Polynomial decay of correlations for almost Anosov diffeomorphisms

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Abstract. We investigate the polynomial lower and upper bounds for decay of correlations of a class of two-dimensional almost Anosov diffeomorphisms with respect to their SRB measures. The degrees of the bounds are determined by the expansion and contraction rates as the orbits approach the indifferent fixed point, and can be expressed by using coefficients of the third order terms in the Taylor expansions of the diffeomorphisms at the indifferent fixed points.

0. Introduction

The purpose of the paper is to obtain polynomial decay of correlations for *diffeomorphisms* on compact manifolds. The systems we consider are C^r, $r \geq 4$, almost Anosov diffeomorphisms f of a two-dimensional manifold M with an indifferent fixed point p at which $Df_p = \mathrm{id}$. We show that under some nondegeneracy conditions, if the coefficients of the third order terms in the Taylor expansions of f at p satisfy certain conditions then f has polynomial decay of correlations, and the degrees of the decay rates are given by the coefficients of the xy^2 and y^3 terms. [3]

Polynomial decay for one-dimensional expanding maps with an indifferent fixed point has been studied extensively (see e.g. [9, 15, 21, 5]). There are some systematic ways developed to obtain polynomial decay rates. The tower structures introduced in [20, 21] are widely used that can apply for both exponential and subexponential decay rates. The renew methods proposed in [17] provide a way to obtain upper and lower bound estimates. For higher-dimensional expanding maps with an indifferent periodic points, upper bounds estimates were made in [15]. Recently both upper and lower bound estimates were obtained in [7] for some non-Morkov maps. Though the methods in both [20] and [17] can be applied to invertible case, there are fewer results in this direction. Liverani and Martens investigated a class of area preserving maps on torus and obtained the upper bounds for the correlation functions [10]. In this work we obtain both upper and lower bound estimates of polynomial decay rates for diffeomorphisms.

Our strategy to prove the results is more or less standard. We first induce two-dimensional almost hyperbolic systems to one-dimensional almost expanding systems by collapsing the stable leaves in a Markov partitions, following the scheme described in [20] in particular. Then we use a corresponding theorem, stated in [17] (and [2] as well), for the induced systems to obtain polynomial decay rates, in which first return maps are used. The last step is to pass the rates we obtained for the induced systems to the original ones.

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[3]We mention here that in the Taylor expansion, the conditions $Df_p = \mathrm{id}$ means that the linear terms are trivial, and hyperbolicity implies that the second order terms must vanish. So under the nondegeneracy conditions the third order terms determine the ergodic properties of the systems.

The most challenging part of the work is to estimate the size of the level sets $[\tau > n]$, where τ is the first return time with respect to the set $M \setminus P$, where P is a rectangle whose interior contains p. Note that restricted to the unstable manifold of the indifferent fixed point p, the map has the form $f(r) \approx r + a_0 r^3$. (See (1.2) and (1.3) with $x = r$ and $y = 0$.) So if we take any point z in the the local unstable manifold of p, then the backward orbit $f^{-n}(z)$ converges to p at a speed proportional to $n^{-1/2}$, that is unsummable. Fortunately, the size of the level sets $[\tau > n]$ is of order between $n^{-1/\alpha}$ and $n^{-1/\beta}$, where $1/\beta > 1/\alpha > 2$, because the stable foliation is *not* Lipschitz continuous near the indifferent fixed point p! (See (1.4) for the value of α and β, and Proposition 4.1 for the estimates.) We obtain such estimates by controlling the slopes of the stable leaves at the points close to the local stable manifold of p.

Another problem comes from the last step, when we use the decay rates of the induced systems to obtain the decay rates of the original ones. In this step we need to estimate of the sizes of the rectangles after nth iteration. We use large deviation estimation to get that most rectangles shrink exponentially fast, and prove directly that other rectangles shrink fast enough, and the measure of the union of such rectangles is small.

