Nonadditive Measure-theoretic Pressure and Applications to Dimensions of an Ergodic Measure

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Abstract. Without any additional conditions on subadditive potentials, this paper defines subadditive measure-theoretic pressure, and shows that the subadditive measure-theoretic pressure for ergodic measures can be described in terms of measure-theoretic entropy and a constant associated with the ergodic measure. Based on the definition of topological pressure on non-compact set, we give another equivalent definition of subadditive measure-theoretic pressure, and obtain an inverse variational principle. This paper also studies the supadditive measure-theoretic pressure which has similar formalism as the subadditive measure-theoretic pressure. As an application of the main results, we prove that an average conformal repeller admits an ergodic measure of maximal Haudorff dimension. Furthermore, for each ergodic measure supported on an average conformal repeller, we construct a set whose dimension is equal to the dimension of the measure.

Key words and phrases. nonadditive, measure-theoretic pressure, variational principle, ergodic measure, Hausdorff dimension.

0 Introduction.

It is well-known that the topological pressure for additive potentials was first introduced by Ruelle for expansive maps acting on compact metric spaces ([24]), furthermore he formulated a variational principle for the topological pressure in that paper. Later, Walters ([26]) generalized these results to general continuous maps on compact metric spaces. In [23], Pesin and Pitskel' defined the topological pressure for noncompact sets which is a generalization of Bowen's definition of topological entropy for noncompact sets ([4]), and they proved the variational principle under some supplementary conditions. The notions of the topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see the books [5, 27]).

Since the work of Bowen ([6]), topological pressure becomes a fundamental tool for study of the dimension theory in conformal dynamical systems (see [22]). Different versions of topological pressure

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were defined in dimension theory and ergodic theory. It has been found out that the dimension of non-conformal repellers can be well estimated by the zero of pressure function (see e.g. [1, 2, 12, 13, 29]). Falconer ([13]) considered the thermodynamic formalism for subadditive potentials on mixing repellers. He proved the variational principle under some Lipschitz conditions and bounded distortion assumptions on subadditive potentials. Barreira ([2]) defined topological pressure for arbitrary sequences of potentials on arbitrary subsets of compact metric spaces, and proved the variational principle under some convergence assumptions on the potentials. However, the conditions given by Falconer and Barreira are not usually satisfied by general subadditive potentials.

In [8], the authors generalized the results of Ruelle and Walters to subadditive potentials in general compact dynamical systems. They defined the subadditive topological pressure and gave a variational principle for the subadditive topological pressure. We mention that their result do not need any additional conditions on either the subadditive potentials or the spaces, as long as they are compact metric spaces. In [30], the authors defined the subadditive measure-theoretic pressure by using spanning sets, and obtained a formalism similar to that for additive measure-theoretic pressure in [15] under tempered variation assumptions on subadditive potentials. Another equivalent definition of subadditive measure-theoretic pressure is given in [31] under the same conditions on the sub-additive potentials by using the definition of topological pressure on noncompact sets. In [28], Zhang studied the local measure-theoretic pressures for subadditive potentials. The pressure is local in the sense that an open cover is fixed.

Part of this paper is a continuation of the work in [30] and [31]. We modify the definition of subadditive measure-theoretic pressure there, and remove the extra condition for the formulas. More precisely, for an ergodic measure μ, the subadditive measure-theoretic pressure defined by using Carathéodory structure can be expressed as the sum of the measure-theoretic entropy and the integral of the limit of the average value of the subadditive potentials. Consequently, this paper gives equivalence to an alternative definition of subadditive measure-theoretic pressure by considering spanning sets about which the Bowen balls only cover a set of measure greater than or equal to $1-\delta$. The results we get here do not need any additional assumptions on the subadditive sequences and the topological dynamical systems, except for compactness of the spaces. Meanwhile, we also define the measure-theoretic pressure for supadditive potentials in a similar way and obtain the same properties.

The present work is also motivated by the dimension theory in dynamical systems. Dimensions of a compact invariant set can often be determined or estimated by a unique root of certain pressure functions. As an application, we proved that the dimension of an average conformal repeller which is introduced in [1] satisfies a variation principle, i.e., the dimension of this repeller is equal to the dimension of some ergodic measure supported on it. Moreover, for each ergodic measure supported on an average conformal repeller, we could construct a certain set with the same dimension of the measure.

The paper is organized in the following manner. The main results, as well as definitions of the measure-theoretic pressure and lower and upper capacity measure-theoretic pressure for subaddtive potentials, are given in Section 1. We prove Theorem A and B, the results related to subadditive potentials, in Section 2. Section 3 is devoted to supadditive potentials, where we give definitions of the pressures, and prove Theorem C and D. Section 4 is for application to dimension of average conformal repellers, where the results are stated in Theorem E and F.

1 Main results

Let (X,T) be a topological dynamical systems(TDS), that is, X is a compact metric space with a metric d, and $T:X\to X$ is a continuous transformation. Denote by $\mathcal{M}_T$ and $\mathcal{E}_T$ the set of all

T-invariant Borel probability measures on X and the set of ergodic measures respectively. For each $\mu \in \mathcal{M}_T$, let $h_\mu(T)$ denote the measure-theoretic entropy of T with respect to μ.

A sequence $\mathcal{F} = \{f_n\}_{n\geq 1}$ of continuous functions on X is a *subadditive potential* on X, if

$$f_{n+m}(x) \leq f_n(x) + f_m(T^n x) \quad \text{for all} \quad x \in X, \; n, m \in \mathbb{N}.$$

For $\mu \in \mathcal{M}_T$, let $\mathcal{F}_*(\mu)$ denote the following limit

$$\mathcal{F}_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int f_n \mathrm{d}\mu = \inf_{n\geq 1} \Big\{ \frac{1}{n} \int f_n \mathrm{d}\mu \Big\}.$$

The limit exists since $\{\int f_n \mathrm{d}\mu\}_{n\geq 1}$ is a subadditive sequence. Also, by subadditive ergodic theorem [18] the limit $\lim\limits_{n\to\infty} (1/n) f_n$ exists μ-almost everywhere for any $\mu \in \mathcal{M}_T$.

Let $d_n(x,y) = \max\{d(T^i(x), T^i(y)) : i = 0, \cdots, n-1\}$ for any $x, y \in X$, and $B_n(x,\epsilon) = \{y \in X : d_n(x,y) < \epsilon\}$. A set $E \subseteq X$ is said to be an (n,ϵ)-separated subset of X with respect to T if $x, y \in E, x \neq y$, implies $d_n(x,y) > \epsilon$. A set $F \subseteq X$ is said to be an (n,ϵ)-spanning subset of X with respect to T if $\forall x \in X$, $\exists y \in F$ with $d_n(x,y) \leq \epsilon$. For each $\mu \in \mathcal{M}_T$, $0 < \delta < 1$, $n \geq 1$ and $\epsilon > 0$, a subset $F \subseteq X$ is an (n,ϵ,δ)-spanning set if the union $\bigcup_{x\in F} B_n(x,\epsilon)$ has μ-measure more than or equal to $1-\delta$.

Recall that the *subadditive topological pressure* of T with respect to a subadditive potential $\mathcal{F} = \{f_n\}_{n\geq 1}$ is give by

$$P(T,\mathcal{F}) = \lim_{\epsilon\to 0} P(T,\mathcal{F},\epsilon),$$

where

$$\begin{aligned} P(T,\mathcal{F},\epsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log P_n(T,\mathcal{F},\epsilon), \\ P_n(T,\mathcal{F},\epsilon) &= \sup\{\sum_{x\in E} e^{f_n(x)} : E \text{ is an } (n,\epsilon)\text{-separated subset of } X\}. \end{aligned}$$

(See e.g. [2], [8].) It satisfies a variational principle (see [8] for a proof and [32] for its random version).

In [17], Katok showed that measure-theoretic entropy can be regarded as the growth rate of the minimal number of ϵ-balls in the d_n metric that cover a set of measure more than or equal to $1-\delta$. Motivated by the observation, the following definition can be given.

