}(z_1)$ be a parametrization by the arc length of the part of the curve $V^s_{P_n}(z_1)$ from $z_1$ to $z_2$ so that $\sigma(0) = z_1$ and $\sigma(1) = z_2$. For each $\tau \in [0,1]$, the new path $\widehat{\gamma}^a_j(\tau) = \{z_1(\tau), \dots, z_5(\tau)\}$ is such that $z_1(\tau) = z_1$, $z_2(\tau) = \sigma(\tau)$ and $z_i(\tau)$ for $i = 3, 4, 5$ are obtained in the way similar to (5.7). Thus we obtain a map $\Theta_\tau = \Theta^a_{\tau,n,\ell,P_n}$, given by $\Theta_\tau(z_1) = z_5$. It maps $W^c_{P_n}(z_0, K, Z^n_\ell)$ into $W^c_{P_n}(z_0)$ and depends continuously on $\tau \in [0,1]$.

Clearly, $\widehat{\gamma}^a_j(1) = \widehat{\gamma}^a_j$ and hence, $\Theta^a_{1,n,\ell,P_n} = \Theta^a_{n,\ell,P_n}$. Furthermore, the path $\widehat{\gamma}^a_j(0)$ degenerates to a path on $V^s_{P_n}(z_1)$ that starts from $z_1 = z_2$, goes to $z_3 = z_4$ and then returns to $z_5 = z_1$. Hence, $\Theta_0 = \mathrm{Id}$.

We stress that $\Theta^a_{n,\ell,P_n}$ depends only on $h^a_{n,\ell}$, since $\widehat{\gamma}^a_j$ consists of stable and unstable leaves of $(q_j, 1/2, y)$ and $(p^a_j, 1/2, y)$ with $y \in Z^{(n)}_\ell$ that are not perturbed by any other perturbations $h^a_{n',\ell'}$ if $(n', \ell') \neq (n, \ell)$. On the other hand, if $\tau \in (0,1)$, then $\Theta^a_{\tau,n,\ell,P_n}$ may depend on other perturbations $h^a_{n',\ell'}$.

By using the paths $\gamma^b_j$ and $\gamma^t_j$ respectively, we can define the maps $\Theta^b_\tau = \Theta^b_{\tau,n,\ell,P_n}$ and $\Theta^t_\tau = \Theta^t_{\tau,n,\ell,P_n}$ for $\tau \in [0,1]$ in a similar way. Furthermore, for any gentle perturbation $P^\natural$ of $P_n$ we can also construct the maps $\Theta^a_{n,\ell,P^\natural}$ and $\Theta^a_{\tau,n,\ell,P^\natural}$ from $W^c_{P^\natural}(z_0, K, Z^n_\ell)$ to itself. Clearly, they have properties similar to those of the maps $\Theta^a_{n,\ell,P_n}$ and $\Theta^a_{\tau,n,\ell,P_n}$. Note that $V^u_{P^\natural}$, $V^s_{P^\natural}$ and $V^c_{P^\natural}$ depend continuously on the perturbation $P^\natural$ as long as $P^\natural$ is a gentle perturbation of $T$ with $P^\natural = T$ outside some fixed $U_n$ and with $\angle(E^i_{P^\natural}(z), E^i_T(z))$ sufficiently small for all $z \in \mathcal{U}_n$ and $i = u, s, c$. It follows that $\Theta^i_{n,\ell,P^\natural}$ and $\Theta^i_{\tau,n,\ell,P^\natural}$, for $i = u, s, c$ depend continuously on $P^\natural$ as well. Since the lengths of all the quadrilateral paths used in the construction of the maps $\Theta^i$ and $\Theta^i_\tau$ are uniformly bounded from above, the continuity is uniform with respect to $z$.

Given $j = 1, \dots 8$ and a point $z = (x, t, y)$, we can find a $(u, s)_T$-path $\gamma_T(z)$ connecting $z$ to the point $z' = (q_j, t, y)$ whose length does not exceed $2d(x, q_j)$ (indeed, such a path can be constructed by using at most three points $z$, $z_1$ and $z'$). This generates a map $\Psi_T = \Psi_{T,j}$ from $\mathcal{G}$ to $\{q_j\} \times K \times G$ given by $\Psi_T(z) = z'$.

Furthermore, given a gentle perturbation $P^\natural$ of $T$ and a point $z \in Z^{(n)}_\ell$, we can find a $(u, s)_{P^\natural}$-path $\gamma_{P^\natural}(z)$, which is close to $\gamma_T(z)$ and connect $z$ to a point $z' = z'(P^\natural) \in W^c_{P^\natural}(z_0(n, \ell), K, Z^{(n)}_\ell)$ and we can then define $\Psi_{P^\natural}(z) = z'(P^\natural)$. Again the path can be chosen to consists of at most three point $z$, $z_1 = z_1(P^\natural)$ and $z' = z'(P^\natural)$, and both $z_1(P^\natural)$ and $z'(P^\natural)$ depend continuously on $P^\natural$. Hence, $\Psi_{P^\natural}$ depends continuously on $P^\natural$ as long as $P^\natural$ is a gentle perturbation of $T$ with $P^\natural = T$ outside some fixed $U_n$ and with $\angle(E^i_{P^\natural}(z), E^i_T(z))$ sufficiently small for all $z \in \mathcal{U}_n$ and $i = u, s, c$. We stress that the lengths of all the paths used in the construction of the map $\Psi$ are uniformly bounded from above for all $z$ and all gentle perturbations $P^\natural$. In particular, the continuity is uniform with respect to $z$.

Given a set $\Gamma \subset \mathcal{M}$ and a gentle perturbation $P^\natural$ of $T$, let

$$\mathcal{A}_{P^\natural}(\Gamma) = \{z \in \mathcal{M} : \text{there is } y \in \Gamma \text{ such that } y \text{ is accessible to } z \text{ via a } (u, s)_{P^\natural}\text{-path}\}. \tag{5.8}$$

For $n \geq 1$ denote by $\epsilon_n = \min\{1/2^{n+5}, \breve{\epsilon}_t(n, \ell), \ell = 1, \dots, k_n\}$ where $\breve{\epsilon}_t(n, \ell)$ is given by (5.2).

We shall now show how to choose the sequence $\{\delta_n\}$. Recall that $U_1 = Z^{(1)}_{i_0 j_0} = Z^{(1)}_1$ and $\widetilde{U}_1 = \widetilde{Z}^{(1)}_1$. We can choose a number $\theta_0 > 0$ such that for any gentle perturbation $P^\natural$ of $T$ with $\angle(E^i_{P^\natural}(z), E^i_T(z)) \leq 2\theta_0(z)$ for $i = s, c, u$ and $z \in \mathcal{U}_1$ the maps $\Psi_{P^\natural}$ and $\Theta^i_{\tau,1,1,P^\natural}$ are well defined. We also assume that the number $\delta_Q$ in Proposition 3.2 is so small that the map $P_0 = Q$ satisfies $\angle(E^i_{P_0}(z), E^i_T(z)) \leq \theta_0$ and $d(\Theta^i_{\tau,1,1,P_0}(z), z) \leq \epsilon_1/4$ for $z \in \mathcal{G}_0$, $\tau \in [0, 1]$ and $i = s, c, u$.

Now we choose a number $\theta_1$ such that $0 < \theta_1' \le \theta_0/2$ and if $\angle(E^i_{P^\natural}(z), E^i_{P_0}(z)) \le 2\theta_1'$ for $i = s, c, u$ and $z \in \mathcal{N} \times Z_1^{(1)}$, then

$$d(\Psi_{P^\natural}(z), \Psi_{P_0}(z)) \le 1/2^8, \quad z \in \mathcal{N} \times Z_1^{(1)}. \tag{5.9}$$

Finally, we may assume that the number $\delta_1'$ in (5.6) is chosen so small that if $\|P_1 - P_0\| \le \delta_1'$, then $\angle(E^i_{P_1}(z), E^i_{P_0}(z)) \le \theta_1'$ for $i = s, c, u$ and $z \in \mathcal{N} \times Z_1^{(1)}$.