It is well known that for almost expanding maps of the interval with indifferent fixed point $p = 0$, if $f(x) \approx x + x^{1+s}$, $s \in (0, 1)$, then the rates of decay of correlations are of the order $n^{-(1/s-1)}$. So faster decay rates are given by stronger expansion near the indifferent fixed point (smaller s). In our case, near the fixed point $f(x, y) \approx \big(x(1 + a_2 y^2), y(1 - b_2 y^2)\big)$, and $a_2/2b_2$ plays the role as $1/s$ in one-dimensional systems. The rates of decay are roughly of the order $n^{-(a_2/2b_2-1)}$. This means that the rates of decay for two-dimensional almost hyperbolic systems are determined by the effect of both contraction and expansion when orbits *approach* the indifferent fixed point, and faster decay rates are given by either stronger expansion (larger a_2) or weaker contraction (smaller b_2) or both. [4]

The rest of the paper is organized as follows. In Section 1, we introduce some related definitions and state the Main Theorem. In Section 2, we give the proof of the theorem. The proof consists of three major steps, which are carried out in three subsections. In Subsection 2.1, we introduce a quotient map by collapsing the map along the stable manifolds. In Subsection 2.2, we obtain both the lower and upper polynomial bounds for the induced systems. In Subsection 2.3, we obtain the polynomial bounds for Hölder continuous observables for the original systems. Section 3 is for distortion estimates, mainly used in Subsection 2.1. The size of the level sets are estimated in Section 4, where quantitative analysis is performed. And the decay rates of the size of rectangles are estimated in Sections 5 and 6.

1. Statement of results

Consider a C^∞ two-dimensional compact Riemannian manifold M without boundary, and the Riemannian measure on M is m. Let $\mathrm{Diff}^4(M)$ be the set of four times differentiable diffeomorphisms.

Definition 1.1. *[[4] Definition 1] A map $f \in \mathrm{Diff}^4(M)$ is called an almost Anosov diffeomorphism, if there exist two continuous families of cones $x \to \mathcal{C}^u_x, \mathcal{C}^s_x$ such that, except for a finite set S,*

[4] We refer Remark 1.9 for the reasons that a_0 and b_0 are not involved here.

(i) $Df_x\mathcal{C}^u_x \subseteq \mathcal{C}^u_{f(x)}$ *and* $Df_x\mathcal{C}^s_x \supseteq \mathcal{C}^s_{f(x)}$*;*
(ii) $|Df_xv| > |v|$ *for any* $v \in \mathcal{C}^u_x$ *and* $|Df_xv| < |v|$ *for any* $v \in \mathcal{C}^s_x$*.*

Since S is a finite set, we only need to consider that S is an invariant set by studying f^n instead of f for some nonnegative integer n. Assume that S consists of a single fixed point p. A fixed point p is called *indifferent* if Df_p has an eigenvalue of modulus 1.

Remark 1.2. (i) *By Proposition 4.2 in* [4]*, there is an invariant decomposition of the tangent bundle into* $TM = E^u \oplus E^s$*, the decomposition is continuous except at the indifferent fixed point. By Definition 1.1, away from the fixed point angle between* E^s *and* E^u *is bounded away from zero.*

(ii) *It follows from Proposition 4.4 in* [4] *local unstable manifolds exist for all* $x \in M$*. Existence of local stable manifolds follows similarly.*

Definition 1.3. *[*[4] *Definition 2] An almost Anosov diffeomorphism* f *is said to be non-degenerate (up to the third order), if there exist constants* $r_0 > 0$ *and* $\kappa^u, \kappa^s > 0$ *such that for any* $x \in B(S, r_0)$*,*

$$\begin{aligned} |Df_xv| &\geq (1 + \kappa^u d(x,S)^2)|v|, \ \forall v \in \mathcal{C}^u_x; \\ |Df_xv| &\leq (1 - \kappa^s d(x,S)^2)|v|, \ \forall v \in \mathcal{C}^s_x. \end{aligned} \tag{1.1}$$

By choosing a suitable coordinate system, there is a neighborhood $B(p, r^*)$ of p such that $p = (0,0)$ and f can be expressed as

$$f(x,y) = \Big(x(1 + \phi(x,y)), y(1 - \psi(x,y))\Big), \tag{1.2}$$

where $(x,y) \in \mathbb{R}^2$ and

$$\begin{aligned} \phi(x,y) &= a_0x^2 + a_1xy + a_2y^2 + O(|(x,y)|^3), \\ \psi(x,y) &= b_0x^2 + b_1xy + b_2y^2 + O(|(x,y)|^3). \end{aligned} \tag{1.3}$$