Definition 1.1. *Given a subadditive potential $\mathcal{F} = \{f_n\}$, for $\mu \in \mathcal{E}_T$, $0 < \delta < 1$, $n \geq 1$, and $\epsilon > 0$, put*

$$\begin{aligned} P_\mu(T,\mathcal{F},n,\epsilon,\delta) &= \inf\{\sum_{x\in F} \exp[\sup_{y\in B_n(x,\epsilon)} f_n(y)] \mid F \textit{ is an } (n,\epsilon,\delta) - \textit{spanning set}\}, \\ P_\mu(T,\mathcal{F},\epsilon,\delta) &= \limsup_{n\to\infty} \frac{1}{n} \log P_\mu(T,\mathcal{F},n,\epsilon,\delta), \\ P_\mu(T,\mathcal{F},\delta) &= \liminf_{\epsilon\to 0} P_\mu(T,\mathcal{F},\epsilon,\delta), \\ P_\mu(T,\mathcal{F}) &= \lim_{\delta\to 0} P_\mu(T,\mathcal{F},\delta). \end{aligned}$$

$P_\mu(T,\mathcal{F})$ is said to be the subadditive measure-theoretic pressure *of T with respect to $\mathcal{F}$.*

Remark 1.1. *It is easy to see that $P_\mu(T,\mathcal{F},\delta)$ increases with δ. So the limit in the last formula exists. In fact, it is proved in [11, Theorem 2.3] that $P_\mu(T,\mathcal{F},\delta)$ is independent of δ. Hence, the limit of $\delta \to 0$ is redundant in the definition. The same phenomenon can also be seen for measure-theoretic entropy (see [17, Theorem 1.1]).*

Remark 1.2. *If $\mathcal{F} = \{f_n\}$ is additive generated by a continuous function φ, that is, $f_n(x) = \sum_{i=0}^{n-1} \varphi(T^i x)$ for some continuous function $\varphi : X \to \mathbb{R}$, then we simply write $P_\mu(T, \mathcal{F})$ as $P_\mu(T, \varphi)$.*

An alternative definition of subadditive measure-theoretic pressure can be given by using the theory of Carathéodory structure (see [22] for more details in additive case).

Let $Z \subseteq X$ be a subset of X, which does not have to be compact nor T-invariant. Fix $\epsilon > 0$, we call $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ a *cover of* Z if $Z \subseteq \bigcup_i B_{n_i}(x_i, \epsilon)$. For $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$, set $n(\Gamma) = \min_i\{n_i\}$.

The theory of Carathéodory dimension characteristic ensures the following definitions.

Definition 1.2. *Let $s \geq 0$, put*

$$M(Z, \mathcal{F}, s, N, \epsilon) = \inf_{\Gamma} \sum_i \exp\big(-sn_i + \sup_{y \in B_{n_i}(x_i, \epsilon)} f_{n_i}(y)\big), \tag{1.1}$$

where the infimum is taken over all covers Γ of Z with $n(\Gamma) \geq N$. Then let

$$m(Z, \mathcal{F}, s, \epsilon) = \lim_{N \to \infty} M(Z, \mathcal{F}, s, N, \epsilon), \tag{1.2}$$

$$P_Z(T, \mathcal{F}, \epsilon) = \inf\{s : m(Z, \mathcal{F}, s, \epsilon) = 0\} = \sup\{s : m(Z, \mathcal{F}, s, \epsilon) = +\infty\}, \tag{1.3}$$

$$P_Z(T, \mathcal{F}) = \liminf_{\epsilon \to 0} P_Z(T, \mathcal{F}, \epsilon), \tag{1.4}$$

where $P_Z(T, \mathcal{F})$ is called a subadditive topological pressure *of T on the set Z (w.r.t. $\mathcal{F}$).*

Further, for $\mu \in \mathcal{M}_T$, put

$$P_\mu^*(T, \mathcal{F}, \epsilon) = \inf\{P_Z(T, \mathcal{F}, \epsilon) : \mu(Z) = 1\},$$

$$P_\mu^*(T, \mathcal{F}) = \liminf_{\epsilon \to 0} P_\mu^*(T, \mathcal{F}, \epsilon), \tag{1.5}$$

where $P_\mu^*(T, \mathcal{F})$ is called a subadditive measure-theoretic pressure *of T with respect to μ.*

It is easy to see that the definition is consistent with that given in [2] by using arbitrary open covers.

Lower and upper capacity topological pressure for additive sequence were defined in [22]. Now we give similar definitions:

Definition 1.3. *Put*

$$\Lambda(Z, \mathcal{F}, N, \epsilon) = \inf_{\Gamma} \sum_i \exp\big(\sup_{y \in B_N(x_i, \epsilon)} f_N(y)\big),$$

where the infimum is taken over all covers Γ of Z with $n_i = N$ for all i. Then we set

$$\underline{CP}_Z(T, \mathcal{F}, \epsilon) = \liminf_{N \to \infty} \frac{1}{N} \log \Lambda(Z, \mathcal{F}, N, \epsilon), \tag{1.6}$$

$$\overline{CP}_Z(T, \mathcal{F}, \epsilon) = \limsup_{N \to \infty} \frac{1}{N} \log \Lambda(Z, \mathcal{F}, N, \epsilon). \tag{1.7}$$

For $\mu \in \mathcal{M}_T$, define

$$\underline{CP}_\mu^*(T, \mathcal{F}, \epsilon) = \lim_{\delta \to 0} \ \inf\{\underline{CP}_Z(T, \mathcal{F}, \epsilon) : \mu(Z) \geq 1 - \delta\},$$

$$\overline{CP}_\mu^*(T, \mathcal{F}, \epsilon) = \lim_{\delta \to 0} \ \inf\{\overline{CP}_Z(T, \mathcal{F}, \epsilon) : \mu(Z) \geq 1 - \delta\}.$$

The subadditive lower *and* upper capacity measure-theoretic pressure of T *with respect to measure μ are defined by*

$$\underline{CP}_\mu^*(T, \mathcal{F}) = \liminf_{\epsilon \to 0} \underline{CP}_\mu^*(T, \mathcal{F}, \epsilon), \tag{1.8}$$

$$\overline{CP}_\mu^*(T, \mathcal{F}) = \liminf_{\epsilon \to 0} \overline{CP}_\mu^*(T, \mathcal{F}, \epsilon). \tag{1.9}$$

Theorem A. Let (X,T) be a TDS and $\mathcal{F}=\{f_n\}_{n\geq 1}$ a subadditive potential on X. For any $\mu\in\mathcal{E}_T$ with $\mathcal{F}_*(\mu)\neq-\infty$, we have

$$P_\mu^*(T,\mathcal{F})=\underline{CP}_\mu^*(T,\mathcal{F})=\overline{CP}_\mu^*(T,\mathcal{F})=P_\mu(T,\mathcal{F})=h_\mu(T)+\mathcal{F}_*(\mu).$$

Remark 1.3. *The results still apply for $\mathcal{F}_*(\mu)=-\infty$ if $h_\mu(T)<\infty$.*

Remark 1.4. *If $\mathcal{F}=\{f_n\}$ is an additive sequence generated by a continuous function $\varphi:X\to\mathbb{R}$, then we have $P_\mu(T,\varphi)=h_\mu(T)+\int\varphi \mathrm{d}\mu$, So Theorem A extends the results in [22] and [15] to the subadditive case. Also, the last equality was proved in [11].*

Remark 1.5. *By the definition of $P_\mu^*(T,\mathcal{F},\epsilon)$ and $P_\mu^*(T,\mathcal{F})$, the theorem gives $h_\mu(T)+\mathcal{F}_*(\mu)=\inf\{P_Z(T,\mathcal{F}):\mu(Z)=1\}$, as in [22]. We call it the* inverse variational principle*.*

The next theorem says that the infimum in the inverse variational principle can be attained on certain sets.

Theorem B. *Let (X,T) be TDS and $\mathcal{F}=\{f_n\}$ a subadditive potential on X. For any $\mu\in\mathcal{E}_T$ with $\mathcal{F}_*(\mu)\neq-\infty$, let*

$$K=\left\{x\in X:\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-\log\mu(B_n(x,\epsilon))}{n}=h_\mu(T)\quad \textit{and}\quad \lim_{n\to\infty}\frac{1}{n}f_n(x)=\mathcal{F}_*(\mu)\right\}.$$

Then we have

$$P_\mu(T,\mathcal{F})=P_K(T,\mathcal{F})=\underline{CP}_K(T,\mathcal{F})=\overline{CP}_K(T,\mathcal{F}).$$

Similar to subadditive sequences, we can study supadditive sequences. A sequence $\Phi=\{\phi_n\}_{n\geq 1}$ of continuous functions on X is a *supadditive potentials* on X, if

$$\phi_{n+m}(x)\geq\phi_n(x)+\phi_m(T^n x)\quad\text{for all}\quad x\in X,\ n,m\in\mathbb{N}. \tag{1.10}$$

Note that if $\Phi=\{\phi_n\}_{n\geq 1}$ is a supadditive sequences, then $-\Phi=\{-\phi_n\}_{n\geq 1}$ is a susadditive sequences. So the limit $\lim\limits_{n\to\infty}(1/n)\phi_n$ exists μ-almost everywhere for any $\mu\in\mathcal{M}_T$. For $\mu\in\mathcal{M}_T$, let

$$\Phi_*(\mu)=\lim_{n\to\infty}\frac{1}{n}\int\phi_n \mathrm{d}\mu=\sup_{n\geq 1}\left\{\frac{1}{n}\int\phi_n \mathrm{d}\mu\right\}.$$

It is always bounded below by $\int\phi_1 \mathrm{d}\mu$.

