Quasi-Shadowing for Partially Hyperbolic Diffeomorphisms

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Abstract: A partially hyperbolic diffeomorphism f has quasi-shadowing property if for any pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, there is a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ tracing it in which y_{k+1} is obtained from $f(y_k)$ by a motion τ along the center direction. We show that any partially hyperbolic diffeomorphism has quasi-shadowing property, and if f has C^1 center foliation then we can require τ to move the points along the center foliation. As applications, we show that any partially hyperbolic diffeomorphism is topologically quasi-stable under C^0-perturbation. When f has uniformly compact C^1 center foliation, we also give partially hyperbolic diffeomorphism versions of some theorems holden for uniformly hyperbolic systems, such as Anosov closing lemma, cloud lemma and spectral decomposition theorem.

0 Introduction

The goal of this paper is to study some shadowing properties for partially hyperbolic systems and to use it to study some topological properties of the systems shared by hyperbolic systems. For partially hyperbolic diffeomorphisms, a center direction is allowed in addition to the hyperbolic directions. The presence of this direction permits a very rich type of structure in these systems. For general theory of partially hyperbolic system, we refer to [8], [11], [1] and [2]. On the other hand, there are still hyperbolic structure in partially hyperbolic systems, and therefore we may see some phenomena similar to that of hyperbolic systems.

It is well known that an Anosov diffeomorphism has the shadowing property. (See [3] for example.)

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A sequence of $\xi = \{x_k\}_{-\infty}^{+\infty}$ is said to be a δ-*pseudo orbit* for f, if

$$\sup_{k\in\mathbb{Z}} d(f(x_k),\ x_{k+1}) \le \delta.$$

If for a δ-pseudo orbit $\xi = \{x_k\}_{-\infty}^{+\infty}$ there is a point $x \in M$ such that

$$d(f^k(x),\ x_k) \le \varepsilon \ \text{ for all } \ k \in \mathbb{Z},$$

then we call that the point x "ε-*shadows*" (or "ε-*traces*") the δ-pseudo orbit ξ. We say that f has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit is ε-shadowed by some point.

In this paper, we shall investigate the "shadowing" property of partially hyperbolic systems. Let f be a partially hyperbolic diffeomorphism. We cannot expect that in general the shadowing property holds for f because of the existence of the center direction. We show in Theorem A that for any pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, there is a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ tracing it in which y_{k+1} is obtained from $f(y_k)$ by a motion τ along center direction. In this case we call that f has *quasi-shadowing property.* Moreover, if center foliation $\mathcal{W}_f^c$ of f exists and is of C^1, then we can choose the motion τ that maps points along the center leaves. This result is given in Theorem B. Theorem B$'$ deal with a particular case, i.e., one dimensional center foliation, in which the map τ can be determined by a flow along the foliation.

In [8] and [11], the notion of pseudo orbits with respect to the plaque of the center foliation is introduced to investigate the robustness of the center foliation for normally hyperbolic and partially hyperbolic systems respectively. Recently, Kryzhevich and Tikhomirov [10] give a version of center shadowing property for dynamically coherent partially hyperbolic diffeomorphisms, that is, any pseudo orbit can be shadowed by a center pseudo orbit. In this paper we show that for any partially hyperbolic diffeomorphism f, without any additional assumption, the quasi-shadowing property holds. In particular, if f has C^1 center foliation, we can also obtain the similar result in [10]. Moreover, the method we use is different from that in [10].

Shadowing property implies some other interesting properties in the study of hyperbolic systems. We can obtain similar results for partially hyperbolic systems from quasi-shadowing property.

It is well known that any Anosov diffeomorphism f on M is *topologically stable* ([15]), that is, for any homeomorphism g C^0-close to f, there exists a surjective continuous map h on M such that $h \circ g = f \circ h$. Topological stability for hyperbolic systems can be obtained by shadowing property ([16], see also [12]). Similarly, as an application of quasi-shadowing property, we show in Theorem C that any partially hyperbolic diffeomorphism has *topological quasi-stability* under C^0 perturbation, that is, for any homeomorphism g C^0-close to f, there exist a surjective continuous h from M to itself and a family of locally defined homeomorphisms $\{\tau_x : x \in M\}$, which map points along the center foliation such that $h \circ g(x) = \tau_{f(x)} \circ f \circ h(x)$ for all $x \in M$. In particular, if center foliation $\mathcal{W}_f^c$ of f exists and is of C^1, then we can choose the motion τ maps points along the center foliation. We can also investigate the quasi-stability of for partially hyperbolic diffeomorphisms under C^0 and C^1-perturbations using a modified version of the method in Theorem A. (see [9] for more details.)

A notable property for hyperbolic systems is Anosov closing lemma, which says that if an orbit returns to a small neighborhood of its initial position, then there is a periodic orbit nearby. Consequently, for an Anosov diffeomorphism, the closure of its periodic orbits is equal to its nonwandering set. It is natural to image that in a partially hyperbolic system, if an orbit returns, then there is a *periodic center leaf* nearby. We prove it by using quasi-shadowing property. Further, we obtain in Theorem D that periodic center leaves are dense in the nonwandering set. If all the center leaves are compact, then the closure of the periodic center leaves is equal to the *center nonwandering set* of the map (see Section 1 for precise meaning).

As the last application in the paper, we give versions of cloud lemma and then spectral decomposition theorem when f has uniformly compact C^1 center foliation, which are generation of the corresponding results for the Axiom A systems (see [3], [14], for example). We use the quasi-shadowing property to show that the center nonwondering set can be uniquely split into finite disjoint *center topologically transitive* closed subsets, and each of which can be uniquely split into finite disjoint sets which are invariant and is *center topologically mixing* under an iteration of f.

This paper is organized as the following. The statements of results are given in Section 1. In Section 2 we deal with the quasi-shadowing property for general case in the proof of Theorem A, where we do not assume existence of center foliation. Section 3 is for the case that the center foliation is of C^1, and proofs of Theorem B and Theorem B′ are given there. The last three sections are concerning applications of of our results. We study topological quasi-stability, denseness of periodic center leaves and spectral decomposition in the center nonwandering sets in Section 4, 5 and 6, respectively.

1 Definition and statement of results

Everywhere in this paper, we assume that M is a smooth m-dimensional compact Riemannian manifold. We denote by $\|\cdot\|$ and $d(\cdot,\cdot)$ the norm on TM and the metric on M induced by the Riemannian metric, respectively.

A diffeomorphism $f : M \to M$ is said to be *(uniformly) partially hyperbolic* if there exist numbers λ, λ', μ and μ' with $0 < \lambda < 1 < \mu$ and $\lambda < \lambda' \le \mu' < \mu$, and an invariant decomposition $T_xM = E^s_x \oplus E^c_x \oplus E^u_x \quad \forall x \in M$, such that for any $n \ge 0$,

$$\begin{aligned}
\|d_x f^n v\| &\le C\lambda^n \|v\| && \text{as } v \in E^s(x),\\
C^{-1}(\lambda')^n\|v\| \le \|d_x f^n v\| &\le C(\mu')^n\|v\| && \text{as } v \in E^c(x),\\
C^{-1}\mu^n \|v\| \le \|d_x f^n v\| & && \text{as } v \in E^u(x)
\end{aligned}$$

hold for some number $C > 0$. The subspaces E^s_x, E^c_x and E^u_x are called *stable, center* and *unstable* subspace, respectively. Via a change of Riemannian metric we always assume that $C = 1$. Moreover, for simplicity of the notation, we assume that $\lambda = \dfrac{1}{\mu}$.

Since M is compact, we can take a constant $\rho_0 > 0$ such that for any $x \in M$, the standard exponential mapping $\exp_x : \{v \in T_xM : \|v\| < \rho_0\} \to M$ is a C^∞ diffeomorphism to the image. Clearly, we have $d(x, \exp_x v) = \|v\|$ for $v \in T_xM$ with $\|v\| < \rho_0$. For any diffeomorphism $f : M \to M$, we take $\rho = \rho_f \in (0, \rho_0/2)$ such that for any $x, y \in M$ with $d(f^{-1}(x), y) \le \rho$, $v \in T_yM$ with $\|v\| \le \rho$,

$$d(x, f \circ \exp_y v) \le \rho_0/2.$$

Reduce ρ if necessary such that both sides in equation (2.3) and (3.2), in the proof of Theorem A and Theorem B respectively, are contained in the set $\{v \in T_xM : \|v\| < \rho_0\}$.