Now we set $\delta_1 = \min\{\delta_1', \delta_1''\}$ and $\theta_1 = \min\{\theta_1', \theta_1''\}$ where the numbers $\delta_1''$ and $\theta_1''$ are given by Lemma 5.2 below. For any gentle perturbation $P^\natural$ of $P_1$ with $\angle(E^i_{P^\natural}(z), E^i_{P_1}(z)) \le \theta_1'$ for $i = s, c, u$ and $z \in \mathcal{N} \times Z_1^{(1)}$, we have $\angle(E^i_{P^\natural}(z), E^i_{P_0}(z)) \le 2\theta_1'$ and therefore (5.9) holds. Since $d(\Theta^i_{\tau,1,1,P_0}(z), z) \le \epsilon_1/4$, we can apply Lemma 5.2 to obtain that $d(\Theta^i_{\tau,2,\ell,P_1}(z), z) \le \epsilon_2/4$ for all $z \in W^c_{P_1}(z_0(2,\ell), K, Z_\ell^{(2)})$, $i = u, s, c$, $\tau \in [0,1]$ and $\ell = 1, \dots, k_2$. Moreover,

$$\mathcal{A}_{P^\natural}(z_0) \supset W^c(z_0(1,1), \bar{K}, \bar{Z}_1^{(1)}).$$

Since the distance between the boundaries $\partial\bar{Z}_1^{(1)}$ and $\partial\widetilde{Z}_1^{(1)}$ is $1/2^6$, (5.9) implies that

$$\Psi_{P^\natural}(\mathcal{N} \times \widetilde{Z}_1^{(1)}) \subset W^c_{P^\natural}(z_0(1,1), \bar{K}, \bar{Z}_1^{(1)}).$$

By definition, $z$ and $\Psi_{P^\natural}(z)$ are $(u,s)_{P^\natural}$-accessible and hence, we have that

$$\mathcal{A}_{P^\natural}(z_0(1,1)) \supset \mathcal{N} \times \widetilde{Z}_1^{(1)}.$$

In particular, for $P^\natural = P_1$, the inclusion holds and so does (5.9).

Proceeding inductively, we assume that for $j = 1, \dots, n-1$, the maps $P_j$, and the numbers $\delta_j$ and $\theta_j$ are chosen such that (5.6) and Statements (5) and (6) of the proposition hold. Moreover, we assume that for all $i = u, s, c$, $\tau \in [0,1]$, $\ell = 1, \dots, k_{j+1}$,

$$d(\Psi_{P_j}(z), \Psi_{P_{j-1}}(z)) \le 1/2^{j+7} \quad \text{for all } z \in \mathcal{N} \times Z_\ell^{(j)}, \tag{5.10}$$

$$d(\Theta^i_{\tau,j+1,\ell,P_j}(z), z) \le \epsilon_{j+1}/4 \quad \text{for all } z \in W^c_{P_j}(z_0(j+1,\ell), K, Z_\ell^{(j+1)}). \tag{5.11}$$

Now we choose $0 < \theta_n' \le \theta_{n-1}/2$ in such a way that for any gentle perturbation $P^\natural$ of $P_{n-1}$, if $\angle(E^i_{P^\natural}(z), E^i_{P_{n-1}}(z)) \le 2\theta_n'$ for $i = u, s, c$, $z \in \mathcal{N} \times Z_\ell^{(n)}$, and $\ell = 1, \dots, k_n$ then

$$d(\Psi_{P^\natural}(z), \Psi_{P_{n-1}}(z)) \le 1/2^{n+7} \tag{5.12}$$

for all $z \in \mathcal{N} \times Z_\ell^{(n-1)}$ and $\ell = 1, \dots, k_n$. Reducing $\delta_n'$ in (5.6) further if necessary we may assume that if $\|P_n - P_{n-1}\| \le \delta_n'$ then $\angle(E^i_{P_n}(z), E^i_{P_{n-1}}(z)) \le \theta_n'$ for $i = u, s, c$ and $z \in \mathcal{U}_n$. Then we take $\delta_n = \min\{\delta_n', \delta_n''\}$ and $\theta_n = \min\{\theta_n', \theta_n''\}$, where $\theta_n''$ and $\delta_n''$ are given in Lemma 5.2.

Since $0 < \theta_n \le \theta_{n-1}/2$, Statement (5) of the proposition holds.

Let $P^\natural$ be a gentle perturbation of $P_n$ such that $\angle(E^i_{P^\natural}(z), E^i_{P_n}(z)) \le \theta_n$ for $i = u, s, c$ and $z \in \mathcal{U}_n$. Then $\angle(E^i_{P^\natural}(z), E^i_{P_{n-1}}(z)) \le 2\theta'_n \le \theta_{n-1}$ for $z \in \mathcal{U}_n$. By Statement (6), we get that $P^\natural$ has the accessibility property on $\widetilde{\mathcal{U}}_{n-1}$.

Since $P_{n-2} = T$ on $\mathcal{N} \times Z^{(n)}_\ell$, applying (5.10) with $j = n-1$, we find that $d(\Psi_{P_{n-1}}(z), \Psi_T(z)) \le 1/2^{n+6}$ for all $z \in \mathcal{N} \times Z^{(n-1)}_\ell$ and $\ell = 1, \dots, k_{n-1}$. Therefore by (5.12), we obtain that

$$d(\Psi_{P^\natural}(z), \Psi_T(z)) \le 1/2^{n+6} + 1/2^{n+7} < 1/2^{n+5}.$$

Applying (5.11) with $j = n-1$, we conclude that the requirement of Lemma 5.2 below holds. Therefore by the lemma and the fact that $d(\partial \widetilde{Z}^{(n)}_\ell, \partial \bar{Z}^{(n)}_\ell) = 1/2^{n+5}$, we obtain following the same line of arguments as in the case $n = 1$ that

$$\mathcal{A}_{P^\natural}(z_0(n, \ell)) \supset \mathcal{N} \times \widetilde{Z}^{(n)}_\ell$$

for all $\ell = 1, \dots, k_n$. In other words, $P^\natural$ has the accessibility property on $\mathcal{N} \times \widetilde{Z}^{(n)}_\ell$ for $\ell = 1, \dots, k_n$. By the construction,

$$\widetilde{\mathcal{U}}_n = \Big(\widetilde{\mathcal{U}}_{n-1}\Big) \bigcup \Big(\bigcup_{l=1}^{k_n} \mathcal{N} \times \widetilde{Z}^{(n)}_\ell\Big).$$

Note that any intersection $\widetilde{Z}^{(n)}_\ell \cap \widetilde{Z}^{(n)}_{\ell'}$ or $\widetilde{Z}^{(n)}_\ell \cap \widetilde{Z}^{(n-1)}_{\ell'}$, if not empty, contains a rectangle of width $1/2^{n+4}$. Hence, the intersection of any two sets among $\widetilde{\mathcal{U}}_{n-1}$ and $\mathcal{N} \times \widetilde{Z}^{(n)}_\ell$, $\ell = 1, \dots, k_n$, contains a nonempy open set whenever they intersect. Since $\widetilde{U}_n$ is connected, we obtain accessiblity of $P^\natural$ on $\widetilde{\mathcal{U}}_n$.

Applying the above result with $P^\natural = P_n$, we obtain that $P_n$ has the accessiblity property on $\widetilde{\mathcal{U}}_n$. Moreover, (5.12) for $P^\natural = P_n$ gives (5.10), and (5.13) below gives (5.11) for $j = n$.