With the sequences, we can define supadditive measure-theoretic pressure $P_\mu(T,\Phi)$ and other quantities $P_\mu^*(T,\Phi),\underline{CP}_\mu^*(T,\Phi)$ and $\overline{CP}_\mu^*(T,\Phi)$, etc. (see Section 3 for precise definitions.)

In [1], the authors gave the variational principle for supadditive topological pressure for C^1 average conformal expanding maps T where the potentials are of the form $\{\phi_n(x)\}=\{-t\log\|DT^n(x)\|\}$ with $t>0$. However, it is still open whether variational principle holds for supadditive topological pressure for a general TDS. Here we show that the supadditive measure-theoretic pressure has similar formalisms as subadditive measure-theoretic pressure.

Theorem C. *Let (X,T) be a TDS and $\Phi=\{\phi_n\}$ a supadditive potential on X. For any $\mu\in\mathcal{E}_T$, we have*

$$P_\mu^*(T,\Phi)=\underline{CP}_\mu^*(T,\Phi)=\overline{CP}_\mu^*(T,\Phi)=P_\mu(T,\Phi)=h_\mu(T)+\Phi_*(\mu).$$

Remark 1.6. *For each $\mu\in\mathcal{E}_T$, from the definition of $P_\mu^*(T,\Phi,\epsilon)$ and $P_\mu^*(T,\Phi)$ given in Section 3, the theorem gives the inverse variational principle $h_\mu(T)+\Phi_*(\mu)=\inf\{P_Z(T,\Phi):\mu(Z)=1\}$.*

Theorem D. *Let (X,T) be TDS, and $\Phi=\{\phi_n\}$ a supadditive potential on X. For any $\mu\in\mathcal{E}_T$, let*

$$K=\{x\in X:\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-\log\mu(B_n(x,\epsilon))}{n}=h_\mu(T)\quad \textit{and}\quad \lim_{n\to\infty}\frac{1}{n}\phi_n(x)=\Phi_*(\mu)\}.$$

Then we have

$$P_\mu(T,\Phi)=P_K(T,\Phi)=\underline{CP}_K(T,\Phi)=\overline{CP}_K(T,\Phi).$$

2 Subadditive measure-theoretic pressures

We start with the section by some properties of pressures for subadditive potentials.

Proposition 2.1. *Let (X, T) be a TDS and $\mathcal{F} = \{f_n\}$ a subadditive potential. Then the following properties hold:*

(i) $\mathcal{P}_{Z_1}(T, \mathcal{F}) \leq \mathcal{P}_{Z_2}(T, \mathcal{F})$ *if* $Z_1 \subset Z_2$, *where* $\mathcal{P}$ *is* P, $\underline{CP}$ *or* $\overline{CP}$;

(ii) $P_Z(T, \mathcal{F}) = \sup_{i\geq 1} P_{Z_i}(T, \mathcal{F})$ *and* $\mathcal{P}_Z(T, \mathcal{F}) \geq \sup_{i\geq 1} \mathcal{P}_{Z_i}(T, \mathcal{F})$, *where* $Z = \bigcup_{i\geq 1} Z_i$, *and* $\mathcal{P}$ *is* $\underline{CP}$ *or* $\overline{CP}$;

(iii) $P_Z(T, \mathcal{F}) \leq \underline{CP}_Z(T, \mathcal{F}) \leq \overline{CP}_Z(T, \mathcal{F})$ *for any subset* $Z \subset X$;

(iv) $P^*_\mu(T, \mathcal{F}, \epsilon) \leq \underline{CP}^*_\mu(T, \mathcal{F}, \epsilon) \leq \overline{CP}^*_\mu(T, \mathcal{F}, \epsilon)$, *and* $P^*_\mu(T, \mathcal{F}) \leq \underline{CP}^*_\mu(T, \mathcal{F}) \leq \overline{CP}^*_\mu(T, \mathcal{F})$.

Proof. (i) and (ii) are dirctly follow from the definition. And (iii) is immediately from similarly arguments as in [2, Theorem 1.4 (a)]. (iv) follows from (iii) immediately by the definition. □

Recall the Brin-Katok's theorem for local entropy (see [7]), which says that if $\mu \in \mathcal{M}_T$, then for μ-almost every $x \in X$,

$$h_\mu(x, T) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)). \tag{2.1}$$

Moreover, if $\mu \in \mathcal{E}_T$, then for $\mu-$almost every $x \in X$, $h_\mu(x, T) = h_\mu(T)$, and for each $\epsilon > 0$, the following two limits are constants almost everywhere:

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)), \quad \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)).$$

Proof of Theorem A. The last equality is a direct consequence of Theorem 2.3 in [11]. By Proposition 2.1, we only need to prove $\overline{CP}^*_\mu(T, \mathcal{F}) \leq h_\mu(T) + \mathcal{F}_*(\mu)$ and $P^*_\mu(T, \mathcal{F}) \geq h_\mu(T) + \mathcal{F}_*(\mu)$.

For $\mu \in \mathcal{E}_T$, we first assume $h_\mu(T)$ is finite and set $h = h_\mu(T) \geq 0$.

Take $\delta > 0$. Fix a positive integer k and a small number $\eta > 0$.

Take $\epsilon_\eta > 0$ such that if $\epsilon \in (0, \epsilon_\eta]$, then for μ-almost every $x \in X$,

$$h - \eta/2 \leq \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)) \leq \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)) \leq h + \eta/2.$$

This is possible because of (2.1). Take $0 < \epsilon \leq \min\{\epsilon_\eta, \epsilon_0\}$, where ϵ_0 is given in Lemma 2.2. Hence, for μ-almost every $x \in X$, there exists a number $N_1(x) > 0$ such that for any $n \geq N_1(x)$,

$$\left| \frac{1}{n} \log \mu(B_n(x, \epsilon/2)) + h \right| \leq \eta. \tag{2.2}$$

By the Birkhoff ergodic theorem, for μ-almost every $x \in X$, there exists a number $N_2(x) > 0$ such that for any $n \geq N_2(x)$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i x) - \int \frac{1}{k} f_k \mathrm{d}\mu \right| \leq \eta. \tag{2.3}$$

Given $N > 0$, set $K_N = \{x \in X : N_1(x), N_2(x) \leq N\}$. We have that $K_N \subset K_{N+1}$, and $\cup_{N\geq 0} K_N$ is a set of full measure. Therefore, one can find $N_0 > 0$ for which $\mu(K_{N_0}) > 1 - \delta$. Fix a number $N > N_0$. Then for any $n > N_0$ and any point $x \in K_N$, by (2.3) and Lemma 2.2 below we have

$$\sup_{y\in B_n(x,\epsilon)} f_n(y) \leq n \int \frac{1}{k} f_k \mathrm{d}\mu + 2n\eta + C,$$

where C is a constant given in Lemma 2.2.

Let E be a maximal (n,ϵ)-separated subset of K_N, then $K_N \subseteq \cup_{x\in E} B_n(x,\epsilon)$. Furthermore, the balls $\{B_n(x,\epsilon/2) : x \in E\}$ are pairwise disjoint and by (2.2) the cardinality of E is less than or equal to $\exp n(h+\eta)$. Therefore, we have

$$\begin{aligned}\Lambda(K_N,\mathcal{F},n,\epsilon) &\leq \sum_{x\in E}\exp(\sup_{y\in B_n(x,\epsilon)} f_n(y))\\ &\leq \exp n(h+\eta)\cdot\exp\left[n(\int\frac{1}{k}f_k\mathrm{d}\mu+2\eta)+C\right]\\ &= \exp\left[n(\int\frac{1}{k}f_k\mathrm{d}\mu+h+3\eta)+C\right].\end{aligned}$$

From (1.7), we have

$$\overline{CP}_{K_N}(T,\mathcal{F},\epsilon)\leq\int\frac{1}{k}f_k\mathrm{d}\mu+h+3\eta.$$

Since $\mu(K_N)\geq 1-\delta$, we have

$$\overline{CP}^*_{\mu}(T,\mathcal{F},\epsilon)\leq\int\frac{1}{k}f_k\mathrm{d}\mu+h+3\eta.$$

Let $\epsilon\to 0$ and $k\to\infty$ in the inequality, and by the arbitrariness of η, we get $\overline{CP}^*_{\mu}(T,\mathcal{F})\leq h+\mathcal{F}_*(\mu)$.