Remark 1.4. *By* (1.1)*, we know that* $\phi(x,y), \psi(x,y) > 0$ *for any* $(x,y) \in B(p, r^*) \setminus \{p\}$*. Hence, we have* $a_0, a_2, b_0, b_2 > 0$*. In this paper, we will consider the case* $a_1 = b_1 = 0$*.*

Given a measurable partition ξ of a measurable space X with a probability measure ν on X, there exists a family of probability measures $\{\nu^\xi_x : \ x \in X\}$ with $\nu^\xi_x(\xi(x)) = 1$, such that for any measurable set $B \subset X$, the map $x \to \nu^\xi_x(B)$ is measurable and

$$\nu(B) = \int_X \nu^\xi_x(B) d\nu(x).$$

This family $\{\nu^\xi_x\}$ is said to be a canonical system of *conditional measures* for ν and ξ [16].

Let $f : (M, \mu) \to (M, \mu)$ be a map with positive Lyapunov exponents almost everywhere. So, the unstable manifold $W^u(x)$ exists almost everywhere and is an immersed submanifold of M ([13]). A measurable partition ξ of M is said to be *subordinate* to unstable manifolds if $\xi(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$ for almost every x with respect to the measure μ. Let m^u_x be the Riemannian measure on $W^u(x)$. The measure μ is said to have absolutely continuous conditional measures on unstable manifolds if for every measurable partition ξ which is subordinate to unstable manifolds, μ^ξ_x is absolutely continuous with respect to m^u_x for μ almost every $x \in M$ ([8]).

Definition 1.5. *An f-invariant Borel probability measure μ on M is said to be an SRB measure if*

(i) *(f,μ) has positive Lyapunov exponents almost everywhere;*
(ii) *μ has absolutely continuous conditional measures on unstable manifolds.*

For any given map f and its invariant probability measure μ, the correlation for two observables Φ and Ψ is defined by

$$\mathrm{Cor}_n(\Phi,\Psi;f,\mu) := \int (\Psi\circ(f^n))\Phi d\mu - \int \Phi d\mu \int \Psi d\mu,$$

where n is a positive integer.

In Lemma 7.1 of [4], it is in fact proved that if f is an almost Anosov diffeomorphism of a torus $M=\mathbb{T}^2$, then for any neighborhood U of p, there exists $\theta^*\in(0,1)$, such that the unstable subspaces are Hölder continuous with Hölder exponent θ^*.

By applying the renewal theory developed by [17] and [2], we could obtain the following results:

Main Theorem. *Let $f\in\mathrm{Diff}^4(M)$ be a topologically mixing almost Anosov diffeomorphism that has an indifferent fixed point p at which (1.1)–(1.3) are satisfied. Suppose $a_0b_2-a_2b_0>0$, $4b_2<a_2$, and $a_1=b_1=0$. Fix any $\alpha,\beta\in(0,1/2)$ with*

$$\frac{\alpha}{1+\alpha}<\beta<\frac{2a_2b_2}{a_2^2+a_2b_2+b_2^2}<\frac{2b_2}{a_2}<\alpha. \tag{1.4}$$

Then for any neighborhood U of p, and any Hölder continuous functions Φ,Ψ with the exponent θ, $\mathrm{supp}\,\Phi,\mathrm{supp}\,\Psi\subset M\setminus U$, and $\int\Phi d\mu\int\Psi d\mu\neq 0$, we have

$$\frac{A'}{n^{\frac{1}{\beta}-1}}\le\left|\mathrm{Cor}_n(\Phi,\Psi;f,\mu)\right|\le\frac{A}{n^{\frac{1}{\alpha}-1}}, \tag{1.5}$$

where μ is an SRB measure, $\theta\in(\max\{(1/\beta-1/\alpha)(3/2+b_0/(2a_0))^{-1},\theta^*\},1]$, and A' and A are positive constants dependent on Φ and Ψ.