For a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ and a sequence of vectors $\{u_k \in E^c_{x_k}\}_{k\in\mathbb{Z}}$ with $\|u_k\| < \rho$ for any $k \in \mathbb{Z}$, we define a family of smooth maps $\tau^{(1)}_{x_k} = \tau^{(1)}_{x_k}(\cdot, u_k)$ on $B(x_k, \rho)$, $k \in \mathbb{Z}$, by

$$\tau^{(1)}_{x_k}(y) = \exp_{x_k}(u_k + \exp^{-1}_{x_k} y). \tag{1.1}$$

Theorem A. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism. Then f has the quasi-shadowing property in the following sense: for any $\varepsilon \in (0, \rho)$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, there exist a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ and a sequence of vectors $\{u_k \in E^c_{x_k}\}_{k\in\mathbb{Z}}$ such that*

$$d(x_k, y_k) < \varepsilon, \tag{1.2}$$

where

$$y_k = \tau^{(1)}_{x_k}(f(x_{k-1})). \tag{1.3}$$

Moreover, $\{y_k\}_{k\in\mathbb{Z}}$ and $\{u_k\}_{k\in\mathbb{Z}}$ can be chosen uniquely so as to satisfy

$$y_k \in \exp_{x_k}(E^s_{x_k} + E^u_{x_k}). \tag{1.4}$$

The above theorem does not require any additional condition, provided that f is a partially hyperbolic diffeomorphism. Here $\tau^{(1)}_{x_k}$ is a motion in the center direction for any $k \in \mathbb{Z}$. If f has C^1 center foliation $\mathcal{W}^c_f$, then we can make τ to move along the center foliation. In this case, we denote for any $\varepsilon > 0$, $\Sigma_\varepsilon(x) = \exp_x(H_x(\varepsilon))$, where $H_x(\varepsilon)$ is the ε-ball in $E^s_x \oplus E^u_x$. Obviously, $\Sigma_\varepsilon(x)$ is a smooth disk transversal to E^c_x at x. Since the center foliation $\mathcal{W}^c_f$ is C^1, we can conclude that if y is close enough to x, then there is a locally defined diffeomorphism $\tau^{(2)}_x$ on some neighborhood $U(x)$ of x and a constant $K_1 > 1$ independent of x such that for any $y \in U(x)$,

$$\tau^{(2)}_x(y) \in \Sigma_\varepsilon(x) \cap \mathcal{W}^c_f(y) \tag{1.5}$$

and

$$d(\tau^{(2)}_x(y), x) < K_1 d(y, x). \tag{1.6}$$

Theorem B. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with C^1 center foliation $\mathcal{W}^c_f$. Then f has the quasi-shadowing property in the following sense: for any $\varepsilon \in (0, \rho)$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, there exists a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ such that*

$$d(x_k, y_k) < \varepsilon, \tag{1.7}$$

where

$$y_k = \tau^{(2)}_{x_k}(f(x_{k-1})). \tag{1.8}$$

Moreover, $\{y_k\}_{k\in\mathbb{Z}}$ can be chosen uniquely so as to satisfy (1.4).

As a particular case, when the center foliation $\mathcal{W}^c_f$ is C^1 and of dimension one then we can define τ more directly. Let u be the vector field consisting of unit vectors in center direction, i.e., $\|u(x)\| = 1$ for any $x \in M$, and φ^t be the flow generated by u. For a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ and a sequence of real numbers $\{\tilde{\tau}_k\}_{k\in\mathbb{Z}}$, denote a sequence of smooth maps $\tau^{(3)}_{x_k} = \tau^{(3)}_{x_k}(\cdot, \tilde{\tau}_k)$ of $B(x_k, \rho)$ for any $k \in \mathbb{Z}$ given by

$$\tau^{(3)}_{x_k}(z) = \varphi^{\tilde{\tau}_k}(z).$$

Theorem B′. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with one dimensional C^1 center foliation $\mathcal{W}^c_f$. Then f has the quasi-shadowing property in the following sense: for any $\varepsilon \in (0, \rho)$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, there exist a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ and a sequence of real numbers $\{\tilde{\tau}_k\}_{k\in\mathbb{Z}}$ such that*

$$d(x_k, y_k) < \varepsilon, \tag{1.9}$$

where

$$y_k = \tau^{(3)}_{x_k}(f(x_{k-1})). \tag{1.10}$$

Moreover, $\{y_k\}_{k\in\mathbb{Z}}$ can be chosen uniquely so as to satisfy (1.4).

Now we consider applications of our results. The first one is about quasi-stability.

Theorem C. *Assume that $f : M \to M$ is a partially hyperbolic diffeomorphism. Then f has topological quasi-stability in the sense that there exists $\varepsilon_0 \in (0, \rho)$ satisfying the following conditions: for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that for any homeomorphism g of M with $d(f, g) < \delta$ there exist a continuous center section $u = \{u_x \in E^c_x : x \in M\}$ and a surjective continuous map $h : M \to M$ such that*

$$h \circ g(x) = \tau^{(1)}_{g(x)} \circ f \circ h(x), \quad x \in M. \tag{1.11}$$

In addition, h can be chosen uniquely so as to satisfy the following conditions:

$$\begin{aligned} &d(h, \mathrm{id}_M) < \varepsilon, \\ &\exp^{-1}_x(h(x)) \in E^s_x \oplus E^u_x \quad \text{for } x \in M. \end{aligned} \tag{1.12}$$

Moreover, if f has C^1 center foliation $\mathcal{W}^c_f$, then there exists h as above such that (1.11) holds with $\tau^{(1)}_{g(x)}$ replaced by $\tau^{(2)}_{g(x)}$. Furthermore, if the above C^1 center foliation $\mathcal{W}^c_f$ is of one dimensional, then there exist h as above and a continuous function $\tilde{\tau}$ on M such that (1.11) holds with $\tau^{(1)}_{g(x)}$ replaced by $\tau^{(3)}_{g(x)}$.

It is well known that for uniformly hyperbolic systems, closing lemma holds and therefore the periodic points are dense in nonwandering set. We can get a similar result for partially hyperbolic systems by using Theorem B. In this case, periodic center leaves and center nonwandering leaves play the role as periodic points and nonwandering points, respectively.

A center leaf $W^c(p)$ is said to be a *periodic center leaf* with period $n \in \mathbb{N}$ if $W^c(p) = W^c(f^n(p))$. Denote

$$P^c(f) = \{p \in M : W^c(p) \text{ is a periodic center leaf}\}.$$

We say that a center leaf $W^c(x)$ is *center nonwandering* if for any neighborhood U of $W^c(x)$ consisting of center leaves, there is $n \geq 1$ such that $f^n U \cap U \neq \emptyset$. We denote the *center nonwandering set* of f by

$$\Omega^c(f) = \{x \in M : W^c(x) \text{ is center nonwandering}\}.$$

It is easy to see that $\Omega^c(f)$ is a closed invariant set and saturated by W^c, i.e., $W^c(x) \subset \Omega^c(f)$ if $x \in \Omega^c(f)$.

Also we denote by $\Omega(f)$ the nonwandering set of f. Clearly, $\Omega(f) \subset \Omega^c(f)$.

We say that the center foliation is *uniformly compact* if

$$\sup\{\mathrm{vol}(W^c(x)) : x \in M\} < +\infty,$$

where $\mathrm{vol}(W^c(x))$ is the Riemannian volume restricted to the submanifold $W^c(x)$ of M. Uniformly compact center foliations for partially hyperbolic systems were studied in [6].