To prove the other inequality, it is sufficient to prove that $P_Z(T,\mathcal{F})\geq h+\mathcal{F}_*(\mu)$ for any subset $Z\subseteq X$ of full μ-measure.

Take $\eta>0$ and $\delta\in(0,1/2)$, and denote $\lambda=h+\mathcal{F}_*(\mu)-2\eta$.

Let

$$K=\left\{x\in X : \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-\log\mu(B_n(x,\epsilon))}{n}=h_\mu(T)\ \text{ and }\ \lim_{n\to\infty}\frac{1}{n}f_n(x)=\mathcal{F}_*(\mu)\right\}.$$

Put $K'=K\cap Z$. By the Brin-Katok's theorem for local entropy and the subadditive ergodic theorem, we have $\mu(K)=1$ and then $\mu(K')=1$. For $\epsilon\in(0,\epsilon_\eta]$, there exists a set $K_1\subset K'$ with $\mu(K_1)>1-\frac{\delta}{2}$ and $N_1>0$ such that for any $x\in K_1$ and $n\geq N_1$, we have

$$\mu(B_n(x,2\epsilon))\leq\exp(-n(h-\eta)).$$

By the subadditive ergodic theorem, there exists a set $K_2\subset K'$ with $\mu(K_2)>1-\frac{\delta}{2}$ and $N_2>0$ such that for any $x\in K_2$ and $n\geq N_2$, we have

$$\left|\frac{1}{n}f_n(x)-\mathcal{F}_*(\mu)\right|<\eta.$$

Put $\widetilde{K}=K_1\cap K_2\subset K'$, $N\geq\max\{N_1,N_2\}$. Clearly $\mu(\widetilde{K})>1-\delta$. We may assume further that $\widetilde{K}$ is compact since otherwise we can approximate it from within by a compact subset. Take an open cover $\Gamma=\{B_{n_i}(x_i,\epsilon)\}_i$ of $\widetilde{K}$ with $n(\Gamma)\geq N$. Since $\widetilde{K}$ is compact, we may assume that the cover is finite and consists of $B_{n_1}(x_1,\epsilon),\cdots,B_{n_l}(x_l,\epsilon)$.

For each $i=1,\cdots,l$, we choose $y_i\in\widetilde{K}\cap B_{n_i}(x_i,\epsilon)$. Hence, $B_{n_i}(x_i,\epsilon)\subset B_{n_i}(y_i,2\epsilon)$, and $\{B_{n_i}(y_i,2\epsilon)\}_i$ form a cover of $\widetilde{K}$ as well. Now we have

$$\sum_{B_{n_i}(x_i,\epsilon)\in\Gamma}\exp\bigl(-n_i\lambda+\sup_{y\in B_{n_i}(x_i,\epsilon)}f_{n_i}(y)\bigr)\geq\sum_{i=1}^{l}\exp\bigl(-n_i\lambda+f_{n_i}(y_i)\bigr)$$
$$\geq\sum_{i=1}^{l}\exp\bigl(-n_i\lambda+n_i(\mathcal{F}_*(\mu)-\eta)\bigr)=\sum_{i=1}^{l}\exp\bigl(-n_i(h-\eta)\bigr)\geq\sum_{i=1}^{l}\mu(B_{n_i}(y_i,2\epsilon))\geq 1-\delta\geq\frac{1}{2}.$$

Note that the inequality holds for any cover $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ of $\widetilde{K}$. Hence

$$M(\widetilde{K}, \mathcal{F}, \lambda, N, \epsilon) \geq \frac{1}{2}.$$

Thus $m(\widetilde{K}, \mathcal{F}, \lambda, \epsilon) \geq 1/2$. It means that

$$P_{\widetilde{K}}(T, \mathcal{F}, \epsilon) \geq \lambda = h + \mathcal{F}_*(\mu) - 2\eta, \quad \text{and} \quad P_{\widetilde{K}}(T, \mathcal{F}) \geq h + \mathcal{F}_*(\mu) - 2\eta.$$

Using Proposition 2.1 and arbitrariness of η, we have

$$P_Z(T, \mathcal{F}) \geq P_{\widetilde{K}}(T, \mathcal{F}) \geq h + \mathcal{F}_*(\mu). \tag{2.4}$$

So by definition we get $P_\mu^*(T, \mathcal{F}) \geq h + \mathcal{F}_*(\mu)$.

When $h_\mu(T) = +\infty$, modify subtly the proof of the second inequality, we can easily have $P_\mu^*(T, \mathcal{F}) = +\infty$. Thus we finish the proof of the theorem. □

Proof of Theorem B. As in the proof of Theorem A, set $h = h_\mu(f) \geq 0$. Fix an integer $k > 0$ and a small real number $\eta > 0$.

For μ-almost every $x \in X$, there exists a number $N_1(x) > 0$ such that for any $n \geq N_1(x)$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i x) - \int \frac{1}{k} f_k \mathrm{d}\mu \right| \leq \eta.$$

For some $\epsilon \in (0, \epsilon_\eta]$, where ϵ_η is chosen in the same way as that in the proof of Theorem A, for μ-almost every $x \in X$ there exists a number $N_2(x) > 0$ such that for any $n \geq N_2(x)$,

$$\left| \frac{1}{n} \log \mu(B_n(x, \epsilon/2)) + h \right| \leq \eta.$$

Given $N > 0$, set $K_N = \{x \in K : N_1(x), N_2(x) \leq N\}$, we have $K_N \subset K_{N+1}$, and $\cup_{N \geq 0} K_N = K$. Similarly, given $\delta > 0$, we can find $N_0 > 0$ for which $\mu(K_{N_0}) > 1 - \delta$.

Fix a number $N \geq N_0$, as in the proof of Theorem A we get that $\overline{CP}_{K_N}(T, \mathcal{F}, \epsilon) \leq \int \frac{1}{k} f_k \mathrm{d}\mu + h + 3\eta$. Letting $k \to \infty$, $\epsilon \to 0$, since η is arbitrary, we get

$$\overline{CP}_{K_N}(T, \mathcal{F}) \leq h + \mathcal{F}_*(\mu).$$

Letting $N \to \infty$, we have that $\overline{CP}_K(T, \mathcal{F}) \leq h + \mathcal{F}_*(\mu)$.

The inequality $P_K(T, \mathcal{F}) \geq h + \mathcal{F}_*(\mu)$ is contained in (2.4) since Z is an arbitrary set with full measure, and here we have $\mu(K) = 1$.

By Theorem A and Proposition 2.1, we get the desired result. □

Lemma 2.2. *Let (X, T) be a TDS and $\mathcal{F} = \{f_n\}$ a subadditive potential. Fix any positive integer k, then for any $\eta > 0$, there exist an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ we have*

$$\sup_{y \in B_n(x, \epsilon)} f_n(y) \leq \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i x) + n\eta + C,$$

where $C = C_k$ is a constant independent of η and ϵ.