Remark 1.6. *The condition on topological mixing seems unnecessary. It can be proved that f is topologically conjugate to an Anosov diffeomorphism on the two-dimensional torus. Hence, f is topologically transitive, and M is the only basic set of f. By the spectral decomposition theorem, f is topologically mixing on M. However, since there is no suitable reference, we put this condition in the theorem.*

Remark 1.7. (i) *Since $\alpha<\dfrac{1}{2}$, the decay rates are faster than n^{-1}.*

(ii) *In inequalities (1.4), we can take $\alpha\gtrsim\dfrac{2b_2}{a_2}$ and $\beta\lesssim\dfrac{2a_2b_2}{a_2^2+a_2b_2+b_2^2}$. Hence $\dfrac{1}{\beta}-\dfrac{1}{\alpha}\gtrsim\dfrac{1}{2}+\dfrac{\alpha}{4}$, while the first inequalities in (1.4) is equivalent to $\dfrac{1}{\beta}-\dfrac{1}{\alpha}<1$. So if $4b_2<a_2$, then we can always choose α and β satisfying (1.4).*

Remark 1.8. *As we see in the above remark, $1/\beta-1/\alpha\gtrsim 1/2+\alpha/4$, and $1/2+\alpha/4<1$. Hence, we can take $\theta\le 1$. In particular, if $2b_2/a_2$ is sufficiently small, then θ can be close to $1/2$.*

Remark 1.9. *To get decay rates of the systems we need to consider first return maps with respect to $M\setminus P$, where P are rectangles with p in its interior. The decay rates are determined by the size of the level sets $[\tau=n]$, where τ is the first return time. For all large n, the sets are in regions close to the local stable manifold of p.*

More precisely, if f has the form given by (1.2) *and* (1.3) *under some coordinate systems, then the level sets $[\tau = n]$ are in regions of the form $\{(x,y):\ 0 < |x| \ll r_1 \le |y| \le r_2\}$ for some $0 < r_1 < r_2$. In the regions a_0x^2 and b_0x^2 are much smaller than a_2y^2 and b_2y^2, and hence we have $f(x,y) \approx (x(1+a_2y^2), y(1-b_2y^2))$. So the degree of the rates of decay only depends on a_2 and b_2.*

2. Proof of the main theorem

In this section, we prove the Main Theorem. The proof consists of three steps, and is carried out in three subsections. In the first step, we induce the system (f, M) to one-dimensional expanding system $(\bar{f}, \overline{M})$ with an indifferent fixed point $\bar{p}$ by taking a Markov partition $\mathcal{P}$ and then collapsing the stable manifolds in each element of the partition. In the second step, we apply a result of Sarig [17] to obtain the lower and upper bounds for the decay of correlations for observable functions on the reduced manifold $\overline{M}$, where the key step is to estimate the measure of the level sets $[\tau = n]$ for the first return time function τ. In the last step, we obtain the decay rates for (f, M) by using the estimates for $(\bar{f}, \overline{M})$, where the main ingredient is to estimate the size of the elements of the partition $\vee_{i=-n}^{n} f^i\mathcal{P}$.

2.1. **Induce to one-dimensional map.** Take a finite Markov partition $\mathcal{P} = \{P_0, P_1, \cdots, P_r\}$ such that $p \in \mathrm{int}P_0 \subset U$, where U is given in the Main Theorem. For any P_i and $x \in P_i$, denote by $\gamma^u(x)$ the connected component of unstable leaf containing x in P_i, and by $W^u(P_i)$ the set of all such leaves. And, $\gamma^s(x)$ and $W^s(P_i)$ are understood in a similar way.

Define an equivalent relation on M by $x \sim y$ if x and y are in the same stable leave $\gamma^s \in W^s(P_i)$ for some P_i. Denote by $\bar{x} = \gamma^s(x)$ the equivalent class that contains x. Denote $\overline{M} = M/\sim$. Let $\pi : M \to \overline{M}$ be the natural projection.

Denote by $\overline{\mathcal{B}}$ the completion of the Borel algebra of $\overline{M}$.

Since $\mathcal{P}$ is a Markov partition, $f(\gamma^s(x)) \subset \gamma^s(f(x))$ for any $x \in P_i$ with $f(x) \in P_j$. Hence, the quotient map $\overline{f} : \overline{M} \to \overline{M}$ given by $\overline{f}(\bar{x}) = \overline{f(x)}$ is well defined. Denote $\overline{P}_i = P_i/\sim$ and $\overline{\mathcal{P}} = \{\overline{P}_0, ..., \overline{P}_r\}$. Since $f(\gamma^u(x)) \supset \gamma^u(f(x))$ for any $x \in P_i$ with $f(x) \in P_j$, $\overline{\mathcal{P}}$ is a Markov partition for $\overline{f}$.