5.4. **A technical lemma.** We prove here some of our main technical statements.

**Lemma 5.2.** *Suppose for some $n > 0$, $d(\Theta^i_{\tau,n,\ell,P_{n-1}}(z), z) \le \epsilon_n/4$ for all $i = u, s, c$, $\tau \in [0,1]$, $z \in W^c_{P_{n-1}}(z_0(n,\ell), K, Z^{(n)}_\ell)$, $\ell = 1, \dots, k_n$. Then there are $\delta''_n > 0$ and $\theta''_n > 0$ such that if $P_n$ satisfies $\|P_n - P_{n-1}\| \le \delta''_n$, then we have*

$$d(\Theta^i_{\tau,n+1,\ell,P_n}(z), z) \le \epsilon_{n+1}/4 \quad \textit{as} \quad z \in W^c_{P_n}(z_0(n+1,\ell), K, Z^{(n+1)}_\ell), \tag{5.13}$$

*for all $i = u, s, c$, $\tau \in [0,1]$, $\ell = 1, \dots, k_{n+1}$; and for any gentle perturbation $P^\natural$ of $P_n$ with*

$$\angle(E^i_{P^\natural}(z), E^i_{P_n}(z)) \le \theta''_n \quad \textit{for all } z \in \mathcal{N} \times Z^{(n)}_\ell,\ i = u, s, c,$$

*we have*

$$\mathcal{A}_{P^\natural}(z_0(n,\ell)) \supset W^c_{P^\natural}(z_0(n,\ell),\bar{K},\bar{Z}^{(n)}_\ell) \quad \textit{for all} \quad \ell = 1,\dots,k_n. \tag{5.14}$$

*In particular,* (5.14) *holds with* $P^\natural = P_n$.

*Proof.* Take $\theta''_n \le \theta_{n-1}/2$ such that for any gentle perturbation $P^\natural$ of $P_{n-1}$, if

$$\angle(E^i_{P^\natural}, E^i_{P_{n-1}}) \le 2\theta''_n \quad \text{on} \quad \mathcal{N}\times Z^{(n)}_\ell,\ i = u,s,c,$$

then (5.13) holds with $P_n = P^\natural$, and

$$d(\Theta^i_{\tau,n,\ell,P^\natural}(z), z) \le \epsilon_n/2 \quad \text{as} \quad z \in W^c_{P^\natural}(z_0(n,\ell),K,Z^{(n)}_\ell), \tag{5.15}$$

for all $i = u,s,c$, $\tau \in [0,1]$, and $\ell = 1,\dots,k_n$. (5.15) is possible because of the assumption of the lemma, while (5.13) is possible because on $\mathcal{N}\times Z^{(n+1)}_\ell$, $P_{n-1} = T$ and therefore $d(\Theta^i_{\tau,n+1,\ell,P_{n-1}}(z), z) = 0$. Then we take $\delta''_n \le \delta_{n-1}/2$ such that if $\|P_n - P_{n-1}\| \le \delta''_n$, then $\angle(E^i_{P_n}, E^i_{P_{n-1}}) \le \theta''_n$ on $\mathcal{N}\times Z^{(n)}_\ell$ for $i = u,s,c$. Hence, (5.15) is satisfied with $P^\natural = P_n$.

Now we only need to prove (5.14) for one square $Z^{(n)}_\ell$.

Define a continuous function $\Phi = \Phi^{(1)}_{P_n} : \mathbb{R} \to W^c_{P_n}(z_0)$ by using $\Theta = \Theta^a_{n,\ell,P_n}$ and $\Theta_\tau = \Theta^a_{\tau,n,\ell,P_n}$ such that the image of $\Phi$ consists of points accessible to $z_0 = (q_j, 1/2, a_0, b_0)$. Namely,

(1) $\Phi(0) = z_0$;
(2) For a positive integer $n$ if $\Phi(n-1) = (q_j, \frac{1}{2}, a, b_0)$ for some $a \in I_n$, then we let $\Phi(n) = \Theta(\Phi(n-1))$;
(3) For a negative integer $-n$ if $\Phi(-n+1) = (q_j, \frac{1}{2}, a, b_0)$ for some $a \in I_n$, then we let $\Phi(-n) = \Theta^{-1}(\Phi(-n+1))$; in other words, $\Phi(-n)$ is the terminate point of the quadrilateral $(u,s)_{P_n}$-path $\widehat{\gamma}^a_j$ with the initial point $\Phi(-n+1)$ such that $\pi_X\widehat{\gamma}^a_j = \gamma^a_j$ with the direction reversed;
(4) For any real number $n+\tau$, where $n \in \mathbb{Z}$ and $\tau \in [0,1)$ if $\Phi(n) = (q_j, \frac{1}{2}, a, b_0)$ then we let $\Phi(n+\tau) = \Theta_\tau(\Phi(n))$.

In fact, if we denote by $\lfloor r \rfloor$ the greatest integer that is less than or equal to $r$, then we have

$$\Phi^{(1)}_{P_n}(r) = \Theta^a_{r-\lfloor r\rfloor} \circ (\Theta^a)^{\lfloor r\rfloor}(z_0).$$

Since, $\lim_{\tau\to 1}\Theta^a_\tau = \Theta^a$ and $\lim_{\tau\to 0}\Theta^a_\tau = \mathrm{Id}$ we have that $\Phi^{(1)}_{P_n}$ is a continuous function of $r$. Furthermore,

$$\Phi^{(1)}_{P_n}(\mathbb{R}) \subset \mathcal{A}_{P_n}(z_0(n,\ell)).$$

By Lemma 5.3 below,

$$\Phi^{(1)}_{P_n}(\mathbb{Z}) \subset \{(q_j, 1/2, a, b_0) : a \in I_n, \} \subset W^c_{P_n}(z_0, K, Z^{(n)}_\ell).$$

Hence, by (5.15) with $P^\natural = P_n$,

$$\Phi^{(1)}_{P_n}(\mathbb{R}) \subset \{(q_j, t, a, b) : |t - 1/2| < \epsilon_n/2,\ a \in I_n(\epsilon_n/2),\ |b - b_0| \le \epsilon_n/2\},$$

where $I_n(\epsilon_n)$ denotes the $\epsilon_n$-neighborhood of $I_n$ in $\mathbb{R}$. It is also clear that

$$\lim_{n\to\pm\infty} \Phi^{(1)}_{P_n}(\pm n) = (q_j, 1/2, a_0 \mp \frac{3}{2^{n+2}}, b_0),$$

where the two points on the right hand side is on the boundary of $Z^{(n)}_\ell$. Hence, we can choose an integer $N = N^a_{n,\ell} > 0$ such that $\Phi^{(1)}_{P_n}(\pm N) \notin \pi_Y^{-1}\breve{Z}^{(n)}_\ell$. In other words, $\Phi^{(1)}_{P_n}([-N, N])$ forms a continuous curve near the line segment $\{(q_j, 1/2, a, b_0) : a \in I_n\}$ and crosses $\breve{Z}^{(n)}_\ell$ in $F^a$ direction.

Now we use the maps $\Theta = \Theta^b_{n,\ell,P_n}$ and $\Theta_\tau = \Theta^b_{\tau,n,\ell,P_n}$ to define a function $\Phi = \Phi^{(2)}_{P_n} : \mathbb{R}^2 \to W^c_{P_n}(z_0)$ such that the image of $\Phi$ consists of the points accessible to $z_0$. Namely, given $r \in \mathbb{R}$, let

(1) $\Phi(r, 0) = \Phi^{(1)}_{P_n}(r)$;
(2) For a positive integer $n$ if $\Phi(r, n-1)$ is defined, we let $\Phi(r, n) = \Theta(\Phi(r, n-1))$;
(3) For a negative integer $-n$ if $\Phi(r, -n+1)$ is defined, we let $\Phi(r, -n) = \Theta^{-1}(\Phi(r, -n+1))$;
(4) For any real number $n + \tau$, where $n \in \mathbb{Z}$ and $\tau \in [0, 1)$ if $\Phi(r, n)$ is defined, we let $\Phi(r, n+\tau) = \Theta_\tau(\Phi(r, n))$.