Proof. Fix a positive integer k, $\frac{1}{k} f_k(x)$ is a continuous function. Hence, for any $\eta > 0$, there exist an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$,

$$d(x, y) < \epsilon \Rightarrow d\left(\frac{1}{k} f_k(x), \frac{1}{k} f_k(y)\right) < \eta. \tag{2.5}$$

For each n, we rewrite n as $n = sk + l$, where $s \geq 0$, $0 \leq l < k$. Then for any integer $0 \leq j < k$, we have

$$f_n(x) \leq f_j(x) + f_k(T^j x) + \cdots + f_k(T^{(s-2)k} T^j x) + f_{k+l-j}(T^{(s-1)k} T^j x),$$

where we take $f_0(x) \equiv 0$. Let $C_1 = \max_{j=1,\cdots 2k} \max_{x\in X} |f_j(x)|$. Summing over j from 0 to $k-1$, we have

$$k f_n(x) \leq 2kC_1 + \sum_{i=0}^{(s-1)k-1} f_k(T^i x).$$

Hence

$$f_n(x) \leq 2C_1 + \sum_{i=0}^{(s-1)k-1} \frac{1}{k} f_k(T^i x) \leq 4C_1 + \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i x). \tag{2.6}$$

Set $C = 4C_1$. By (2.5) we have that

$$\sup_{y\in B_n(x,\epsilon)} f_n(y) \leq \sup_{y\in B_n(x,\epsilon)} (C + \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i y)) \leq \sum_{i=0}^{n-1} \frac{1}{k} f_k(T^i x) + n\eta + C.$$

This completes the proof of the lemma. □

3 Supadditive measure-theoretic pressures

Recall that supadditive sequence is defined in (1.10)

Definition 3.1. *Let $\Phi = \{\phi_n\}$ be a given supadditive potential. For $\mu \in \mathcal{E}_T$, $0 < \delta < 1$, $n \geq 1$, and $\epsilon > 0$, put*

$$\begin{aligned}
P_\mu(T, \Phi, n, \epsilon, \delta) &= \inf\{\sum_{x\in F} e^{\phi_n(x)} \mid F \text{ is an } (n, \epsilon, \delta)\text{-spanning set}\},\\
P_\mu(T, \Phi, \epsilon, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log P_\mu(T, \Phi, n, \epsilon, \delta),\\
P_\mu(T, \Phi, \delta) &= \lim_{\epsilon\to 0} P_\mu(T, \Phi, \epsilon, \delta),\\
P_\mu(T, \Phi) &= \lim_{\delta\to 0} P_\mu(T, \Phi, \delta),
\end{aligned}$$

where $P_\mu(T, \Phi)$ is called the supadditive measure-theoretic pressure *of T with respect to Φ.*

Note that there is small difference between the definitions of $P_\mu(T, \mathcal{F}, n, \epsilon, \delta)$ and $P_\mu(T, \Phi, n, \epsilon, \delta)$. This difference makes it possible to remove the tempered variation assumptions on the potentials as in [30, 31].

Remark 3.1. *Similar to Remark 1.1, $P_\mu(T, \Phi, \delta)$ also increases with δ and therefore the limit in the last formula exists. Moreover, by Proposition 3.2 below $P_\mu(T, \Phi, \delta)$ is independent of δ.*

Recall that if $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ is a cover of a subset $Z \subseteq X$, where $\epsilon > 0$, then $n(\Gamma) = \min_i\{n_i\}$.

Definition 3.2. *Let $\Phi = \{\phi_n\}$ be a given supadditive potential. For $s \geq 0$, define*

$$M(Z, \Phi, s, N, \epsilon) = \inf_\Gamma \sum_i \exp(-sn_i + \phi_{n_i}(x_i)),$$

where the infimum is taken over all covers $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ *of* Z *with* $n(\Gamma) \geq N$. *Then let*

$$m(Z, \Phi, s, \epsilon) = \lim_{N\to\infty} M(Z, \Phi, s, N, \epsilon), \tag{3.1}$$

$$P_Z(T, \Phi, \epsilon) = \inf\{s : m(Z, \Phi, s, \epsilon) = 0\} = \sup\{s : m(Z, \Phi, s, \epsilon) = +\infty\}, \tag{3.2}$$

$$P_Z(T, \Phi) = \liminf_{\epsilon\to 0} P_Z(T, \Phi, \epsilon), \tag{3.3}$$

where $P_Z(T, \Phi)$ *is called a* supadditive topological pressure *of* T *on the set* Z *(w.r.t.* Φ*).*

Further, for $\mu \in \mathcal{M}_T$, *put*

$$\begin{aligned} P_\mu^*(T, \Phi, \epsilon) &= \inf\{P_Z(T, \Phi, \epsilon) : \mu(Z) = 1\}, \\ P_\mu^*(T, \Phi) &= \liminf_{\epsilon\to 0} P_\mu^*(T, \Phi, \epsilon), \end{aligned} \tag{3.4}$$

where $P_\mu^*(T, \Phi)$ *is called a* supadditive measure-theoretic pressure *of* T *with respect to* μ.

Definition 3.3. *Put*

$$\Lambda(Z, \Phi, N, \epsilon) = \inf_\Gamma \sum_{B_N(x,\epsilon)\in\Gamma} \exp(\phi_N(x)),$$

where the infimum is taken over all covers Γ *of* Z *with* $n_i = N$ *for all* i. *And then we set*

$$\underline{CP}_Z(T, \Phi, \epsilon) = \liminf_{N\to\infty} \frac{1}{N} \log \Lambda(Z, \Phi, N, \epsilon), \tag{3.5}$$

$$\overline{CP}_Z(T, \Phi, \epsilon) = \limsup_{N\to\infty} \frac{1}{N} \log \Lambda(Z, \Phi, N, \epsilon). \tag{3.6}$$

For $\mu \in \mathcal{M}_T$, *define*

$$\underline{CP}_\mu^*(T, \Phi, \epsilon) = \lim_{\delta\to 0} \inf\{\underline{CP}_Z(T, \Phi, \epsilon) : \mu(Z) \geq 1 - \delta\},$$

$$\overline{CP}_\mu^*(T, \Phi, \epsilon) = \lim_{\delta\to 0} \inf\{\overline{CP}_Z(T, \Phi, \epsilon) : \mu(Z) \geq 1 - \delta\}.$$

The supadditive lower *and* upper capacity measure-theoretic pressure of T *with respect to measure* μ *are defined by*

$$\underline{CP}_\mu^*(T, \Phi) = \liminf_{\epsilon\to 0} \underline{CP}_\mu^*(T, \Phi, \epsilon), \tag{3.7}$$

$$\overline{CP}_\mu^*(T, \Phi) = \liminf_{\epsilon\to 0} \overline{CP}_\mu^*(T, \Phi, \epsilon). \tag{3.8}$$

It is easy to see that all the pressures of supadditive potentials defined as above have the same properties of the corresponding pressures of subadditive potentials. We state it here, whose proof is similar and left to the reader.

Proposition 3.1. *All the properties stated in Proposition 2.1 are true if we replace the subadditive potential* $\mathcal{F}$ *by a supadditive potential* Φ.

Proof of Theorem C. The last equality is proved in Proposition 3.2 below. So by Proposition 3.1, we only need to prove $\overline{CP}_\mu^*(T, \Phi) \leq h_\mu(T) + \mathcal{F}_*(\mu)$ and $P_\mu^*(T, \Phi) \geq h_\mu(T) + \mathcal{F}_*(\mu)$.

For $\mu \in \mathcal{E}_T$, we first assume $h_\mu(T)$ is finite and set $h = h_\mu(T) \geq 0$.

Fix a small number $\eta > 0$ and an $\epsilon \in (0, \epsilon_\eta]$, where ϵ_η is determined in a similar way as in the proof of Theorem A. Hence, by the Brin-Katok theorem (see [7]) for local entropy, for μ-almost every $x \in X$ there exists a number $N_1(x) > 0$ such that for any $n \geq N_1(x)$, we have

$$\left| \frac{1}{n} \log \mu(B_n(x, \epsilon/2)) + h \right| \leq \eta. \tag{3.9}$$

Since $-\Phi = \{-\phi_n\}$ is a subadditive sequence, by the subadditive ergodic theorem, for μ-almost every $x \in X$ there exists a number $N_2(x) > 0$ such that for any $n \geq N_2(x)$, we have

$$\left| \frac{1}{n}\phi_n(x) - \Phi_*(\mu) \right| \leq \eta \tag{3.10}$$

Given $N > 0$, set $K_N = \{x \in X : N_1(x), N_2(x) \leq N\}$. We have that $K_N \subset K_{N+1}$, and $\cup_{N \geq 0} K_N$ is a set of full measure. Therefore, given $\delta > 0$, we can find $N_0 > 0$ for which $\mu(K_{N_0}) > 1 - \delta$.