It is easy to see that if the center foliation is uniformly compact, then a center leaf $W^c(x)$ is center nonwandering if and only if for any $\delta > 0$, there is $y \in M$ and $n \in \mathbb{N}$ such that

$$\max\{d_H(W^c(x), W^c(y)), d_H(W^c(x), W^c(f^n y))\} < \delta, \tag{1.13}$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff distance given by $d_H(A, B) = \max_{a \in A} \min_{b \in B} d(a, b)$ for subsets $A, B \subset M$.

For any set $S \subset M$, denote $W^c(S) = \cup_{x \in S} W^c(x)$.

With the notions $P^c(f)$ and $\Omega^c(f)$, we can get an analogues of Anosov closing lemma for partially hyperbolic diffeomorphisms (see Lemma 5.1 for details). Based on the results, we have the following theorem.

Theorem D. *For any partially hyperbolic diffeomorphism $f : M \to M$ with C^1 center foliation $\mathcal{W}^c_f$, $\Omega(f) \subset \overline{P^c(f)}$.*

Moreover, if the center foliation of f is uniformly compact, then

$$\overline{P^c(f)} = \Omega^c(f) = W^c\big(\Omega(f)\big). \tag{1.14}$$

Further, if a partially hyperbolic diffeomorphism has uniformly compact C^1 center foliation, then we have cloud lemma (see Lemma 6.3), and therefore we can get spectral decomposition for $\Omega^c(f)$.

The substitutes of topological transitivity and topological mixing are center topological transitivity and center topological mixing. An f-invariant set S is said to be *center topologically transitive*, if for any two nonempty open sets U, V in S, there is $n \in \mathbb{N}$ such that

$$f^n(W^c(U)) \cap V \neq \emptyset.$$

S is said to be *center topologically mixing*, if for any two nonempty open sets U, V in S, there is $n_0 \in \mathbb{N}$ such that

$$f^n(W^c(U)) \cap V \neq \emptyset \qquad \forall n \geq n_0.$$

Theorem E. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with uniformly compact C^1 center foliation. Then $\Omega^c(f)$ is a union of finite pairwise disjoint closed sets*

$$\Omega^c(f) = \Omega^c_1 \cup \cdots \cup \Omega^c_k.$$

Moreover, for each $i = 1, 2, \cdots, k$, Ω^c_i satisfies that

(a) $f(\Omega^c_i) = \Omega^c_i$ and $f|_{\Omega^c_i}$ is center topologically transitive;

(b) $\Omega_i^c = X_{1,i} \cup \cdots \cup X_{n_i,i}$ *such that the* $X_{j,i}$ *are disjoint closed sets,* $f(X_{j,i}) = X_{j+1,i}$ *for* $1 \le j \le n_i - 1$*,* $f(X_{n_i,i}) = X_{1,i}$*, and* $f^{n_i}|_{X_{j,i}}$ *is center topologically mixing.*

We call $\Omega_i^c, i = 1, 2, \cdots, k$ the *center basic sets* of f.

Remark 1.1. *We mention that if* f *is dynamically coherent and plaque expansive with respect to the center foliation then the similar results in Theorem B hold ([10]). Therefore, if we replace the* C^1 *smoothness of the center foliation by the weaker conditions, we can obtain the same results in Theorem C, D and E in a similar strategy.*

2 Quasi-shadowing for the general case

We prove Theorem A in this section.

Recall that $\|\cdot\|$ is the norm on TM. We define the norm $\|\cdot\|_1$ on TM by $\|w\|_1 = \|u\| + \|v\|$ if $w = u + v \in T_xM$ with $u \in E_x^c$ and $v \in E_x^u \oplus E_x^s$. For any sequence $\{x_k\}_{k\in\mathbb{Z}}$, Denote

$$\mathfrak{X} = \{w = \{w_k\}_{k\in\mathbb{Z}} : w_k \in T_{x_k}M, k \in \mathbb{Z}\},$$

$$\mathfrak{X}^c = \{u = \{u_k\}_{k\in\mathbb{Z}} : u_k \in E^c_{x_k}, k \in \mathbb{Z}\}$$

and

$$\mathfrak{X}^{us} = \{v = \{v_k\}_{k\in\mathbb{Z}} : v_k \in E^u_{x_k} \oplus E^s_{x_k}, k \in \mathbb{Z}\}.$$

For any $w = u + v \in \mathfrak{X}$, where $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$, we also define

$$\|w\| = \sup_{k\in\mathbb{Z}} \|w_k\|$$

and

$$\|w\|_1 = \|u\| + \|v\|.$$

By triangle inequality and the fact that the angles between E^c and $E^u \oplus E^s$ are uniformly bounded away from zero, we know that there exists a constant L such that

$$\|w\| \le \|w\|_1 \le L\|w\|. \tag{2.1}$$

For any $\varepsilon > 0$, we denote

$$\mathfrak{B}(\varepsilon) = \{w \in \mathfrak{X} : \|w\| \le \varepsilon\}, \qquad \mathfrak{B}^{us}(\varepsilon) = \{w \in \mathfrak{X}^{us} : \|w\| \le \varepsilon\},$$

$$\mathfrak{B}_1(\varepsilon) = \{w \in \mathfrak{X} : \|w\|_1 \le \varepsilon\}.$$

We denote $\Pi_x^s : T_xM \to E_x^s$ be the projection onto E_x^s along $E_x^c \oplus E_x^u$. Π_x^c and Π_x^u are defined in a similar way.

Proof of Theorem A. Given a δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f. To find a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ and a sequence of vectors $\{u_k \in E^c_{x_k}\}_{k\in\mathbb{Z}}$ satisfying (1.2), (1.3) and (1.4), we shall try to solve the equations

$$y_k = \tau^{(1)}_{x_k}(f(x_{k-1})), \quad k \in \mathbb{Z}, \tag{2.2}$$

for unknown $\{y_k\}_{k\in\mathbb{Z}}$ and $\{u_k \in E^c_{x_k}\}_{k\in\mathbb{Z}}$, where $\tau^{(1)}_x$ is defined in (1.1). Put $v_k = \exp^{-1}_{x_k} y_k, k \in \mathbb{Z}$. Then the equations (2.2) can be written as

$$v_k = \exp_{x_k} \tau^{(1)}_{x_k}(f \circ \exp_{x_{k-1}} v_{k-1}), \quad k \in \mathbb{Z}.$$

By (1.1), the equations are equivalent to

$$v_k = u_k + \exp^{-1}_{x_k} \circ f \circ \exp_{x_{k-1}} v_{k-1}, \quad k \in \mathbb{Z}. \tag{2.3}$$

Define an operator $\beta : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}$ and a linear operator $A : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ by

$$(\beta(v))_{k-1} = \exp^{-1}_{x_{k-1}} \circ f \circ \exp_{x_{k-1}} v_{k-1}, \tag{2.4}$$

and

$$(Av)_k = ((A^s + A^u)v)_k = (A^s_{k-1} + A^u_{k-1})v_{k-1}, \tag{2.5}$$

where

$$\begin{aligned} A^s_{k-1} &= \Pi^s_{x_k} \circ d_0(\exp^{-1}_{x_k} \circ f \circ \exp_{x_{k-1}}) \circ \Pi^s_{x_{k-1}}, \\ A^u_{k-1} &= \Pi^u_{x_k} \circ d_0(\exp^{-1}_{x_k} \circ f \circ \exp_{x_{k-1}}) \circ \Pi^u_{x_{k-1}}. \end{aligned} \tag{2.6}$$

Let $\eta = \beta - A$. By (2.4) and (2.5), (2.3) is equivalent to

$$v = u + Av + \eta(v),$$

or

$$v - u - Av = \eta(v).$$

Define a linear operator P from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$Pw = -u + (\mathrm{id}_{\mathfrak{X}^{us}} - A)v, \tag{2.7}$$

and then define an operator Φ from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$\Phi(w) = P^{-1}\eta(v)$$

for $w = u + v$ in the neighborhood of $0 \in \mathfrak{X}$, where $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$.

Hence, the equations (2.3) are equivalent to

$$\Phi(w) = w, \tag{2.8}$$

namely, w is a fixed point of Φ.