In other words,

$$\Phi^{(2)}_{P_n}(r, r') = \Theta^b_{r'-\lfloor r'\rfloor} \circ (\Theta^b)^{\lfloor r'\rfloor}(\Phi^{(1)}_{P_n}(r))$$

or equivalently,

$$\Phi^{(2)}_{P_n}(r, r') = \Theta^b_{r'-\lfloor r'\rfloor} \circ (\Theta^b)^{\lfloor r'\rfloor} \circ \Theta^a_{r-\lfloor r\rfloor} \circ (\Theta^a)^{\lfloor r\rfloor}(z_0).$$

It is clear that $\Phi^{(2)}_{P_n}$ is continuous, and $\Phi(\mathbb{R}^2) \subset \mathcal{A}(q_j, 1/2, a_0, b_0)$. Furthermore, for $r \in \mathbb{R}$,

$$\Phi(r, \mathbb{Z}) \subset \{\Phi^{(1)}_{P_n}(r) + (0, 0, 0, 0, b) :\ \pi_{F^b}\Phi^{(1)}_{P_n}(r) + b \in J_n\}.$$

Hence, by (5.15) with $P^\natural = P_n$,

$$\Phi(r, \mathbb{R}) \subset \{(q_j, t, a, b) : |t - 1/2| \le \epsilon_n, (a, b) \in Z^{(n)}_\ell(\epsilon_n)\},$$

where $Z^{(n)}_\ell(\epsilon_n)$ denotes the $\epsilon_n$-neighborhood of $Z^{(n)}_\ell$ in $Y$. This means that $\Phi(\mathbb{R}^2)$ is contained in the $\epsilon_n$-neighborhood of the set $\{q_j\} \times \{1/2\} \times Z^{(n)}_\ell$.

Similarly, for every $r \in \mathbb{R}$ there exists $N(r) = N^b_{n,\ell}(r)$ such that the set $\Phi(r, [-N(r), N(r)])$ forms a continuous curve near $J_n(r)$ and crosses $\breve{Z}^{(n)}_\ell$. By continuity, we can take $N = N^b_{n,\ell}$ such that $\Phi(r, [-N, N])$ crosses $\breve{Z}^{(n)}_\ell$ for all

$r \in [-N, N]$. Moreover, we may assume that $N^a_{n,\ell} = N^b_{n,\ell}$, since otherwise we may use the larger one instead. By continuity, we obtain that the four curves

$$\Phi(-N, [-N, N]),\ \Phi(N, [-N, N]),\ \Phi([-N, N], -N),\ \Phi([-N, N], N)$$

form a closed curve. The projection of the curves under $\mathrm{Proj}_Y$ is outside $\breve{Z}^{(n)}_\ell$. Hence, by Sublemma 5.4, we get that $\mathrm{Proj}_Y\{\Phi(r, r') : r, r' \in [-N, N]\}$ covers $\breve{Z}^{(n)}_\ell$.

Finally, we use the maps $\Theta = \Theta^t_{n,\ell,P_n}$ and $\Theta_\tau = \Theta^t_{\tau,n,\ell,P_n}$ to define a function $\Phi = \Phi^{(3)}_{P_n} : \mathbb{R}^3 \to W^c_{P_n}(z_0)$ such that the image of $\Phi$ consists of the points accessible to $z_0$. See Figures 3 and 4. Namely, given $r, r' \in \mathbb{R}$, let

(1) $\Phi(r, r', 0) = \Phi^{(2)}_{P_n}(r, r')$;
(2) For a positive integer $n$ if $\Phi(r, r', n-1)$ is defined, we let $\Phi(r, r', n) = \Theta(\Phi(r, r', n-1))$;
(3) For a negative integer $-n$ if $\Phi(r, r', -n+1)$ is defined, we let $\Phi(r, r', -n) = \Theta^{-1}(\Phi(r, r', -n+1))$;
(4) For any real number $n+\tau$, where $n \in \mathbb{Z}$ and $\tau \in [0, 1)$ if $\Phi(r, r', n)$ is defined, we let $\Phi(r, r', n+\tau) = \Theta_\tau(\Phi(r, r', n))$.

We have

$$\Phi^{(3)}_{P_n}(r, r', r'') = \Theta^t_{r''-\lfloor r''\rfloor} \circ (\Theta^t)^{\lfloor r''\rfloor} \circ \Theta^b_{r'-\lfloor r'\rfloor} \circ (\Theta^b)^{\lfloor r'\rfloor} \circ \Theta^a_{r-\lfloor r\rfloor} \circ (\Theta^a)^{\lfloor r\rfloor}(z_0).$$

The function $\Phi^{(3)}_{P_n}$ is continuous and $\Phi(\mathbb{R}^3) \subset \mathcal{A}(z_0)$.

We also have that there exists $N > 0$ such that $\Phi$ maps the surfaces of the cube $[-N, N] \times [-N, N] \times [-N, N]$ into outside the corresponding surfaces of $W^c_{P_n}(z_0, \breve{K}, \breve{Z}^{(n)}_\ell)$ and inside the corresponding surfaces of the $2\epsilon_n$-neighborhood of $W^c_{P_n}(z_0, K, Z^{(n)}_\ell)$. By Sublemma 5.4, $\{\Phi(r, r', r'') : r, r', r'' \in [-N, N]\}$ covers $W^c_{P_n}(z_0, \breve{K}, \breve{Z}^{(n)}_\ell)$, and we obtain that

$$\mathcal{A}(z_0) \supset W^c_{P_n}(z_0, \breve{K}, \breve{Z}^{(n)}_\ell).$$

We may reduce $\delta''_n$ again if neccesary such that any gentle perturbation $P^\natural$ of $P_n$ satisfying $\|P^\natural - P_n\| \le \delta''_n$ is so close to the unperturbed map $P_n$ that the map $\Theta^i_{P^\natural} = \Theta^i_{n,\ell,P^\natural}$ and $\Theta^i_{\tau,P^\natural} = \Theta^i_{\tau,n,\ell,P^\natural}$ are well defined on for $i = u, s, c$ and $\tau \in [0, 1]$, and close to $\Theta^i_{P_n} = \Theta^i_{n,\ell,P_n}$ and $\Theta^i_{\tau,P_n} = \Theta^i_{\tau,n,\ell,P_n}$ respectively. Then we define $\Phi^{(3)}_{P^\natural} : \mathbb{R}^3 \to W^c_{P^\natural}(z_0, K, Z^{(n)}_\ell)$ by

$$\Phi^{(3)}_{P^\natural}(r, r', r'') = \Theta^t_{\{r''\},P^\natural} \circ (\Theta^t_{P^\natural})^{\lfloor r''\rfloor} \circ \Theta^b_{\{r'\},P^\natural} \circ (\Theta^b_{P^\natural})^{\lfloor r'\rfloor} \circ \Theta^a_{\{r\},P^\natural} \circ (\Theta^a_{P^\natural})^{\lfloor r\rfloor}(z_0),$$

where $\{r\} = r - \lfloor r\rfloor$ denotes the fractional part of $r$. If $\theta''_n$ is small enough and $\angle(E^i_{P_n}(z), E^i_{P^\natural}(z)) \le \theta''_n$ for $i = u, s, c$ and all $z \in \mathcal{N} \times Z^{(n)}_\ell$, then $\Theta^i_{P^\natural}$ and $\Theta^i_{\tau,P^\natural}$