Fix an arbitrary $\hat{\gamma}_i^u \in W^u(P_i)$, $0 \le i \le r$. By abuse of notation we also let $\pi : P_i \to \hat{\gamma}_i^u$ be the sliding map along stable leaves such that for any $x \in P_i$, $\pi(x) = \gamma^s(x) \cap \hat{\gamma}_i^u = \hat{x}$, where $\gamma^s(x) \in W^s(P_i)$.

Now, we define a reference measure $\overline{\nu}$ on $\overline{M}$. For each $\gamma \in W^u(P_i)$, denote by m_γ the Lebesgue measure restricted to γ. We introduce the following function

$$u_n(x) := \sum_{i=0}^{n-1} \big(\log |Df_{x_i}|_{E^u_{x_i}}| - \log |Df_{\widehat{x}_i}|_{E^u_{\widehat{x}_i}}|\big),$$

where $x_i = f^i(x)$. By Lemma 3.1 in the next section, one has that u_n converges uniformly to some function u. We define ν by $d\nu_\gamma(x) := e^{u(x)}dm_\gamma(x)$. By (1) of Lemma 3.3 in the next section, we can define a measure $\overline{\nu}$ on $\overline{M}$ satisfying $\overline{\nu}|_{\overline{P}_i} = \nu_{\hat{\gamma}_i^u}$.

Note that the Jacobian of f with respect to ν is given by

$$J(f)(x) = |D(f)|_{E^u_x}| \cdot e^{u(f(x))} \cdot e^{-u(x)}$$

for ν_γ almost every $x \in M$. By (2) of Lemma 3.3, we have that $J(\overline{f})(\bar{x})$ can be defined as $J(f)(y)$ for any $y \in \gamma^s(x)$.

By Theorem B in [4], f has an SRB measure μ under our assumption. And, μ induces an invariant measure $\overline{\mu}$ on $\overline{M}$ in an obvious way. The estimates for bounded distortion given by Proposition 7.5 in [4] imply that the conditional measure is equivalent to the Lebesgue measure, when the measure is restricted to any unstable curve γ^u away from the indifferent fixed point p. Hence, $\overline{\mu}$ is an absolutely continuous invariant measure with respect to $\overline{\nu}$, and is equivalent to $\overline{\nu}$ away from $\bar{p}$.

Now, we obtain a Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ in the following sense (see [1, 17]):

(i) (Generator property) $\overline{\mathcal{B}}$ is complete and is the smallest σ-algebra containing $\cup_{n\geq 0}\overline{f}^{-n}(\overline{\mathcal{P}})$;

(ii) (Markov property) $\overline{\mathcal{P}}$ is a Markov partition, that is, for any $\overline{P}_i, \overline{P}_j \in \overline{\mathcal{P}}$, if $\bar{\mu}(\overline{f}(\overline{P}_i) \cap \overline{P}_j) > 0$, then $\overline{f}(\overline{P}_i) \supset \overline{P}_j \pmod{\bar{\mu}}$;

(iii) (Local invertibility) for any $\overline{P}_i \in \overline{\mathcal{P}}$ with $\bar{\mu}(\overline{P}_i) > 0$, $\overline{f} : \overline{P}_i \to \overline{f}(\overline{P}_i)$ is invertible with measurable inverse.

By the assumption that f is topologically mixing, the Markov map is irreducible.

2.2. **Polynomial decay rates.** Recall that the indifferent fixed point $p \in \operatorname{int} P_0$, and hence, $\bar{p} \in \operatorname{int} \overline{P}_0$. Denote $\widetilde{M} = \overline{M} \setminus \overline{P}_0$.

Take the first return map $\widetilde{f} = f^\tau$ of f with respect to $M \setminus P_0$, that is, $\widetilde{f}(x) = f^{\tau(x)}(x)$, where τ is the first return time, $\tau(x) = \min\{n > 0 : f^n(x) \in M \setminus P_0\}$. Clearly $\widetilde{f} : M \setminus P_0 \to M \setminus P_0$ induces a first return map from $\widetilde{M}$ to itself. For the sake of simplicity of notation we also denote it by $\widetilde{f}$.

Let $\mathcal{T}' = \{[\tau = n] : n = 1, 2, \ldots\}$ be a partition into the level sets. Then let $\mathcal{T} = \mathcal{T}' \vee \overline{\mathcal{P}}_0$, where $\overline{\mathcal{P}}_0 = \overline{\mathcal{P}} \setminus \{\overline{P_0}\}$ is the Markov partition of $\widetilde{M}$. It is clear that $\mathcal{T}$ is a Markov partition of $\widetilde{M}$.