By Lemma (2.1) bellow, we know that for any $\varepsilon \in (0, \rho)$ there exists $\delta = \delta(\varepsilon)$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, the operator $\Phi : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ defined as above is a contracting map, and therefore has a fixed point in $\mathfrak{B}_1(\varepsilon)$. Hence, (2.3) has a unique solution. □

Lemma 2.1. *For any $\varepsilon \in (0, \rho)$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, $\Phi(\mathfrak{B}_1(\varepsilon)) \subset \mathfrak{B}_1(\varepsilon)$ and for any $w, w' \in \mathfrak{B}_1(\varepsilon)$,*

$$\|\Phi(w) - \Phi(w')\|_1 \le \frac{1}{2}\|w - w'\|_1.$$

Proof. Recall that $\lambda \in (0,1)$ is given in the definition of partially hyperbolic diffeomorphism. For any $\widetilde{\lambda} \in (\lambda, 1)$ and $\varepsilon \in (0, \rho)$, we take $\delta > 0$ and $C(\delta) > 0$ such that $d(f(y), x) < \delta$ implies

$$\left\|\Pi_x^s \circ d_0(\exp_x^{-1} \circ f \circ \exp_y)|_{E_y^s}\right\| \leq \widetilde{\lambda}, \tag{2.9}$$

$$\left\|\left[\Pi_x^u \circ d_0(\exp_x^{-1} \circ f \circ \exp_y)|_{E_y^u}\right]^{-1}\right\| \leq \widetilde{\lambda}, \tag{2.10}$$

$$\sum_{i,j=s,c,u,\ i\neq j} \left\|\Pi_x^i \circ d_0(\exp_x^{-1} \circ f \circ \exp_y)|_{E_y^j}\right\| \leq \frac{C(\delta)}{2} \tag{2.11}$$

and for any $v', v'' \in H_x(\varepsilon)$ and any $t \in [0,1]$,

$$\|d_{v''+t(v'-v'')}(\exp_x^{-1} \circ f \circ \exp_y) - d_0(\exp_x^{-1} \circ f \circ \exp_y)\| \leq \frac{C(\delta)}{2}. \tag{2.12}$$

We can take $C(\delta) > 0$ in a way such that $C(\delta) \to 0$ as $\delta \to 0$.

Note that if δ satisfies (2.9)–(2.12), then Sublemma 2.2 and 2.3 below can be applied.

Further, we assume δ and $C(\delta)$ are small enough such that

$$\frac{L}{1-\tilde{\lambda}}\delta < \frac{1}{2}\varepsilon, \qquad \frac{L}{1-\tilde{\lambda}}C(\delta) < \frac{1}{2}. \tag{2.13}$$

Take $w = u + v \in \mathfrak{B}_1(\varepsilon)$ with $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$. Note that for any $k \in \mathbb{Z}$, $\|(\eta(0))_k\| = \|(\beta(0))_k\| = \|\exp_{x_k}^{-1} f(x_{k-1})\| \leq \delta$. and hence $\|\eta(0)\| \leq \delta$. So by Sublemma 2.2 and 2.3 below, and (2.13), we can get

$$\begin{aligned}
&\|\Phi(w)\|_1 \leq \|P^{-1}\|_1 \cdot \|\eta(v)\|_1 \leq \frac{1}{1-\tilde{\lambda}} \cdot L\|\eta(v)\| \\
\leq &\frac{L}{1-\tilde{\lambda}}(\|\eta(v) - \eta(0)\| + \|\eta(0)\|) \leq \frac{L}{1-\tilde{\lambda}}(C(\delta)\|v\|_1 + \delta) < \frac{1}{2}\|w\|_1 + \frac{1}{2}\varepsilon \leq \varepsilon,
\end{aligned}$$

which implies that $\Phi(\mathfrak{B}_1(\varepsilon)) \subset \mathfrak{B}_1(\varepsilon)$.

Similarly, for two elements $w = u + v,\ w' = u' + v' \in \mathfrak{B}_1(\varepsilon)$ with $u, u' \in \mathfrak{X}^c$ and $v, v' \in \mathfrak{X}^{us}$, we have

$$\begin{aligned}
&\left\|\Phi(w) - \Phi(w')\right\|_1 \leq \frac{1}{1-\tilde{\lambda}}\left(\|\eta(v) - \eta(v')\|_1\right) \\
\leq &\frac{L}{1-\tilde{\lambda}}\left(\|\eta(v) - \eta(v')\|\right) \leq \frac{L}{1-\tilde{\lambda}}\left(C(\delta)\|w - w'\|_1\right) \leq \frac{1}{2}\|w - w'\|_1.
\end{aligned}$$

This proves that $\Phi : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ is a contraction. □

Sublemma 2.2. *For $\delta > 0$ satisfying (2.9)–(2.12) and any $v, v' \in \mathfrak{B}^{us}(\varepsilon)$,*

$$\|\eta(v') - \eta(v)\| \leq C(\delta)(\|v' - v\|),$$

where $C(\delta)$ is chosen in the beginning of the proof of Lemma 2.1.

Proof. Denote $\eta_k(v_k) = (\eta(v))_{k+1}$ for $v = \{v_k\}_{k\in\mathbb{Z}}$ in a neighborhood of $0 \in \mathfrak{X}^{us}$. By the definition of η, we can write

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)},$$

where

$$\eta_k^{(1)}(v_k) = \exp_{x_{k+1}}^{-1} \circ f \circ \exp_{x_k}(v_k) - d_0(\exp_{x_{k+1}}^{-1} \circ f \circ \exp_{x_k})v_k$$

and

$$\eta_k^{(2)}(v_k) = \sum_{i=s,c,u,\ j=s,u,\ i\neq j} \Pi^i_{x_{k+1}} \circ d_0(\exp^{-1}_{x_{k+1}} \circ f \circ \exp_{x_k}) \circ \Pi^j_{x_k} v_k.$$

Note that for $v', v'' \in H_k(\varepsilon)$, we have

$$\begin{aligned} & \left\|\eta_k^{(1)}(v') - \eta_k^{(1)}(v'')\right\| \\ = & \left\| \int_0^1 \left[d_{v''+t(v'-v'')}(\exp^{-1}_{x_{k+1}} \circ f \circ \exp_{x_k}) - d_0(\exp^{-1}_{x_{k+1}} \circ f \circ \exp_{x_k})\right](v'-v'')dt \right\| \\ \leq & \sup_{t\in[0,1]} \left\|d_{v''+t(v'-v'')}(\exp^{-1}_{x_{k+1}} \circ f \circ \exp_{x_k}) - d_0(\exp^{-1}_{x_{k+1}} \circ f \circ \exp_{x_k})\right\| \cdot \left\|v'-v''\right\|. \end{aligned}$$

Therefore, from (2.12) we have

$$\|\eta_k^{(1)}(v') - \eta_k^{(1)}(v'')\| \leq \frac{C(\delta)}{2}\|v'-v''\|. \tag{2.14}$$

By (2.11), we have for $v', v'' \in H_k(\varepsilon)$

$$\|\eta_k^{(2)}(v') - \eta_k^{(2)}(v'')\| \leq \frac{C(\delta)}{2}\|v'-v''\|. \tag{2.15}$$

Combining (2.14) and (2.15), for $v', v'' \in H_k(\varepsilon)$ we have

$$\|\eta_k(v') - \eta_k(v'')\| \leq C(\delta)\|v'-v''\|. \tag{2.16}$$

Hence, we can get the result we need immediately. $\square$

Sublemma 2.3. *For any $\delta > 0$ satisfying (2.9)–(2.12) and any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, the operator P defined as (2.7) is invertible and*

$$\|P^{-1}\|_1 \leq \frac{1}{1-\tilde{\lambda}}.$$

Proof. By the definition of P, we have $P|_{\mathfrak{X}^i} = id_{\mathfrak{X}^i} - A^i,\ i = s, u$, and $P|_{\mathfrak{X}^c} = \mathrm{id}_{\mathfrak{X}^c}$. So $P(\mathfrak{X}^i) = \mathfrak{X}^i,\ i = u, s, c$.