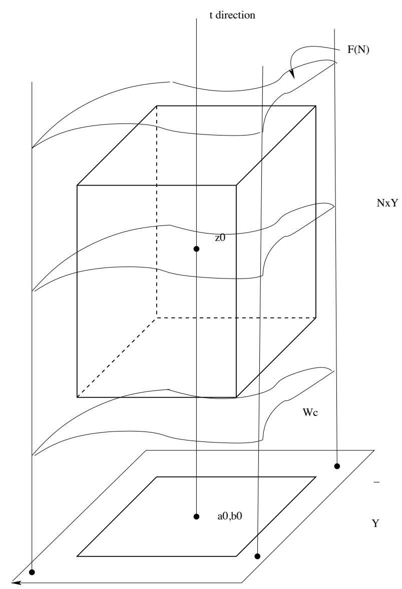


FIGURE 3. The action of the function $\Phi^{(3)}$ on the central direction of $z_0$

are sufficiently close to $\Theta^i_{P_n}$ and $\Theta^i_{\tau,P_n}$ respectively for $i = a, b, t$. Thus we can obtain that

$$d(\Phi^{(3)}_{P^\natural}(r, r', r''), \Phi^{(3)}_{P_n}(r, r', r'')) \le 1/2^{n+4},$$

which is the distance between $\partial \breve{Z}^{(n)}_k$ and $\partial \bar{Z}^{(n)}_k$, for $r, r', r'' \in [-N, N]$. In other words, $\Phi^{(3)}_{P^\natural}(r, r', r'')$ maps the surface of the cube $[-N, N]\times[-N, N]\times[-N, N]$ to the surfaces that are close to and outside the corresponding surfaces of $W^c_{P^\natural}(z_0, \bar{K}, \bar{Z}^{(n)}_\ell)$. Hence, by Sublemma 5.4, the set

$$\{\Phi^{(3)}_{P^\natural}(r, r', r'') : r, r', r'' \in [-N, N]\}$$

covers $W^c_{P^\natural}(z_0, \bar{K}, \bar{Z}^{(n)}_\ell)$ implying that

$$\mathcal{A}_{P^\natural}(z_0(n, \ell)) \supset W^c_{P^\natural}(z_0(n, \ell), \bar{K}, \bar{Z}^{(n)}_\ell).$$

The desired result follows. □

**Sublemma 5.3.** *For each $n > 0$, there exists $\delta''_n > 0$ such that if $\|h_n - \mathrm{Id}\|_{C^n} \le \delta''_n > 0$, then for any $a \in I_n$,*

(1) $\Theta^a((q_j, 1/2, a, b_0)) = (q_j, 1/2, a', b_0)$ *with* $a' < a$;
(2) $b \in J_n$, $t \in (1/2 - \epsilon_t, 1/2 + \epsilon_t)$, $\Theta^b((q, t, a, b)) = (q, t, a, b')$ *with* $b' < b$;
(3) $b \in J_n$, $t \in K$, $\Theta^t((q, t, a, b)) = (q, t', a, b)$ *with* $t' < t$.

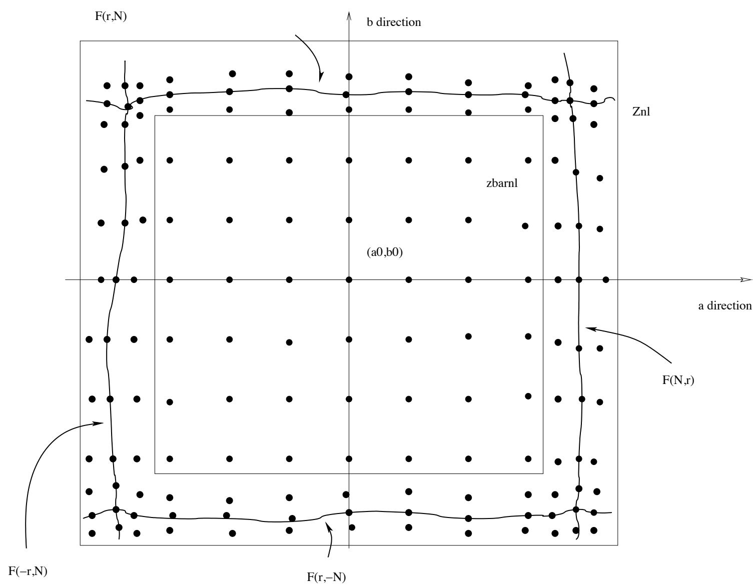


FIGURE 4. The function $\Phi$

*Proof.* The proof is similar to that of Lemma B.4 in [8].

We prove the first statement. Consider the coordinate system in $\Omega^a$ with the origin at $(p_j^a, 1/2, a_0, b_0)$ which therefore has local coordinates $(0,0,0,0,0)$. We may assume that the local coordinates of the points $q_j$, $[q, p_j^a]$ and $[p_j^a, q]$ are $(u_0, s_0)$, $(0, s_0)$ and $(u_0, 0)$ respectively, where $u_0 = \alpha_u^a$ and $s_0 = \alpha_s^a$ are given by (5.1).

We first consider the case $n > 1$ and note that the path $\widehat{\gamma}_j^a$ is contained in the closure of $\Omega_{n,\ell}^a$ (see Subsection 5.2) for $n > 1$ and $\ell = 1, \dots k_n$. We have that $P_n|\Omega_{n,\ell}^a = h_{n,\ell}^a \circ T$. Furthermore, since $h_{n,\ell}^a = \mathrm{Id}$ on the curve $V_T^u(q_j, t, y)$ for $t \in K$ and $y \in Z_\ell^{(n)}$, we have that $V_{P_n}^u(q_j, t, y) = V_T^u(q_j, t, y)$. It follows that if $(u_0, s_0, 0, a_1, 0)$ are the local coordinates of the point $z_1 = (q_j, 1/2, a_1, b_0)$, then $(0, s_0, 0, a_2, 0)$ are the local coordinates of the point $z_2 = ([q_j, p_j^a], 1/2, a_2, b_0)$ with $a_2 = a_1$.

Recall that by (5.3), the $a$-component of the vector field $X^a(z)$ is equal to $\beta\phi^a(u)\psi^a(s)\zeta_t(t)\zeta_Y(b)\xi_Y(a)$ and that $\phi^a(u)$, $\psi^a(s)$, $\zeta_t(t)$ and $\zeta_Y(b)$ are constants for $|u| \le \breve{\alpha}_u^a$, $|s| \le \breve{\alpha}_s^a$, $|t| \le \breve{\epsilon}^t$ and $b \in \breve{J}_n$ respectively. Recall also that the map $h^a$ preserves the $s$-, $t$- and $b$-coordinates. Therefore, if $|u| \le \breve{\alpha}_u^a$, $|s| \le \breve{\alpha}_s^a$, $|t| \le \breve{\epsilon}^t$, $a \in I_n$, and $b \in \breve{J}_n$, then

$$h^a(u, s, t, a, b) = (u', s, t, a + c(a, t), b), \tag{5.16}$$

where $u'$ is close to $u$ provided $\beta$ is sufficiently small, and $c(a,t) > 0$ if $|t| \le \epsilon_t$ and $c(a,t) = 0$ otherwise. Moreover, if $|t| \le \check{\epsilon}^t$, then $c(a,t) = c(a)$ is independent of $t$. Also, if $u = 0$ then $u' = 0$ since in this case $\int_0^u \phi^a(r)dr = 0$ and therefore the $u$-component of $X^a$ is zero. Since $\alpha_s^a/\eta\kappa(a,b) \le \breve{\alpha}_s^a$, we have that for $|s| \le \alpha_s^a$ and $b \in \breve{J}_n$,

$$\begin{aligned} P_n(0,s,t,a,b) &= h^a\big(T(0,s,t,a,b)\big) \\ &=(0,\ s/\eta\kappa(a,b),\ t+\kappa(a,b),\ a+c(a,t), b). \end{aligned}$$

Note that $P_n = T$ near the orbit of $(p_j^a, t, y)$ and outside $\Omega^a$. Hence, under the iterations of $P_n$ all points of the set $\{(0,s,t,a,b) : |s| \le \alpha_s^a\}$ have fixed $u$- and $b$-coordinates and the same $t$- and $a$-coordinates. Therefore, this set belongs to $V^s(p_j^a, t, a, b)$. Since $z_2 \in V^s(z_3)$, the fact that$(0, s_0, 0, a_2, 0)$ are the local coordinates of the point $z_2$ yields that $(0,0,0,a_3,0)$ are the local coordinates of the point $z_3 = (p_j^a, 1/2, a_3, b_0)$ with $a_3 = a_2$.