Fix a number $N \geq N_0$. Let E be a maximal (n, ϵ)-separated subset of K_N, then $K_N \subseteq \cup_{x \in E} B_n(x, \epsilon)$. Furthermore, the balls $\{B_n(x, \epsilon/2) : x \in E\}$ are pairwise disjoint. By (3.9) the cardinality of E is less than or equal to $\exp n(h + \eta)$. Therefore, by (3.10) we have

$$\begin{aligned}\Lambda(K_N, \Phi, n, \epsilon) &\leq \sum_{x \in E} \exp(\phi_n(x)) \leq \exp n(h + \eta) \cdot \exp n(\Phi_*(\mu) + \eta) \\ &= \exp n(\Phi_*(\mu) + h + 2\eta).\end{aligned}$$

It follows

$$\overline{CP}_{K_N}(T, \Phi, \epsilon) \leq \Phi_*(\mu) + h + 2\eta.$$

Since $\mu(K_N) \geq 1 - \delta$ and $\mu(K_N) \to 1$ as $N \to \infty$, we have

$$\overline{CP}^*_\mu(T, \Phi, \epsilon) \leq \Phi_*(\mu) + h + 2\eta.$$

Let $\epsilon \to 0$ and take limit. By arbitrariness of η, we have $\overline{CP}^*_\mu(T, \Phi) \leq h + \Phi_*(\mu)$.

To prove the other inequality, it is sufficient to prove that $P_Z(T, \Phi) \geq h + \Phi_*(\mu)$ for any subset $Z \subseteq X$ of full μ-measure. Fix a subset Z with full measure and a positive integer k. Note that for an additive sequence $\mathcal{F} = \{f_n\} = \{\sum_{i=1}^{n} \frac{1}{k}\phi_k \circ T^{i-1}\}$, $P_Z(T, \mathcal{F})$ becomes the pressure $P_Z(T, f_1)$ of $f_1 = (1/k)\phi_k$. So if we apply the same arguments as in the proof of Theorem A for the sequence, we get the inequality as in (2.4), that is,

$$P_Z(T, \frac{1}{k}\phi_k) \geq h + \int \frac{1}{k}\phi_k \mathrm{d}\mu.$$

Using lemma 3.3 below, we have

$$P_Z(T, \Phi) \geq h + \int \frac{1}{k}\phi_k \mathrm{d}\mu.$$

The arbitrariness of k implies that $P_Z(T, \Phi) \geq h + \Phi_*(\mu)$.

If $h_\mu(T) = +\infty$, we can easily have $P^*_\mu(T, \Phi) = +\infty$. Thus we finish the proof of the theorem. □

Proof of Theorem D. As in the proof of Theorem C, set $h = h_\mu(f) \geq 0$. By the same arguments, there is a sequence of subsets $\{K_N\}_{N \geq 1}$ such that $K_N \subset K_{N+1}$ and $\cup_{N \geq 0} K_N = K$. Moreover,

$$\overline{CP}_{K_N}(T, \Phi) \leq h + \Phi_*(\mu)$$

for all sufficiently large N. Letting $N \to \infty$, we get that $\overline{CP}_K(T, \Phi) \leq h + \Phi_*(\mu)$.

By Theorem C, the reverse inequality $P_K(T, \Phi) \geq h + \Phi_*(\mu)$ is immediate since $\mu(K) = 1$. Hence Theorem C and Proposition 3.1 implies the desired results. □

Proposition 3.2. *Let (X, T) be a TDS, and $\Phi = \{\phi_n\}$ a supadditive potential on X. For $\mu \in \mathcal{E}_T$, we have*

$$P_\mu(T, \Phi) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(T, \Phi, n, \epsilon, \delta) = h_\mu(T) + \Phi_*(\mu).$$

Proof. Fix a positive integer k and a small number $\delta > 0$. Take $\eta > 0$. Let ϵ_0 be as in Sublemma 3.5. Then by Lemma 3.4, for any $\epsilon \in (0, \epsilon_0]$, $n > 0$, we can get

$$P_\mu(T, \Phi, n, \epsilon, \delta) \geq e^{-n\eta - C} P_\mu(T, \frac{\phi_k}{k}, n, \epsilon, \delta).$$

By Theorem 2.1 in [15],

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(T, \frac{\phi_k}{k}, n, \epsilon, \delta) = h_\mu(T) + \int_X \frac{\phi_k}{k} \mathrm{d}\mu.$$

Hence by definition, we have

$$P_\mu(T, \Phi) \geq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(T, \Phi, n, \epsilon, \delta) - \eta \geq h_\mu(T) + \int_X \frac{\phi_k}{k} \mathrm{d}\mu - \eta.$$

Note that η is arbitrary. By letting $k \to \infty$ we get

$$P_\mu(T, \Phi) \geq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(T, \Phi, n, \epsilon, \delta) \geq h_\mu(T) + \Phi_*(\mu). \tag{3.11}$$

Now we only need to prove the reversed inequality $P_\mu(T, \Phi) \leq h_\mu(T) + \Phi_*(\mu)$.

Take $\eta > 0$. For each N, take a set K_N as in the proof of Theorem C. Then take $N_0 > 0$ such that for any $N \geq N_0$, $\mu(K_N) > 1 - \delta$. For any $n \geq N$, let F_n be a maximal $(n, \epsilon/2)$-separated subset of K_N. Then $K_N \subseteq \cup_{x \in F_n} B_n(x, \epsilon)$. It means that F_n is a (n, ϵ, δ)-spanning set.

By (3.9), if $x \in K_N$, then $\mu B_n(x, \epsilon/2) \geq \exp[-n(h + \eta)]$. Hence, F_n contains at most $\exp[n(h + \eta)]$ elements. By (3.10), if $x \in K_N$, then $\phi_n(x) \leq n(\Phi_*(\mu) + \eta)$. Now we get

$$\begin{aligned}\sum_{x \in F_n} \exp[\phi_n(x)] &\leq \sum_{x \in F_n} \exp[n(\Phi_*(\mu) + \eta)] \leq \exp[n(h + \eta)] \cdot \exp[n(\Phi_*(\mu) + \eta)] \\ &= \exp[n(h + \Phi_*(\mu) + 2\eta)].\end{aligned}$$

Therefore,

$$P_\mu(T, \Phi, n, \epsilon, \delta) \leq \exp[n(h + \Phi_*(\mu) + 2\eta)].$$

Consequently,

$$P_\mu(T, \Phi, \delta) \leq h + \Phi_*(\mu) + 2\eta. \tag{3.12}$$

Since δ and η are arbitrary, we have

$$P_\mu(T, \Phi) \leq h_\mu(T) + \Phi_*(\mu),$$

which is the desired inequality. ◻

Lemma 3.3. *Let (X, T) be a TDS and $\Phi = \{\phi_n\}$ a supadditive potential. Fix any positive integer k. For each subset Z we have*

$$P_Z(T, \Phi) \geq P_Z(T, \frac{1}{k}\phi_k).$$

Proof. Fix a positive integer k. By Sublemma 3.5 below we have

$$M(Z, \Phi, s, N, \epsilon) \geq e^{-C} M(Z, \frac{1}{k}\phi_k, s + \eta, N, \epsilon).$$

Therefore

$$P_Z(T, \frac{1}{k}\phi_k) \leq P_Z(T, \Phi) + \eta.$$

This immediately implies the desired result. ◻

Lemma 3.4. *Let (X,T) be a TDS and $\Phi=\{\phi_n\}$ a supadditive potential. Let k be any positive integer and $\mu\in\mathcal{E}_T$. Then for any $\eta>0$, there exist an $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$, $n>0$, $\delta\in(0,1]$, we have*

$$P_\mu(T,\Phi,n,\epsilon,\delta)\geq e^{-n\eta-C}P_\mu(T,\frac{\phi_k}{k},n,\epsilon,\delta),$$

and therefore

$$P_\mu(T,\mathcal{F})\geq P_\mu(T,\frac{f_k}{k}).$$

Remark 3.2. *For a subadditive potential $\mathcal{F}=\{f_n\}$, the result becomes $P_\mu(T,\mathcal{F})\leq P_\mu(T,\frac{f_k}{k})$.*

Proof of Lemma 3.4. By Sublemma 3.5 below, for any small number $\eta>0$, there exist an $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$, $n>0$, $\delta\in(0,1]$, we have the first inequality of the lemma. Therefore

$$P_\mu(T,\Phi)\geq P_\mu(T,\frac{\phi_k}{k})-\eta.$$

The arbitrariness of η immediately yields the desired result. $\square$

Sublemma 3.5. *Let (X,T) be a TDS and $\Phi=\{\phi_n\}$ a supadditive potential. Fix any positive integer k. Then for any small number $\eta>0$, there exist an $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$ we have*

$$\phi_n(x)\geq\sup_{y\in B_n(x,\epsilon)}\sum_{i=0}^{n-1}\frac{1}{k}\phi_k(T^iy)-n\eta-C,$$

where C is a constant independent of η and ϵ.