By (2.10) and (2.9), $\|A^s\|, \|(A^u)^{-1}\| \leq \tilde{\lambda} < 1$. Hence, both $P|_{\mathfrak{X}^s}$ and $P|_{\mathfrak{X}^u}$ are invertible and

$$(P|_{\mathfrak{X}^s})^{-1} = (id_{\mathfrak{X}^s} - A^s)^{-1} = \sum_{k=0}^{\infty} A^s_k,$$
$$(P|_{\mathfrak{X}^u})^{-1} = (id_{\mathfrak{X}^u} - A^u)^{-1} = -\sum_{k=1}^{\infty} (A^u_k)^{-1}.$$

It follows that

$$\|(P|_{\mathfrak{X}^{us}})^{-1}\| \leq \max\left\{\|(P|_{\mathfrak{X}^s})^{-1}\|, \|(P|_{\mathfrak{X}^u})^{-1}\|\right\} \leq \frac{1}{1-\tilde{\lambda}}.$$

It is obvious that

$$\|(P|_{\mathfrak{X}^c})^{-1}\| = 1.$$

So we obtain that

$$\|P^{-1}\|_1 \leq \max\left\{\|(P|_{\mathfrak{X}^{us}})^{-1}\|, \|(P|_{\mathfrak{X}^c})^{-1}\|\right\} \leq \frac{1}{1-\tilde{\lambda}}.$$

This is what we need. $\square$

3 Quasi-shadowing for the system with C^1 center foliation

3.1 The general case

Recall that $\mathfrak{X}^{us}$ and $\mathfrak{B}^{us}(\rho)$ are defined in the beginning of the previous section.

Proof of Theorem B. The proof is similar to that of Theorem A.

Given a δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f. To find a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ satisfying (1.7), (1.8) and (1.4), we shall try to solve the equations

$$y_k = \tau^{(2)}_{x_k}(f(x_{k-1})) \tag{3.1}$$

for unknown $\{y_k\}_{k\in\mathbb{Z}}$. Put $v_k = \exp^{-1}_{x_k} y_k$. Then the equations (3.1) are equivalent to

$$v_k = \exp^{-1}_{x_k} \circ \tau^{(2)}_{x_k} \circ f \circ \exp_{x_{k-1}} v_{k-1}, \qquad k \in \mathbb{Z}. \tag{3.2}$$

Define an operator $\beta : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ and a linear operator $A : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ by

$$(\beta(v))_k = \exp^{-1}_{x_k} \circ \tau^{(2)}_{x_k} \circ f \circ \exp_{x_{k-1}} v_{k-1}, \quad k \in \mathbb{Z}, \tag{3.3}$$

and

$$(Av)_k = (A^u_{k-1} + A^s_{k-1})v_{k-1}, \tag{3.4}$$

where

$$A^s_{k-1} = \Pi^s_{x_k} \circ d_0(\exp^{-1}_{x_k} \circ \tau^{(2)}_{x_k} \circ f \circ \exp_{x_{k-1}}) \circ \Pi^s_{x_{k-1}},$$
$$A^u_{k-1} = \Pi^u_{x_k} \circ d_0(\exp^{-1}_{x_k} \circ \tau^{(2)}_{x_k} \circ f \circ \exp_{x_{k-1}}) \circ \Pi^u_{x_{k-1}}.$$

Let $\eta = \beta - A$. By (3.3) and (3.4), (3.2) is equivalent to

$$v = Av + \eta(v),$$

further, is equivalent to

$$v - Av = \eta(v).$$

Define a linear operator P from a neighborhood of $0 \in \mathfrak{X}^{us}$ to $\mathfrak{X}^{us}$ by

$$Pv = (id_{\mathfrak{X}^{us}} - A)v, \tag{3.5}$$

and then define an operator Φ from a neighborhood of $0 \in \mathfrak{X}^{us}$ to $\mathfrak{X}^{us}$ by

$$\Phi(v) = P^{-1}\eta(v) \tag{3.6}$$

for v in a neighborhood of $0 \in \mathfrak{X}^{us}$.

Hence, the equations (3.2) are equivalent to

$$\Phi(v) = v, \tag{3.7}$$

namely, v is a fixed point of Φ.

The remaining work is to show that for any $\varepsilon \in (0, \rho)$ there exists $\delta = \delta(\varepsilon)$ such that for a δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, $\Phi : \mathfrak{B}^{us}(\varepsilon) \to \mathfrak{B}^{us}(\varepsilon)$ is a contracting map, and therefore has a fixed point in $\mathfrak{B}^{us}(\varepsilon)$. Hence, (3.2) has a unique solution. To this end we only need to modify the proof of Lemma 2.1 to a easer version since in this case we do not need to consider the center direction. We leave the details to the reader. $\square$

3.2 $\mathcal{W}_f^c$ is of one dimensional

Proof of Theorem B′. The proof is also similar to that of Theorem A.

Given a δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f. To find a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ and and a sequence of real numbers $\{\tilde{\tau}_k\}_{k\in\mathbb{Z}} \in \mathfrak{C}(\rho)$, where $\mathfrak{C}(\rho) = \{\tilde{\tau} = \{\tilde{\tau}_k\}_{k\in\mathbb{Z}} : \tilde{\tau}_k \in \mathbb{R}, |\tilde{\tau}_k| \leq \rho, k \in \mathbb{Z}\}$, satisfying (1.9), (1.10) and (1.4), we shall try to solve the equations

$$y_k = \tau_{x_k}^{(3)}(f(x_{k-1})) \tag{3.8}$$

for unknown $\{y_k\}_{k\in\mathbb{Z}}$ and $\{\tilde{\tau}_k\}_{k\in\mathbb{Z}}$. Putting $v_k = \exp_{x_k}^{-1} y_k$, then the equations (3.8) are equivalent to

$$v_{k+1} = \exp_{x_{k+1}}^{-1} \circ \tau_{x_{k+1}}^{(3)} \circ f \circ \exp_{x_k} v_k, \quad k \in \mathbb{Z},$$

i.e.,

$$v_{k+1} = \exp_{x_{k+1}}^{-1} \circ \varphi^{\tilde{\tau}_{k+1}} \circ f \circ \exp_{x_k} v_k, \quad k \in \mathbb{Z}. \tag{3.9}$$

Define an operator $\beta : \mathfrak{B}^{us}(\rho) \times \mathfrak{C}(\rho) \to \mathfrak{X}$ and a linear operator $A : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ by

$$\beta(v, \tilde{\tau})_{k+1} = \exp_{x_{k+1}}^{-1} \circ \varphi^{\tilde{\tau}_{k+1}} \circ f \circ \exp_{x_k}(v_k). \tag{3.10}$$

and

$$(Av)_k = (A_{k-1}^u + A_{k-1}^s) v_{k-1}, \tag{3.11}$$

where

$$A_{k-1}^s = \Pi_{x_k}^s \circ (d_{(0,0)}\beta(v, \tilde{\tau}))_k \circ \Pi_{x_{k-1}}^s,$$
$$A_{k-1}^u = \Pi_{x_k}^u \circ (d_{(0,0)}\beta(v, \tilde{\tau}))_k \circ \Pi_{x_{k-1}}^u.$$

Let $\eta = \beta - A$. Let u be a vector field consisting of unit vectors tangent to $\mathcal{W}_f^c$. Then by (3.10) and (3.11), (3.9) is equivalent to

$$v = \tilde{\tau} u + Av + \eta(v),$$

for some $\tilde{\tau} \in \mathfrak{C}(\rho)$. Further, the equations are equivalent to

$$-\tilde{\tau} u + v - Av = \eta(v).$$

Define a linear operator P from a neighborhood of $0 \in \mathfrak{X}(\rho)$ to $\mathfrak{X}$ by

$$Pv = \tilde{\tau} u + (id_{\mathfrak{X}^{us}} - A)v, \tag{3.12}$$

and then define an operator Φ from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$\Phi(w) = P^{-1}\eta(v),$$

where $w = \tilde{\tau} \cdot u + v \in \mathfrak{X}$ with $v \in \mathfrak{B}^{us}(\rho)$ and $\tilde{\tau} \in \mathfrak{C}(\rho)$.