By similar arguments, we can show that if $|u| \le \breve{\alpha}_u^a$, $a \in I_n$ and $b \in \breve{J}_n$, then

$$\begin{aligned} P_n^{-1}(u,0,t,a,b) &= T^{-1}\big((h^a)^{-1}(u,0,t,a,b)\big) \\ &=(u''/\eta\kappa(a,b), 0, t-\kappa(a',b), a', b), \end{aligned} \tag{5.17}$$

for some $u''$ close to $u$, where by (5.16), $a'$ satisfies $a'+c(a',t) = a$. If we choose $\delta_n'' > 0$ small enough, then $\|h_n^a - \mathrm{Id}\,\| \le \delta_n''$ implies that $u''$ is sufficiently close to $u$ and therefore $|u''|/\eta\kappa(a,b) \le \breve{\alpha}_u^a$. Hence, under the iterations of $P_n^{-1}$ all points of the set $\{(u,0,t,a',b) : |u| \le \breve{\alpha}_u^a\}$ have fixed $s$- and $b$-coordinates and the same $t$- and $a$-coordinates. Therefore, this set belongs to $V^u(p_j^a, t, a, b)$. On the other hand, by the definition of $h^a$ and the choice of $\alpha_u^a$, we have $h^a = \mathrm{Id}$ if $|u| \ge \alpha_u^a$. Therefore, since $u_0 = \alpha_u^a$,

$$\begin{aligned} P_n^{-1}(u_0,0,t,a',b) &= T^{-1}(u_0,0,t,a',b) \\ &=(u_0/\eta\kappa(a,b), 0, t-\kappa(a',b), a', b). \end{aligned} \tag{5.18}$$

Comparing (5.18) with (5.17) we obtain that the point with local coordinates $(u_0, 0, t, a', b)$ is on the unstable local manifold of the point with local coordinates $(0,0,t,a,b)$, where $a' + c(a',t) = a$ and $c(a',t) > 0$. So if $z_4 = ([p_j^a, q_j], t, a_4, b_0) \in V^u(z_3)$, then $z_4$ has local coordinates $(u_0, 0, 0, a_4, b_0)$ with $a_4 < a_3$.

Since the path on $V^s(q_j)$ is unperturbed, the fact that $z_4 \in V^s(z_5)$ yields that the point $z_5 = (q_j, 1/2, a_5, b_0)$ has local coordinates $(u_0, s_0, 0, a_5, b_0)$ with $a_5 = a_4$. This implies that in the the case $n > 1$ we have $a_1 = a_2 = a_3 > a_4 = a_5$.

In the case $n = 1$ similar arguments can be used with the following modification. To obtain the $a$-coordinate of the points on $V^s(p_j^a, 1/2, a, b_0)$ and

$V^u(p_j^a, 1/2, a, b_0)$, we need to consider $P_1^{\ell^u} = h^a \circ T^{\ell^u}$ and $P_1^{\ell^s} = h^a \circ T^{\ell^s}$ respectively, (recall that near the pathes $\gamma_1^a$ and on the set $\Omega^a$, the map $T$ is unperturbed, and hence $Q = T$,) and therefore get $a_1 = a_2 \geq a_3 > a_4 = a_5$. The assumption $\kappa = \kappa_0$ on $U_1$ guarantees that on these local manifolds the $t$-coordinates are the same. This implies Statement (1).

In the above arguments, we can actually replace $b_0$ by any $b \in J_n$ and the number $1/2$ by any $t \in (1/2-\epsilon_n, 1/2+\epsilon_n)$ and can still obtain the same results. Therefore, Statement (2) can be proved by switching the roles of $a$ and $b$.

Statement (3) can be proved in the same way. In particular, since $h^t$ preserves $a$- and $b$-coordinates and the stable and unstable local manifolds for $T$ at $(q_j, t, y)$ and $(p_j^t, t, y)$ are unperturbed except by $h^t$, the arguments can be carried over on the submanifold $\mathcal{N}_y$ (see (3.1)) for each $y \in Z_\ell^{(n)}$. □

**Sublemma 5.4.** *Let $\Phi : I^n \to I^n$ be a homeomorphism of the $n$-dimensional cube $I^n$ and $\partial_i I^n$ be the faces of $I^n$, $i = 1, \dots, 2n$. Assume that $\Phi(\partial_i I^n) \subset B(\partial_i I^n, \epsilon) \setminus I^n$ for $i = 1, \dots, 2n$, where $B(\cdot, \epsilon)$ is the $\epsilon$-neighborhood of the set. Then $I^n \subset \Phi(I^n)$.*

*Proof.* This is a variation of a general topology thoerem, which says that in the setting if $\Phi(\partial_i I^n) \subset B(\partial_i I^n, \epsilon)$ for $i = 1, \dots, 2n$, then $I^n \setminus B(\partial_i I^n, \epsilon) \subset \Phi(I^n)$. □

## Appendix A

Let $\mathcal{M}$ be a compact smooth Riemannian manifold and $\mathcal{S} \subset \mathcal{M}$ an open subset. Let also $h$ be a $C^1$ diffeomorphism that is pointwise partially hyperbolic on $\mathcal{S}$. Further, let $\mathcal{U}_n \subset \mathcal{S}$, $n \geq 1$ be a sequence of open subsets such that:

(1) $\mathcal{U}_n \subset \overline{\mathcal{U}}_n \subset \mathcal{U}_{n+1}$ and $\bigcup \mathcal{U}_n = \mathcal{S}$;
(2) each $\mathcal{U}_n$ is $h$-invariant;
(3) $h|\overline{\mathcal{U}}_n$ is uniformly partially hyperbolic.

The goal of this Appendix is to prove the following statement. Suppose there is a sequence of diffeomorphisms $h_n$ such that $h_0 = h$, $h_n = h_{n-1}$ on $\mathcal{M} \setminus \overline{\mathcal{U}}_n$. Clearly, $\overline{\mathcal{U}}_n$ is $h_n$-invariant, and $h_n = h$ on $\mathcal{M} \setminus \overline{\mathcal{U}_n}$.

**Theorem A.1.** *Let $h_n$ be a sequence of diffeomorphisms for which $h_0 = h$ and $h_n = h_{n-1}$ on $\mathcal{M} \setminus \overline{\mathcal{U}}_n$, so that $\overline{\mathcal{U}}_n$ is $h_n$-invariant and $h_n = h$ on $\mathcal{M} \setminus \overline{\mathcal{U}}_n$. Then there exists a sequence of positive numbers $\varepsilon_n$ such that if $\|h_n - h_{n-1}\|_{C^1} \leq \varepsilon_n$, then*

(1) *each map $h_n$ is uniformly partially hyperbolic on $\overline{\mathcal{U}}_n$ and hence pointwise partially hyperbolic on $\mathcal{S}$;*
(2) *the limit $H = \lim_{n\to\infty} h_n$ exists and is a $C^1$ pointwise partially hyperbolic diffeomorphism of $\mathcal{S}$.*

We need the following technical statements.