Proof. Fix a positive integer k. Since $\frac{1}{k}\phi_k(x)$ is a continuous function, for any $\eta>0$, there exist $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$,

$$d(x,y)<\epsilon\Rightarrow d(\frac{1}{k}\phi_k(x),\frac{1}{k}\phi_k(y))<\eta.$$

Using the supadditivity of Φ, as (2.6) we have

$$\phi_n(x)\geq\sum_{i=0}^{n-1}\frac{1}{k}\phi_k(T^ix)-C,$$

where C is a constant. Thus

$$\phi_n(x)\geq\sup_{y\in B_n(x,\epsilon)}\sum_{i=0}^{n-1}\frac{1}{k}\phi_k(T^iy)-n\eta-C,$$

which is the desired result. $\square$

4 Dimensions of ergodic measure on an average conformal repeller

In this section we give an application of the nonadditive measure-theoretic pressures to average conformal repellers defined in [1]. We use the relations proved in the previous sections among the pressures, entropy, and limits of the nonadditive potentials to prove that the Hausdorff and box dimension of an average conformal repeller is equal to the Hausdorff dimension of an ergodic measure support on it.

Let M be an m-dimensional smooth Riemannian manifold. Let U be an open subset of M and $f:U\to M$ be a C^1 map. Suppose $J\subset U$ is a compact f-invariant subset. And let $\mathcal{M}(f|_J)$ and

$\mathcal{E}(f|_J)$ denote the set of all $f-$invariant measures and the set of all ergodic measures supported on J respectively.

For $x \in M$ and $v \in T_xM$, the Lyapunov exponent of v at x is the limit

$$\chi(x,v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\|$$

if the limit exists. By the Oseledec multiplicative ergodic theorem [21], for μ-almost every point x, every vector $v \in T_xM$ has a Lyapunov exponent, and they can be denoted by $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_m(x)$. Since the Lyapunov exponents are f-invariant, if μ is ergodic, then we can define Lyapunov exponents $\lambda_1(\mu) \le \lambda_2(\mu) \le \cdots \le \lambda_m(\mu)$, where $m = \dim M$, for the measure.

A compact invariant set $J \subset M$ is an *average conformal repeller* if for any $\mu \in \mathcal{E}(f|_J)$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_m(\mu) > 0$. For simplicity we denote by $\lambda(\mu)$ the unique Lyapunov exponent with respect to μ.

Remark 4.1. *If a compact f-invariant set J is an average conformal repeller, it is indeed a repeller in the usual way ([9]), that is, f is uniformly expanding on J.*

On the other hand, there are average conformal repellers which are not conformal repellers (see an example in [33]).

Remark 4.2. *The notion of average conformal repellers is a generalization of the quasi-conformal and asymptotically conformal repellers in [2, 22]. In [1], the authors studied the dimensions of average conformal repellers by using thermodynamic formalism.*

Given a set $Z \subset M$, its *Hausdorff dimension* is defined by

$$\dim_{\mathrm{H}}(Z) = \inf\{s : \lim_{\epsilon\to 0} \inf_{\mathrm{diam}\,\mathcal{U}<\epsilon} \sum_{U\in\mathcal{U}} (\mathrm{diam}\, U)^s = 0\},$$

where $\mathcal{U}$ is a cover of Z and $\mathrm{diam}\,\mathcal{U} = \sup\{\mathrm{diam}\, U : U \in \mathcal{U}\}$. If ν is a probability measure on M, then the Hausdorff dimension of the measure ν is given by

$$\dim_{\mathrm{H}}(\nu) = \inf\{\dim_{\mathrm{H}}(Z) : Z \subset M, \ \nu Z = 1 \}.$$

The *upper* and *lower box dimensions* of Z are defined by

$$\overline{\dim}_{\mathrm{B}}(Z) = \limsup_{\epsilon\to 0} \frac{\log N(\epsilon)}{-\log \epsilon} \qquad \text{and} \qquad \underline{\dim}_{\mathrm{B}}(Z) = \liminf_{\epsilon\to 0} \frac{\log N(\epsilon)}{-\log \epsilon}$$

respectively, where $N(\epsilon)$ denotes the minimum number of balls of radius ϵ which cover Z. $\overline{\dim}_{\mathrm{B}}(Z)$ and $\underline{\dim}_{\mathrm{B}}(Z)$ are also called *upper* and *lower capacities.* If $\overline{\dim}_{\mathrm{B}}(Z) = \underline{\dim}_{\mathrm{B}}(Z)$, then we simply call this number the *box dimension*, and denoted it by $\dim_{\mathrm{B}}(Z)$. Similarly, for any probability measure ν, we have

$$\overline{\dim}_{\mathrm{B}}(\nu) = \lim_{\delta\to 0} \inf\{\overline{\dim}_{\mathrm{B}}(Z) : Z \subset M, \ \nu Z > 1-\delta\},$$
$$\underline{\dim}_{\mathrm{B}}(\nu) = \lim_{\delta\to 0} \inf\{\underline{\dim}_{\mathrm{B}}(Z) : Z \subset M, \ \nu Z > 1-\delta\}.$$

The *upper* and *lower Ledrappier dimensions* ([19, 22]) is given by

$$\overline{\dim}_{\mathrm{L}}(\nu) = \lim_{\delta\to 0} \limsup_{\epsilon\to 0} \frac{\log N(\epsilon,\delta)}{-\log \epsilon} \qquad \text{and} \qquad \underline{\dim}_{\mathrm{L}}(\nu) = \lim_{\delta\to 0} \liminf_{\epsilon\to 0} \frac{\log N(\epsilon,\delta)}{-\log \epsilon},$$

where $N(\epsilon,\delta)$ is the minimum number of balls of diameter ϵ covering a subset in M of measure greater than $1-\delta$.

It is known that if a compact invariant set $J \subset M$ is a conformal repeller for a C^1 map, then J support an ergodic measure μ such that the Hausdorff or box dimension of measure μ are equal to the Hausdorff or box dimension of the set J, which are known as the variational principle for dimension. And these dimensions can be given by the ratio of the measure-theoretic entropy and the Lyapunov exponent (see e.g. [4, 25, 14]). See also [3, 20] for some recent progresses of variational principle of dimension for non-conformal maps.

Our next theorem is a generalization of the results to average conformal repeller of a C^1 map f.

Theorem E. *Suppose J is an average conformal repeller of a C^1 map f, then there exists an f-invariant ergodic measure μ supported on J such that*

$$D(J) = D(\mu) = \frac{h_\mu(f)}{\lambda(\mu)},$$

where $D(\mu)$ is $\dim_{\mathrm{H}} \mu$, $\overline{\dim}_{\mathrm{L}}\mu$, $\underline{\dim}_{\mathrm{L}}\mu$, $\overline{\dim}_{\mathrm{B}}\mu$ or $\underline{\dim}_{\mathrm{B}}\mu$, and $D(J)$ is $\dim_{\mathrm{H}} J$, $\underline{\dim}_{\mathrm{B}}J$ or $\overline{\dim}_{\mathrm{B}}J$.

In the following, for each f-invariant ergodic measure μ supported on J, we will construct a certain set K whose dimension is equal to the dimension of the measure.

We denote by $m(Df(x))$ the minimal norm of $Df(x)$, that is, $m(Df(x)) = \min\{\|Df(x)u\| : u \in T_xM, \|u\| = 1\}$.

Theorem F. *Suppose J is an average conformal repeller of a C^1 map f, and $\mu \in \mathcal{E}(f|_J)$. Let $\mathcal{F} = \{\log m(Df^n(x))\}_{n\geq 1}$ and*

$$K = \big\{x \in M : \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{-\log \mu(B_n(x,\epsilon))}{n} = h_\mu(T) \text{ and } \lim_{n\to\infty}\frac{1}{n}\log m(Df^n(x)) = \mathcal{F}_*(\mu)\big\}.$$

Then we have

$$D(K) = D(\mu),$$

where $D(\mu)$ is $\dim_{\mathrm{H}} \mu$, $\overline{\dim}_{\mathrm{L}}\mu$, $\underline{\dim}_{\mathrm{L}}\mu$, $\overline{\dim}_{\mathrm{B}}\mu$ or $\underline{\dim}_{\mathrm{B}}\mu$, and $D(K)$ is $\dim_{\mathrm{H}} K$, $\underline{\dim}_{\mathrm{B}}K$ or $\overline{\dim}_{\mathrm{B}}K$.