Hence, the equations (3.9) are equivalent to

$$\Phi(\tilde{\tau} \cdot u + v) = \tilde{\tau} \cdot u + v, \tag{3.13}$$

namely, $\tilde{\tau} \cdot u + v$ is a fixed point of Φ.

The remaining work is to show that for any $\varepsilon \in (0, \rho)$ there exists $\delta = \delta(\varepsilon)$ such that for δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, $\Phi : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ is a contracting map, and therefore has a fixed point in $\mathfrak{B}_1(\varepsilon)$. Hence, (3.9) has a unique solution. We leave the details to the reader.

□

4 Quasi-stability

It is well known that for any homeomorphism f on a compact metric space, shadowing property together with expansiveness implies topological stability (see [16] for example). In the case of partially hyperbolic diffeomorphism, we can get topological quasi-stability from quasi-shadowing property.

Proof of Theorem C. For the simplicity of the notation, we only prove this theorem under the condition that f is a partially hyperbolic diffeomorphism with C^1 center foliation.

Choose $\varepsilon_0 \in (0, \rho)$ small enough such that any continuous map h with $d(h, \mathrm{id}_M) < \varepsilon_0$ must be surjective (see e.g Lemma 3 of [15] for existence of such ε_0).

Let $\varepsilon \in (0, \varepsilon_0)$. From Theorem B, there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of f, there exist a unique pseudo orbit $\{y_k\}_{k\in\mathbb{Z}}$ ε-quasi-shadowing it that satisfies (1.4) and $y_{k+1} \in W^c(y_k)$ for all $k \in \mathbb{Z}$. Let g be any homeomorphism with $d(f, g) < \delta$. It is obvious that for any $x \in M$, its orbit $\mathrm{orb}_g(x) = \{x_k = g^k(x)\}_{k\in\mathbb{Z}}$ is a δ-pseudo orbit of f, hence, there exists a unique corresponding pseudo orbit $\{y_k\}_{k\in\mathbb{Z}}$ ε-quasi-shadowing it. Let $h(x) = y_0$.

Now we consider continuity of h. Notice that the sequence $\{y_k\}_{k\in\mathbb{Z}}$, which is ε-quasi-shadowing the orbit of x, is defined by the sequence $v = \{\exp^{-1}_{g^k(x)} y_k\}_{k\in\mathbb{Z}}$, and v is the fixed point of the operator $\Phi_{\mathrm{orb}_g(x)} : \mathfrak{B}^{us}_{\mathrm{orb}_g(x)}(\varepsilon) \to \mathfrak{B}^{us}_{\mathrm{orb}_g(x)}(\varepsilon)$ in the proof of Theorem B (here we use the notions $\Phi_{\mathrm{orb}_g(x)}$ and $\mathfrak{B}^{us}_{\mathrm{orb}_g(x)}(\varepsilon)$ since they all depend on $\mathrm{orb}_g(x)$). Given x' near x, denote by $v' = \{v'_k \in E^{us}_{g^k(x')}\}$ the unique fixed point of the operator $\Phi_{\mathrm{orb}_g(x')} : \mathfrak{B}^{us}_{\mathrm{orb}_g(x')}(\varepsilon) \to \mathfrak{B}^{us}_{\mathrm{orb}_g(x')}(\varepsilon)$. By the definition of h, $h(x') = \exp_{x'}(v'_0)$. By continuity of the distribution E^{us}, continuity of the differential of f and the construction of the operator Φ, we can see that as x' approaches x, v'_0 approaches v_0 in the tangent bundle TM. Therefore, $h(x')$ arbitrarily approaches $h(x)$ as x' sufficiently close to x. This means that the map h is continuous. □

5 Center Nonwandering Sets

It is well known that for a uniformly hyperbolic system $f : M \to M$, if x is close to $f^n x$ for some $x \in M$ and $n > 0$, then there is a periodic point $y \in M$ of period n close to x. The result is the main part of Anosov closing lemma (see e.g. [3, 14]), and sometimes is directly called Anosov closing lemma (see e.g. [7]).

The next lemma is an analogue of the result for partially hyperbolic diffeomorphisms.

Lemma 5.1. *Suppose $f : M \to M$ is a partially hyperbolic diffeomorphism with C^1 center foliation $\mathcal{W}^c_f$. For any $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that for any $x \in M$ and $n \in \mathbb{N}$ with $d(x, f^n x) < \delta$, there exists a periodic center leaf $W^c(p)$ of period n satisfying $d(p, x) \le \varepsilon$.*

Moreover, if $W^c(x)$ is compact and $d_H\big(W^c(x), f^n(W^c(x))\big) < \delta$, then there exists a periodic center leaf $W^c(p)$ of period n such that $d(p, x') \le \varepsilon$ for some $x' \in W^c(x)$.

Proof. By Theorem B, there is $\delta \in (0, \varepsilon)$ such that any δ-pseudo orbit can be ε-quasi-shadowed. Since $d(x, f^n x) < \delta$, we can repeat the orbit segment $\{x, fx, \cdots, f^{n-1}x\}$ to get a δ-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, where $x_k = f^i x$ if $k \equiv i (\mathrm{mod}\, n)$. By Theorem B, there is a sequence $\{y_k\}_{k\in\mathbb{Z}}$ ε-quasi-shadows $\{x_k\}_{k\in\mathbb{Z}}$. Note that $x_{n+i} = x_i$ for all $i \in \mathbb{Z}$, $\{y_{n+k}\}_{k\in\mathbb{Z}}$ also ε-quasi-shadows $\{x_k\}_{k\in\mathbb{Z}}$.

By uniqueness of the quasi-shadowing point, $y_n = y_0$. Note that if $\{y_k\}_{k\in\mathbb{Z}}$ ε-quasi-shadows $\{x_k\}_{k\in\mathbb{Z}}$, then $f(y_{k-1}) \in W^c(y_k)$ and therefore $fW^c(y_{k-1}) = W^c(y_k)$. So $y_n = y_0$ implies $f^n W^c(y_0) = W^c(y_n) = W^c(y_0)$, that is, $W^c(y_0)$ is a periodic center leaf. So the result of the first part of the lemma follows with $p = y_0$.

To prove the second part, we use $d_H\big(W^c(x), W^c(f^n x)\big) < \delta$. Take $x_0 = x$ and then take $x_1 \in W^c(x)$ such that $d(x_1, f^n(x_0)) < \delta$. Inductively, for any $i \geq 1$, if $x_{i-1} \in W^c(x)$ is taken, then we can choose $x_i \in W^c(x)$ such that $d(x_{i-1}, x_i) < \delta$. Since $W^c(x)$ is compact, there exist $i < j$ such that $d(x_i, x_j) < \delta$. Repeating the pseudo orbit segment

$$x_i, f(x_i), \cdots, f^{n-1}(x_i), x_{i+1}, f(x_{i+1}), \cdots, f^{n-1}(x_{i+1}), \cdots, x_{j-1}, f(x_{j-1}), \cdots, f^{n-1}(x_{j-1})$$

we get a δ-pseudo orbit $\{x^{(k)}\}_{k\in\mathbb{Z}}$ satisfying that $x^{(0)} = x_i$ and $x^{(k)} = x^{(\ell)}$ if $k \equiv \ell \ (\mathrm{mod}\, n(j-i))$.

By the same arguments as above, there is a periodic center leaf $W^c(y)$ with $d(x^{(0)}, y) \leq \varepsilon$. Hence, we get the result if we take $p = y$ and $x' = x^{(0)}$. $\square$

Proof of Theorem D. The first result of the theorem follows from the first result of Lemma 5.1 immediately. This is because for any $x \in \Omega(f)$ and $\varepsilon > 0$, we can find $y \in \Omega(f)$ and $n > 0$ such that $d(x, y) \leq \varepsilon$ and $d(y, f^n y) \leq \delta$, and therefore there exists $p \in P^c(f)$ with $d(x, p) < \varepsilon$. Hence, $d(x, p) \leq 2\varepsilon$. It means that $x \in \overline{P^c(f)}$. We get $\Omega(f) \subset \overline{P^c(f)}$.