**Lemma A.2.** *Given a sequence of positive numbers* $\{a_n\}_{n\geq 1}$ *satisfying*

$$\sum_{n=1}^{\infty} a_n \leq \frac{1}{4},$$

*we have*

$$\prod_{n=1}^{\infty}(1+a_n) \leq 1+2\sum_{n=1}^{\infty} a_n \text{ and } \prod_{n=1}^{\infty}(1-a_n) \geq 1-2\sum a_n.$$

**Lemma A.3.** *Set*

$$M_n = \sup_{x\in\mathcal{M}} \|d_x h_n\| \text{ and } m_n = \inf_{x\in\mathcal{M}} m(d_x h_n).$$

*If* $\varepsilon_n < m_0/2^{n+4}$*, then* $M_n \leq 2M_0$ *and* $m_n \geq 0.5m_0$*.*

*Proof.* Note that $|M_n - M_{n-1}| \leq \varepsilon_n$ and $|m_n - m_{n-1}| \leq \varepsilon_n$. Applying Lemma A.2, one can show by induction that

$$1-\frac{1}{2^{n+2}} \leq \frac{M_n}{M_{n-1}}, \text{ and } \frac{m_n}{m_{n-1}} \leq 1+\frac{1}{2^{n+2}}.$$

The desired result follows. □

Given two diffeomorphisms $f$ and $g$ with invariant distributions $E_f$ and $E_g$ on $\mathcal{S}$ respectively, let

$$\begin{aligned}\Delta_{f,g,E_f,E_g}(x) =& \max\left\{\left|\frac{\|d_x g|E_g(x)\|}{\|d_x f|E_f(x)\|}-1\right|, \left|\frac{m(d_x g|E_g(x))}{m(d_x f|E_f(x))}-1\right|\right\}, \\ \varepsilon_{f,g}(x) =& \|d_x g - d_x f\|, \quad \theta_{E_f,E_g}(x) = \angle(E_f(x), E_g(x)).\end{aligned} \tag{A.19}$$

**Lemma A.4.** *Assume that*

$$\sup_{x\in\mathcal{M}} \|d_x f\| \leq M := 2M_0, \quad \inf_{x\in\mathcal{M}} m(d_x f) \geq m := 0.5m_0.$$

*Then for any* $x \in \mathcal{S}$,

$$\Delta_{f,g,E_f,E_g}(x) \leq \frac{1}{m}[\varepsilon_{f,g}(x) + CM\theta_{E_f,E_g}(x)],$$

*where* $C > 0$ *is a constant which depends only on the Riemannian metric of* $\mathcal{M}$*.*

*Proof.* We have that

$$\begin{aligned} |\|d_x g|E_g(x)\| - \|d_x f|E_f(x)\|| &\leq |\|d_x g|E_g(x)\| - \|d_x f|E_g(x)\|| \\ &+ |\|d_x f|E_g(x)\| - \|d_x f|E_f(x)\|| \\ &\leq \|d_x g - d_x f\| + \|d_x f\| \mathrm{dist}(E_g(x), E_f(x)) \\ &\leq \|d_x g - d_x f\| + C\|d_x f\| \angle(E_g(x), E_f(x)), \end{aligned}$$

for some constant $C > 0$ depending only on the Riemannian metric of $\mathcal{M}$. Dividing both sides of the inequality by $\|d_x f|E_f(x)\|$ and noting that $\|d_x f|E_f(x)\| \geq m(d_x f)$, we obtain that

$$\begin{aligned} \left| \frac{\|d_x g|E_g(x)\|}{\|d_x f|E_f(x)\|} - 1 \right| &\leq \frac{\|d_x g - d_x f\|}{m(d_x f)} + C \frac{\|d_x f\|}{m(d_x f)} \angle(E_g(x), E_f(x)) \\ &\leq \frac{1}{m}[\varepsilon_{f,g}(x) + CM\theta_{E_f, E_g}(x)] \end{aligned}$$

Similarly one can show that $\left| \frac{m(d_x g|E_g(x))}{m(d_x f|E_f(x))} - 1 \right|$ has the same upper bound. $\square$

**Lemma A.5.** *Suppose that $f$ is uniformly partially hyperbolic on a compact invariant subset $\Lambda \subset \mathcal{S}$. Pick numbers $0 < \lambda < \tilde{\lambda} \leq 1 \leq \tilde{\mu} < \mu$ such that*

$$\begin{aligned} \lambda \geq \lambda(f, \Lambda) = \sup_{x \in \Lambda} \|d_x f^s\|, &\qquad \widetilde{\lambda} \leq \widetilde{\lambda}(f, \Lambda) = \inf_{x \in \Lambda} m(d_x f^c), \\ \widetilde{\mu} \geq \widetilde{\mu}(f, \Lambda) = \sup_{x \in \Lambda} \|d_x f^c\|, &\qquad \mu \leq \mu(f, \Lambda) = \inf_{x \in \Lambda} m(d_x f^u). \end{aligned}$$

*Given $\Delta > 0$, there is $\varepsilon = \varepsilon(\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu) < \frac{m\Delta}{2}$ such that if $\|g - f\|_{C^1} < \varepsilon$ and $g = f$ on $\mathcal{S} \backslash \Lambda$, then $g|\Lambda$ is also uniformly partially hyperbolic and*

$$\Delta^\omega_{f,g}(x) := \Delta_{f,g,E^\omega_f,E^\omega_g}(x) \leq \Delta, \quad \omega = s, c, u, \ x \in \Lambda. \tag{A.20}$$

*In particular,*

$$1 - \Delta \leq \frac{\lambda(g, \Lambda)}{\lambda(f, \Lambda)}, \ \frac{\widetilde{\lambda}(g, \Lambda)}{\widetilde{\lambda}(f, \Lambda)}, \ \frac{\widetilde{\mu}(g, \Lambda)}{\widetilde{\mu}(f, \Lambda)}, \ \frac{\mu(g, \Lambda)}{\mu(f, \Lambda)} \leq 1 + \Delta.$$

*Proof.* Note that the set of uniformly partially hyperbolic diffeomorphisms is $C^1$-open, and the invariant distributions $E^\omega_g$ depend continuously on $g$, $\omega = s, c, u$ (see [**?**]). More precisely, there is $\varepsilon < \frac{m\Delta}{2}$ depending on $\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu$ such that if $\|g - f\|_{C^1} < \varepsilon$ and $g = f$ on $\mathcal{S} \backslash \Lambda$, then $g|\Lambda$ is uniformly partially hyperbolic with

$$\sup_{x\in\Lambda} \angle(E_g^\omega(x), E_f^\omega(x)) < \frac{m\Delta}{2CM}. \tag{A.21}$$

Then by Lemma A.4, it is immediate that $\Delta^\omega_{f,g}(x)\lambda\Delta$. □

We shall now specify how to choose the sequence of numbers $\varepsilon_n$ in the theorem. First choose four sequences of numbers $0 < \lambda_n < \widetilde{\lambda}_n \le 1 \le \widetilde{\mu}_n < \mu_n$ such that

(1) $\lambda_n \ge \lambda(h, \overline{\mathcal{U}_n})$, $\widetilde{\lambda}_n \le \widetilde{\lambda}(h, \overline{\mathcal{U}_n})$, $\widetilde{\mu}_n \ge \widetilde{\mu}(h, \overline{\mathcal{U}_n})$, $\mu_n \le \mu(h, \overline{\mathcal{U}_n})$;
(2) $\lambda_n, \widetilde{\mu}_n$ are strictly increasing while $\widetilde{\lambda}_n, \mu_n$ are strictly decreasing.