For any point $\bar{x}, \bar{y} \in \widetilde{M}$, the separation time is defined by

$$s(\bar{x}, \bar{y}) := \sup\{n \geq 0 : \ \widetilde{f}^i(\bar{y}) \in \mathcal{T}(\widetilde{f}^i(\bar{x})), \ 0 \leq i \leq n\}.$$

We may also regard $s(x, y) = s(\bar{x}, \bar{y})$ if $x \in \bar{x}$ and $y \in \bar{y}$.

Let

$$\lambda = \sup\{\|Df_x|_{E^u_x}\|^{-1}, \ \|Df_x|_{E^s_x}\| : \ x \in M \setminus P_0\}. \tag{2.1}$$

Clearly $\lambda \in (0, 1)$. Let $\theta^* \in (0, 1)$ as in Lemma 3.1, and then take $\theta \in [\theta^*, 1)$.

For any function Φ defined on $\overline{M}$, take a semi-norm by

$$D\Phi := \sup_{\bar{x}, \bar{y} \in \widetilde{M}} \frac{|\Phi(\bar{x}) - \Phi(\bar{y})|}{\sqrt{\lambda}^{\ \theta s(\bar{x}, \bar{y})}}.$$

Then we consider the Banach space

$$\mathcal{L} := \{\Phi : \operatorname{supp} \Phi \subset \widetilde{M}, \ \|\Phi\|_\infty + D\Phi < \infty\}. \tag{2.2}$$

and take the norm in $\mathcal{L}$ by $\|\Phi\|_{\mathcal{L}} = \|\Phi\|_\infty + D\Phi$.

It is clear that $\mathcal{L}$ contains Hölder functions with Hölder exponent θ supported on $\widetilde{M}$. If $\Phi \in \mathcal{L}$, then for any $\bar{x}, \bar{y}$ with $s(\bar{x}, \bar{y}) \geq n$, we have

$$|\Phi(\bar{x}) - \Phi(\bar{y})| \leq (D\Phi)\lambda^{\theta s(\bar{x}, \bar{y})} \leq (D\Phi)(\lambda^\theta)^n \leq (D\Phi)(\sqrt{\lambda}^{\theta})^n.$$

That is, Φ is *locally Hölder continuous* in the sense given in [17] (see also [1]).

By Lemma 3.4, we know that $\log J(\widetilde{f}) \in \mathcal{L}$. By standard arguments, it is easy to know (e.g. see Lemma 2 in Subsection 3.1 in [20]) that $\widetilde{f}$ admits an absolutely continuous invariant measure $\widetilde{\mu}$ on $\widetilde{M}$ with the density function $\widetilde{h}$ with respect to $\widetilde{\nu}$, and the density function satisfies $\log \widetilde{h} \in \mathcal{L}$ and is bounded away from 0 and infinity. By uniqueness we know that $\widetilde{\mu}$ is the conditional measure mentioned in the last subsection with respect to $\widetilde{M}$.

The Jacobian of $\widetilde{f}$ with respect to $\widetilde{\mu}$ is given by

$$J_{\widetilde{\mu}}(\widetilde{f}) = J(\widetilde{f})\frac{\widetilde{h}\circ\widetilde{f}}{\widetilde{h}}.$$

Since both $\log J(\widetilde{f})$ and $\log \widetilde{h}$ are in $\mathcal{L}$, so is $-\log J_{\widetilde{\mu}}(\widetilde{f})$. Hence, $-\log J_{\widetilde{\mu}}(\widetilde{f})$ is locally Hölder continuous.

Now we are ready to apply the following theorem that is directly derived from Theorem 2 in [17].