Now we consider the second part of the theorem. Note that if f has uniformly compact center foliation, then by (1.13), for any $x \in \Omega^c(f)$ there is $y \in M$ such that $d_H\big(W^c(x), W^c(y)\big) < \varepsilon$ and $d_H\big(W^c(y), f^n(W^c(y))\big) < \delta$. By the same arguments we get that there exists $p \in P^c(f)$ with $d(y', p) < \varepsilon$ for some $y' \in W^c(y)$. Hence $d(x', p) < \varepsilon$ for some $x' \in W^c(x)$. And we get $\Omega^c(f) \subset \overline{P^c(f)}$.

By Lemma 5.2 below, we know that $\Omega^c(f) \subset W^c(\Omega(f))$. Since it is obvious that $\overline{P^c(f)} \subset \Omega^c(f)$ and $W^c(\Omega(f)) \subset \Omega^c(f)$, we get the equality (1.14). $\square$

Lemma 5.2. *Suppose $f : M \to M$ is a partially hyperbolic diffeomorphism with compact C^1 center foliation $\mathcal{W}^c_f$. Then for any $x \in \Omega^c(f)$, there exists $x' \in \Omega(f) \cap W^c(x)$.*

Proof. Suppose $\Omega(f) \cap W^c(x) = \emptyset$. Then any point $y \in W^c(x)$ is a wandering point. Hence, there is a neighborhood U_y of y such that $f^n(U_y) \cap U_y = \emptyset$ for any $n > 0$. Clearly $\{U_y : y \in W^c(x)\}$ form a open cover of $W^c(x)$. Let $\{U_1, \ldots, U_k\}$ be a subcover of $W^c(x)$, and let $U = \cup_{i=1}^k U_i$. Then U is a neighborhood of $W^c(x)$. By Lemma 5.1, U contains a periodic leaf $W^c(z)$.

Suppose $W^c(z)$ has period ℓ. Then $f^{j\ell}(z) \in U$ for any $j > 1$. Since $U = \cup_{i=1}^k U_i$, there are $j_1 < j_2$ such that $f^{j_1\ell}(z), f^{j_2\ell}(z) \in U_i$ for some U_i. That is, $f^{(j_2-j_1)\ell} U_i \cap U_i \neq \emptyset$, contrdicting the fact that $f^n(U_y) \cap U_y = \emptyset$ for any $n > 0$. $\square$

6 A spectral decomposition theorem

In this section, we assume that $f : M \to M$ is a partially hyperbolic diffeomorphism with uniformly compact C^1 center foliation.

Denote by $W^u_\varepsilon(x)$ and $W^s_\varepsilon(x)$ the local unstable and stable manifolds of size ε at x respectively.

We recall that f is dynamically coherent since the center foliation W^c of f is C^1 ([13]).

The next lemma gives the local product structure.

Lemma 6.1. *There are $\varepsilon, \delta > 0$ such that for any $x, y \in M$ with $d(x, y) < \delta$, for any $x_1 \in W^c(x)$, there is $y_1 \in W^c(y)$ such that $W^s_\varepsilon(x_1) \cap W^u_\varepsilon(y_1)$ contains exact one point.*

Proof. Since W^s and W^{cu} are uniformly transversal, and W^u subfoliate W^{uc}, it is obvious that there are $\varepsilon, \delta' > 0$ such that if $w, z \in M$ with $d(w, z) < \delta'$, then $W^s_\varepsilon(w) \cap W^u_\varepsilon(z_1)$ contains exact one point for some $z_1 \in W^u(z)$.

Since f has uniformly compact C^1 center foliation, we can take $\delta \in (0, \delta')$ such that if $d(x, y) < \delta$, then $d_H\big(W^c(x), W^c(y)\big) < \delta'$. Then for any $x_1 \in W^c(x)$, we can find $y_0 \in W^c(y)$ such that $d(x_1, y_0) < \delta'$. Thus the result follows. □

For uniformly hyperbolic systems, the cloud lemma gives that for any periodic points p and q, any point $x \in W^u(p) \cap W^s(q)$ is contained in the nonwandering set of the map (see e.g. [14]). The next lemma can be regarded as a local version of the cloud lemma for partially hyperbolic diffeomorphisms with uniformly compact C^1 center foliation.

Lemma 6.2. *Let $p, q \in P^c(f)$ with $d(p, q) < \delta$. If $x \in W^s_\varepsilon(p_1) \cap W^u_\varepsilon(q_1)$ for some $p_1 \in W^c(p)$ and $q_1 \in W^c(q)$, then $x \in \Omega^c(f)$.*

Proof. By the definition of $\Omega^c(f)$ and uniform compactness of the center foliation, it is sufficient to prove that for any $\alpha > 0$, there are a point y and a number $n \in \mathbb{N}$ such that

$$d(x, y) < \alpha \qquad \text{and} \qquad d(f^n(y), W^c(y)) < \alpha. \tag{6.1}$$

By uniform compactness of the center foliation, there exists $\beta \in (0, \alpha)$ such that

$$d(x, y) < \beta \quad \Longrightarrow \quad d_H(W^c(x), W^c(y)) < \frac{\alpha}{2} \qquad \forall x, y \in M. \tag{6.2}$$

Since $x \in W^s(p_1)$ and $p_1 \in P^c(f)$, $\{f^i(x)\}_{i\geq 0}$ has an accumulated point $p_2 \in W^c(p)$ (See Figure 1). Note that $d(p_2, W^c(q)) < \delta'$ by the choice of δ in the proof of Lemma 6.1. Hence, we can find $i_0 > 0$ such that $d(f^{i_0}(x), W^c(q)) < \delta'$. By Lemma 6.1, there are $q_2 \in W^c(q)$ and $z_0 \in M$ such that $z_0 \in W^u_\varepsilon(f^{i_0}(x)) \cap W^s_\varepsilon(q_2)$. Set $z = f^{-i_0}(z_0)$. We can choose i_0 large enough such that

$$d(z, x) < \frac{\beta}{2},$$

where β is given in (6.2). Note that $\{f^n(z_0)\}_{n\geq 0}$ has an accumulated point $q_3 \in W^c(q)$ since $z \in W^{cs}(q_2) = W^{cs}(q)$ and $W^c(q)$ is compact. We may assume $f^{n_j}(z_0) \to q_3$ for some $n_j \to +\infty$. This implies that

$$\lim_{j\to\infty} f^{n_j - i_0}(z) = q_3. \tag{6.3}$$

Recall that $x \in W^u_\varepsilon(q_1) \subset W^{cu}(q)$. There is a point $x' \in W^c(x) \cap W^u(q_3)$ since W^c and W^u subfoliate W^{cu}.

Note that $z \in W^u_\varepsilon(x)$ and $x' \in W^u(q_3)$. By continuity of the unstable foliation, (6.3), implies that there are a point $y \in W^u_{\frac{\beta}{2}}(z)$ and an integer $j_0 \in \mathbb{N}$ such that $d(f^{j_0}(y), x') < \frac{\alpha}{2}$, and therefore

$$d(f^{j_0}(y), W^c(x)) < \frac{\alpha}{2}.$$

Also we have that $d(x,y) \le d(x,z) + d(z,y) < \frac{\beta}{2} + \frac{\beta}{2} = \beta$. So (6.2) can be applied, and we have

$$d(f^{j_0}(y), W^c(y)) \le d(f^{j_0}(y), W^c(x)) + d(W^c(x), W^c(y)) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Now we get (6.1) with $n = j_0$, and complete the proof. □