For all $x \in \mathcal{S}$, let

$$\gamma(x) = \min\left\{\frac{\min\{1, m(d_xh^c)\}}{\|d_xh^s\|},\ \frac{m(d_xh^u)}{\max\{1, \|d_xh^c\|\}}\right\},$$

and choose a strictly decreasing sequence of numbers $\gamma_n$ such that

$$0 < \gamma_n \le \inf_{x\in\overline{\mathcal{U}_n}} \frac{\gamma(x) - 1}{8}. \tag{A.22}$$

Now choose a sequence of positive numbers $\Delta_n$ such that

$$\max\{\frac{\widetilde{\lambda}_{n+1}}{\widetilde{\lambda}_n},\ \frac{\mu_{n+1}}{\mu_n}\} \le 1 - \Delta_n < 1 + \Delta_n \le \min\{\frac{\lambda_{n+1}}{\lambda_n},\ \frac{\widetilde{\mu}_{n+1}}{\widetilde{\mu}_n}\}; \tag{A.23}$$

$$\Delta_n < \frac{1}{2^{n+2}}, \quad \sum_{k=n}^{\infty} \Delta_k < \gamma_n. \tag{A.24}$$

Finally, choose

$$\varepsilon_n < \frac{1}{2}\min\{\frac{m_0}{2^{n+4}},\ \ \varepsilon(\Delta_n, \lambda_n, \widetilde{\lambda}_n, \widetilde{\mu}_n, \mu_n)\},$$

where $\varepsilon(\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu)$ is given by Lemma A.5.

*Proof of Theorem A.1.* First we shall show that for every $n > 0$ the map $h_n$ is uniformly partially hyperbolic on $\overline{\mathcal{U}_n}$. It is clearly true for $h_0$ and we shall use induction assuming that $h_k|\overline{\mathcal{U}_k}$ for $k = 1, \dots, n$ are uniformly partially hyperbolic. By Lemma A.5 we obtain that

$$1 - \Delta_k \le \frac{\lambda(h_k, \overline{\mathcal{U}_k})}{\lambda(h_{k-1}, \overline{\mathcal{U}_k})},\ \frac{\widetilde{\lambda}(h_k, \overline{\mathcal{U}_k})}{\widetilde{\lambda}(h_{k-1}, \overline{\mathcal{U}_k})},\ \frac{\widetilde{\mu}(h_k, \overline{\mathcal{U}_k})}{\widetilde{\mu}(h_{k-1}, \overline{\mathcal{U}_k})},\ \frac{\mu(h_k, \overline{\mathcal{U}_k})}{\mu(h_{k-1}, \overline{\mathcal{U}_k})} \le 1 + \Delta_k.$$

Note that

$$\begin{aligned}\lambda(h_k, \overline{\mathcal{U}_{k+1}}) &\leq \max\{\lambda(h, \overline{\mathcal{U}_{k+1}}), \lambda(h_k, \overline{\mathcal{U}_k})\}\\ &\leq \max\{\lambda_{k+1}, \lambda(h_k, \overline{\mathcal{U}_k})\}\\ &\leq \max\{\lambda_{k+1}, \lambda(h_{k-1}, \overline{\mathcal{U}_k})(1+\Delta_k)\}.\end{aligned}$$

The fact that $\lambda(h_0, \overline{\mathcal{U}_1}) \leq \lambda_1$ and the choice of $\Delta_n$ in (A.23) guarantee that

$$\lambda'_n := \lambda(h_n, \overline{\mathcal{U}_{n+1}}) \leq \lambda_{n+1}.$$

Similarly, we have

$$\widetilde{\lambda}'_n := \widetilde{\lambda}(h_n, \overline{\mathcal{U}_{n+1}}) \geq \widetilde{\lambda}_{n+1}, \quad \widetilde{\mu}'_n := \widetilde{\mu}(h_n, \overline{\mathcal{U}_{n+1}}) \leq \widetilde{\mu}_{n+1},$$

$$\mu'_n := \mu(h_n, \overline{\mathcal{U}_{n+1}}) \geq \mu_{n+1}.$$

It follows that

$$\varepsilon_n \leq \varepsilon(\Delta_n, \lambda_n, \widetilde{\lambda}_n, \widetilde{\mu}_n, \mu_n) \leq \varepsilon(\Delta_n, \lambda'_n, \widetilde{\lambda}'_n, \widetilde{\mu}'_n, \mu'_n).$$

Since $\|h_{n+1} - h_n\|_{C^1} \leq \varepsilon_n$, by Lemma A.5 we obtain that $h_{n+1}|\overline{\mathcal{U}_{n+1}}$ is uniformly partially hyperbolic.

Next we shall show that $H = \lim_{n\to\infty} h_n$ exists and is indeed pointwise partially hyperbolic on $\mathcal{S}$. Since $\varepsilon_n < m_0/2^{n+4}$, the sequence of maps $h_n$ is a Cauchy sequence and hence it converges in the $C^1$ topology. Moreover, as shown in (A.21), given $x \in \mathcal{U}_k$ and $n > k$, we have

$$\angle(E^\omega_{h_n}(x), E^\omega_{h_{n-1}}(x)) < \frac{m\Delta_n}{2CM} \leq \frac{m}{2^{n+3}CM}, \quad \omega = s, c, u.$$

Hence, the sequence of subspaces $E^\omega_{h_n}(x)$ is a Cauchy sequence, and thus there is a limit

$$E^\omega_H(x) = \lim_{n\to\infty} E^\omega_{h_n}(x).$$

Fix $n > 0$. We now wish to estimate $\Delta^\omega_{H,h}(x)$ for $x \in \mathcal{U}_n \backslash \overline{\mathcal{U}_{n-1}}$ (see (A.19) and (A.20)). We have

$$\Delta^\omega_{h_k,h_{k-1}}(x) \begin{cases} = 0, & k < n,\\ \leq \Delta_k, & k \geq n.\end{cases}$$

Note that

$$\frac{\|d_x h^\omega_l\|}{\|d_x h^\omega\|} = \prod_{k=1}^{l} \frac{\|d_x h^\omega_k\|}{\|d_x h^\omega_{k-1}\|}, \qquad \frac{m(d_x h^\omega_l)}{m(d_x h^\omega)} = \prod_{k=1}^{l} \frac{m(d_x h^\omega_k)}{m(d_x h^\omega_{k-1})},$$

and by (A.24), $\sum \Delta_k < 1/4$. It follows from Lemma A.2 that

$$\Delta^{\omega}_{h_l,h}(x) \le \prod_{k=1}^{l}(1+\Delta^{\omega}_{h_k,h_{k-1}}(x)) - 1 \le \prod_{k=n}^{\infty}(1+\Delta_k) - 1 \le 2\sum_{k=n}^{\infty}\Delta_k.$$

Letting $l \to \infty$, we find that

$$\Delta^{\omega}_{H,h}(x) \le 2\sum_{k=n}^{\infty}\Delta_k, \quad \omega = s, c, u.$$

Therefore by (A.22),

$$\begin{aligned} \frac{\|d_x H^s\|}{\min\{1, m(d_x H^c)\}} &\le \frac{1+2\sum_{k=n}^{\infty}\Delta_k}{1-2\sum_{k=n}^{\infty}\Delta_k}\frac{\|d_x h^s\|}{\min\{1, m(d_x h^c)\}} \\ &< (1+8\gamma_n)\frac{\|d_x h^s\|}{\min\{1, m(d_x h^c)\}} \\ &\le \gamma(x)\frac{\|d_x h^s\|}{\min\{1, m(d_x h^c)\}} < 1 \end{aligned}$$

Similarly, one can show $m(d_x H^u) > \max\{1, \|d_x H^c\|\}$. It follows that $H$ is pointwise partially hyperbolic on $\mathcal{S}$. □

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