We end the paper by provide a proof of these two theorems.

Proof of Theorem E. In [1], the authors proved that $D(J) = s_0$, where s_0 is the unique root of the equation $P(f, -s\mathcal{F}) = 0$, where $\mathcal{F} = \{\log m(Df^n(x))\}_{n\geq 1}$. Next, we show that the subadditive topological pressure $P(f, -s\mathcal{F})$ can be attained by an ergodic measure.

By Remark 4.1, we know that the entropy map $\nu \mapsto h_\nu(f)$ is upper-semicontinuous on $\mathcal{M}(f|_J)$. It is also easy to check that $\nu \mapsto -s\mathcal{F}_*(\nu)$ is upper-semicontinuous on $\mathcal{M}(f|_J)$. Hence by compactness of $\mathcal{M}(f|_J)$, and the variational principle for subadditive topological pressure [8], we have $P(f, -s\mathcal{F}) = h_{\mu_0}(f) - s\mathcal{F}_*(\mu_0)$ for some invariant measure $\mu_0 \in \mathcal{M}(f|_J)$. By ergodic decomposition theorem (see e.g. [27]), we can obtain that $P(f, -s\mathcal{F}) = h_\mu(f) - s\mathcal{F}_*(\mu)$ for some f-invariant ergodic measure μ.

Note that $\mathcal{F}_*(\mu) = \lambda(\mu)$ is the unique Lyapunov exponent with respect to μ. Hence $D(J) = s_0 = \frac{h_\mu(f)}{\lambda(\mu)}$. On the other hand, by ([10, Corollary 2]), we can get $D(\mu) = \frac{h_\mu(f)}{\lambda(\mu)}$. Thus we obtain the equality of the theorem. □

Proof of Theorem F. For an f-invariant ergodic measure μ and for each $s > 0$, by Theorem 4.2 in [1], the set K is equal to the set

$$\big\{x \in M : \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{-\log \mu(B_n(x,\epsilon))}{n} = h_\mu(T) \text{ and } \lim_{n\to\infty}\frac{-s}{n}\sum_{i=0}^{n-1}\phi(f^i x) = -s\int \phi \mathrm{d}\mu\big\},$$

where $\phi(x) = \frac{1}{m}\log|\det(Df(x))|$. Using Theorem B, we have

$$P_K(T, -s\phi) = \overline{CP}_K(T, -s\phi) = h_\mu(f) - s\int \phi \mathrm{d}\mu. \tag{4.1}$$

For every subset $Z \subset J$, it is proved in [10] that

$$\dim_{\mathrm{H}} Z \geq t^* \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} Z \leq s^*,$$

where t^* and s^* are the unique root of the Bowen's equation $P_Z(f, -t\phi) = 0$ and $\overline{CP}_Z(f, -s\phi) = 0$ respectively. Using this fact and (4.1), we have

$$\dim_{\mathrm{H}} K \geq \frac{h_\mu(f)}{\mathcal{F}_*(\mu)} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} K \leq \frac{h_\mu(f)}{\mathcal{F}_*(\mu)}$$

since $\mathcal{F}_*(\mu) = \int \phi \mathrm{d}\mu$. Note that $\mathcal{F}_*(\mu)$ is equal to the unique Lyapunov exponent $\lambda(\mu)$ because J is an average conformal repeller. Combining the above inequalities and the result in Theorem E, we have

$$D(K) = \frac{h_\mu(f)}{\lambda(\mu)} = D(\mu).$$

$\square$

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References

[1] J. Ban, Y. Cao and H. Hu, Dimensions of average conformal repeller, *Trans. Amer. Math. Soc.*, **362** (2010), 727-751.

[2] L. Barreira, A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems, *Ergodic Theory and Dynamical Systems*, **16** (1996), 871-927.

[3] L. Barreira and C. Wolf, Measures of maximal dimension for hyperbolic diffeomorphisms, *Commun. Math. Phys.*, **239** (2003), 93-113.

[4] R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.*, **184** (1973), 125-136.

[5] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture notes in Math.*, 470, Springer-Verlag, 1975.

[6] R. Bowen, Hausdorff dimension of quasicircles, *Inst. Haustes Études Sci. Publ. Math.*, **50** (1979), 11-25.

[7] M. Brin and A. Katok, On local entropy, *Lecture Notes in Mathematics*, 1007, Springer-Verlag, 1983.

[8] Y. Cao, D. Feng and W. Huang, The thermodynamic formalism for sub-multiplicative potentials, *Discrete Contin. Dynam. Syst. Ser. A*, **20** (2008), 639-657.

[9] Y. Cao, Nonzero Lyapunov exponents and uniform hyperbolicity, *Nonlinearity*, **16** (2003), 1473-1479.

[10] Y. Cao, Dimension estimate and multifractal analysis of C^1 average conformal repeller, *Preprint.*

[11] W. Cheng, Y. Zhao and Y. Cao, Pressures for asymptotically subadditive potentials under a mistake funciton, *Discrete Contin. Dynam. Syst. Ser. A*, **32** (2012), 487-497.

[12] V. Climenhaga, Bowen's equation in the non-uniform setting, *Ergodic Theory and Dynamical Systems*, **31** 2011, 1163-1182.

[13] K. Falconer, A subadditive thermodynamic formalsim for mixing repellers, *J. Phys. A*, **21** (1988), 737-742.

[14] D. Gatzouras and Y. peres, The variational principle for Hausdorff dimension: a survay, *London Math. Soc. Lecture Note Series* 228, Cambridge University Press, (1996), 113-126.

[15] L. He, J. Lv and L. Zhou, Definition of measure-theoretic pressure using spanning sets, *Acta Math. Sinica(English Series)*, **20** (2004), 709-718.

[16] W. Huang and P. Zhang, Pointwise dimension, entropy and Lyapunov exponents for C^1 map, To appear in *Trans. Amer. Math. Soc.*

[17] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, **51** (1980), 137-173.

[18] J. F.C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Royal Stat. Soc. Ser B*, **30** (1968), 499-510.

[19] F. Ledrappier, Some relations between dimension and Lyapunov exponent, *Commun. Math. Phys.*, **81** (1981), 229-238.

[20] N. Luzia, A variational principle for the dimension for a class of non-conformal repellers, *Ergodic Theory and Dynamical Systems*, **26** (2006), 821-845.

[21] V. Oseledec, A multiplicative ergodic theorem, characteristic Lyapnov exponents of dynamical systems, *Transactions of the Moscow Mathematical Society*, **19** (1968), American Mathematical Society, Providence, R.I..

[22] Ya. Pesin, Dimension theory in dynamical systems, Contemporary Views and Applications, *University of Chicago Press*, Chicago, 1997.

[23] Ya. Pesin and B. Pitskel', Topological pressure and the variational principle for noncompact sets, *Functional Anal. Appl.*, **18** (1984), 307-318.

[24] D. Ruelle, Statistical mechanics on a compact set with Z^ν action satisfying expansiveness and specification, *Trans. Amer. Math. Soc.*, **187** (1973), 237-251.

[25] D. Ruelle, Repellers for real analytic maps, *Ergodic Theory Dynamical Systems*, **2** (1982), 99-107.

[26] P. Walters, A variational principle for the pressure of continuous transformations, *Amer. J. Math.*, **97** (1975), 937-971.

[27] P. Walters, An introduction to ergodic theory, *Springer-Verlag*, New York, 1982.

[28] G. Zhang, Variational principles of pressure, *Discrete Continuous Dynam. Systems A*, **24** (2009), 1409-1435.

[29] Y. Zhang, Dynamical upper bounds for Hausdorff dimension of invariant sets, *Ergodic Theory and Dynamical Systems*, **17** (1997), 739-756.

[30] Y. Zhao and Y. Cao, Measure-theoretic pressure for subadditive potentials, *Nonlinear analysis*, **70** (2009), 2237-2247.

[31] Y. Zhao, A note on the measure-theoretic pressure in subadditive case, *Chinese Annals of Math., Series A*, **No. 3** (2008), 325-332.

[32] Y. Zhao and Y. Cao, On the topological pressure of random bundle transformations in subadditive case, *J. Math. Anal. Appl.*, **342** (2008), 715-725.

[33] Y. Zhao, Y. Cao and J. Ban, The Hausdorff dimension of average conformal repellers under random perturbation, *Nonlinearity*, **22** (2009), 2405-2416.