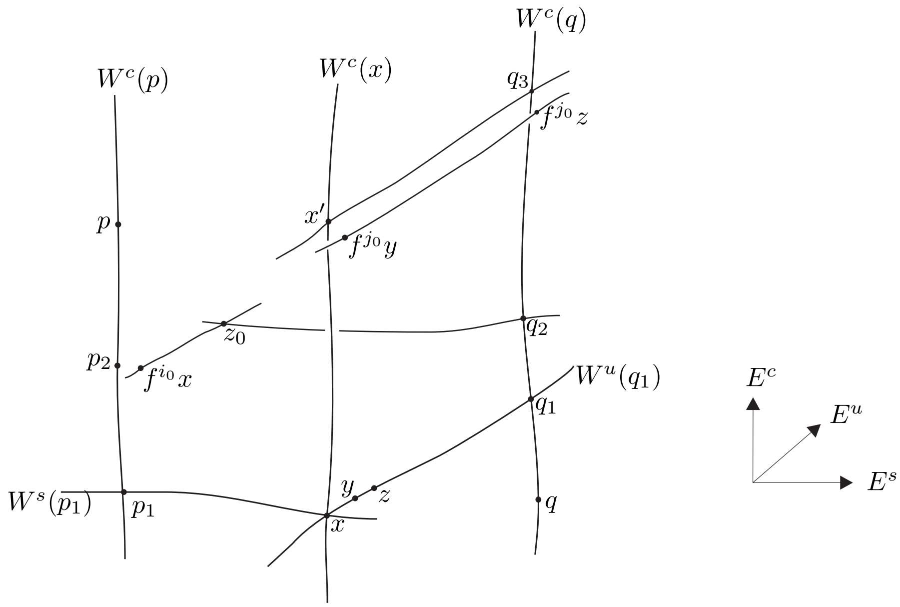

Figure 1: intersection of stable and unstable manifolds

Proof of Theorem E. For any $p \in P^c(f)$, we set

$$X_p = \overline{W^{cu}(p) \cap \Omega^c(f)}.$$

By Lemma 6.3 below, X_p is both open and closed in $\Omega^c(f)$. By Lemma 6.4 below, $\{X_p : p \in P^c(f)\}$ are either identical or disjoint. Since $\Omega^c(f)$ is compact, there are finitely many points $p_1, \cdots, p_n \in P^c(f)$ such that

$$\Omega^c(f) = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_n},$$

where X_{p_i} are pairwise disjointed. Then $f(X_{p_j}) = X_{f(p_j)}$ and hence equals to some X_{p_i}. So f permutes these X_{p_j}'s. We set Ω_i^c as the union of the X_{p_j}'s in the various cycles of permutation. Then we can get

$$\Omega^c(f) = \Omega_1^c \cup \cdots \cup \Omega_k^c.$$

Center topological transitivity in (a) is implied by center topological mixing in (b). For finishing the proof, we only need to prove that $f^N : X_p \to X_p$ is center mixing whenever $p \in P^c(f)$ and $N \in \mathbb{N}$ satisfying $f^N(X_p) = X_p$.

Suppose U, V are nonempty open subsets in X_p. We choose a point $q \in P^c(f) \cap U$, and assume that $W^c(q)$ has period n. Note that $n = tN$ for some $t \ge 1$.

We firstly prove that there is $i_0 \in \mathbb{N}$ such that

$$f^{in}(W^c(U)) \cap V \neq \emptyset \qquad \forall i \geq i_0. \tag{6.4}$$

In fact, since U is open, there exists $\varepsilon > 0$ such that

$$B_{\varepsilon,\Omega^c(f)}(W^c(q)) = \{x \in \Omega^c(f) : d(x, W^c(q)) < \varepsilon\} \subset W^c(U).$$

On the other hand, since $W^{cu}(q)$ is dense in $X_q = X_p$, we can select a point $z \in W^{cu}(q) \cap \Omega^c(f) \cap V$. Then there is $i_0 \in \mathbb{N}$ such that

$$f^{-in}(z) \in B_{\varepsilon,\Omega^c(f)}(W^c(q)) \qquad \forall i \geq i_0$$

and hence this proves (6.4).

Similarly to (6.4), for any $j = 1, \cdots, t-1$, there is $i_j \in \mathbb{N}$ such that

$$f^{in}(f^{jN}(W^c(U))) \cap V \neq \emptyset \qquad \forall i \geq i_j. \tag{6.5}$$

Set $i_* = t \cdot \max\{i_0, i_1, \cdots, i_{t-1}\}$. Then, for any $i \geq i_*$, we can write

$$iN = ln + jN,$$

where $l \geq \max\{i_0, i_1, \cdots, i_{t-1}\}$ and $1 \leq j \leq t-1$. So, by (6.4) and (6.5)

$$f^{iN}(W^c(U)) \cap V = f^{ln}(f^{jN}(W^c(U))) \cap V \neq \emptyset \qquad \forall i \geq i_*.$$

That is to say, $f^N|_{X_p}$ is center mixing. We complete the proof of Theorem E. $\square$

Lemma 6.3. *There exists $\delta > 0$ such that for any $p \in P^c(f)$,*

$$B_{\delta,\Omega^c(f)}(X_p) := \{x \in \Omega^c(f) : \ d(x, X_p) < \delta\} = X_p,$$

where $d(x, X_p) = \min_{y \in X_p} d(x, y)$.

Proof. Let $\delta > 0$ as in Lemma 6.1. By Theorem D, we only need to prove that $q \in X_p$ for any $q \in P^c(f)$ with $d(q, X_p) < \delta$.

By the definition of X_p, we can find a point $x \in W^{cu}(p) \cap \Omega^c(f)$ such that $d(x, q) < \delta$. By Lemma 6.1, we can take $z \in W^u(x) \cap W^s(q)$. Since $x \in \Omega^c(f)$, Theorem D implies that there are infinitely many points $p_n \in P^c(f)$ such that $p_n \to x$ as $n \to \infty$. Hence, $d(p_n, q) < \delta$ for all n large enough. By Lemma 6.1, there exist $q_n \in W^c(q)$ and $z_n \in M$ such that $z_n \in W^u_\varepsilon(p_n) \cap W^s_\varepsilon(q_n)$. By Lemma 6.2, $z_n \in \Omega^c(f)$. Note that by continuity if $p_n \to x$, then $q_n \to q$ and $z_n \to z$. We get $z \in \Omega^c(f)$. Since $z \in W^u(x)$ and $x \in W^{cu}(p)$, we get $z \in X_p$ by the definition of X_p. Further, since $z \in W^s(q)$ and $q \in P^c(f)$, $\{f^n(z)\}_{n \geq 0}$ has at least one accumulated point in $W^c(q)$. This implies that $W^c(q) \subset X_p$ and we complete the proof. $\square$

Lemma 6.4. *Let $p, q \in P^c(f)$ and $X_p \cap X_q \neq \emptyset$. Then $X_p = X_q$.*

Proof. Since $X_p \cap X_q \neq \emptyset$, there are points $x \in W^{cu}(p) \cap \Omega^c(f)$ and $q' \in X_q \cap P^c(f)$ such that $d(x, q') < \delta$. By Lemma 6.1, there exists a point $z \in \Omega^c(f)$ such that $z \in W^u_\varepsilon(x) \cap W^s_\varepsilon(q'_1)$ for some $q'_1 \in W^c(q')$. Let n be the period of $W^c(q')$. Then

$$\lim_{i \to +\infty} d\big(f^{in}(z), W^c(q')\big) = 0.$$

By Lemma 6.3, $f^{in}(z) \in X_{q_1}$ for i large enough and hence $z \in X_q$.

At the same time, we have

$$\lim_{i \to +\infty} d\big(f^{-in}(z), W^c(p)\big) = 0.$$

So, $W^c(p) \subset X_q$.

For any $y \in W^{cu}(p) \cap \Omega^c(f)$, one has

$$\lim_{i \to +\infty} d\big(f^{-im}(y), W^c(p)\big) = 0,$$

where m is the period of $W^c(p)$. So $f^{-im}(y) \in B_{\delta, \Omega^c(f)}(W^c(p)) \subset B_{\delta, \Omega^c(f)}(X_q)$ for i large enough and hence $y \in X_q$. Noting that $X_p = \overline{W^{cu}(p) \cap \Omega^c(f)}$, we have $X_p \subset X_q$.

Similarly, one can get $X_q \subset X_p$. This completes the proof. $\square$

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