Theorem. *Let $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ be an irreducible measure preserving Markov map with $\bar{\mu}(\overline{M}) = 1$, and assume that $-\log|J_{\widetilde{\mu}}(\widetilde{f})|$ has a $(\widetilde{f}, \mathfrak{T})$-locally Hölder continuous version for $\overline{M}$. If g.c.d. $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\} = 1$, and $\bar{\mu}[\tau > n] = O(1/n^{\varrho})$ with $\varrho > 2$, then there exists $C > 0$ such that for any $\Phi \in \mathcal{L}$ and $\Psi \in L^{\infty}$ with* $\operatorname{supp}\Phi, \operatorname{supp}\Psi \subset \widetilde{M}$, *one has*

$$\left|\mathrm{Cor}_n(\Phi, \Psi; \overline{f}, \bar{\mu}) - \left(\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]\right)\int \Phi \int \Psi\right| \le C F_{\varrho}(n)\|\Psi\|_{\infty}\|\Phi\|_{\mathcal{L}},$$

where $F_{\varrho}(n) = O(1/n^{\varrho})$.

We have an irreducible measure preserving Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ by the previous subsection. By above arguments we know that $-\log|J_{\widetilde{\mu}}(\widetilde{f})|$ has a $(\widetilde{f}, \mathfrak{T})$-locally Hölder continuous version. It is clear that $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\} = 1$ by our construction. So, what we need to do is to estimate $\bar{\mu}[\tau > n]$, that is, to estimate the exponent ϱ.

Recall that $P = P_0$ is the element of the Markov Partition $\mathcal{P}$ with $p \in \operatorname{int} P$. Denote $Q = f^{-1}P \setminus P$. Clearly Q is a rectangle and the set of points $x \in M$ with $\tau(x) > 1$, where τ is the first return time given at the beginning of this subsection. Denote $Q_k = [\tau \ge k]$. Clearly $Q = Q_2$ and $Q_{k+1} \subset Q_k$ for any $k \ge 2$. Moreover, Q_k are rectangles such that for any $x \in Q_k$, $W^s(x, Q_k) = W^s(x, Q)$ and $W^u(x, Q_k) \subset W^u(x, Q)$.

For any unstable curve $\gamma^u \in W^u(Q)$, let $\gamma_k^u = \gamma^u \cap Q_k$. By Proposition 4.1, we know that there exist $D_\alpha > 0$ and $D_\beta > 0$ such that

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \le m_\gamma^u(\gamma_k^u) \le \frac{D_\alpha}{k^{\frac{1}{\alpha}}},$$

where α and β are given in the Main Theorem, and m_γ^u is the Lebesgue measure restricted to γ^u.

Denote by μ_γ^u the conditional measure of the SRB measure μ on γ^u. Since the distortion of f along any unstable curve is uniformly bounded above and below away from p (see Lemma 3.1, also Proposition 7.5 in [4]), so is the density function $\dfrac{d\mu_\gamma^u}{dm_\gamma^u}$. Hence, there exist $C_\alpha, C_\beta > 0$ such that

$$\frac{C_\beta}{k^{\frac{1}{\beta}}} \le \mu_\gamma^u(\gamma_k^u) \le \frac{C_\alpha}{k^{\frac{1}{\alpha}}}.$$

By integration, we get that similar inequalities are true for $\mu Q_k = \mu[\tau > k]$ with different constant coefficients, that is, there exist two positive constants $B_\alpha, B_\beta > 0$ such that

$$\frac{B_\beta}{k^{\frac{1}{\beta}}} \le \mu(Q_k) \le \frac{B_\alpha}{k^{\frac{1}{\alpha}}}. \tag{2.3}$$

It gives that $\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]$ has the order between $n^{-(\frac{1}{\alpha}-1)}$ and $n^{-(\frac{1}{\beta}-1)}$.

By (2.3), we can take $\varrho = 1/\alpha$. Since $F_\varrho(n)$ is of order of $n^{-\varrho}$ and $\varrho > \dfrac{1}{\beta} - 1$, we get that there exist $A_\alpha, A_\beta > 0$ such that

$$\frac{A_\beta}{n^{\frac{1}{\beta}-1}} \le \mathrm{Cor}_n(\Phi, \Psi; \overline{f}, \bar{\mu}) \le \frac{A_\alpha}{n^{\frac{1}{\alpha}-1}}. \tag{2.4}$$

2.3. **Polynomial decay rates for diffeomorphisms.** In this subsection, we establish polynomial decay of correlations for almost Anosov diffeomorphisms using the results we obtained in the reduced systems.

Recall that $\mathcal{P}$ is a Markov partition, and $P = P_0$ is the element of $\mathcal{P}$ containing p, and $M_0 = M \setminus P_0$.

We introduce a type of Hölder functions:

$$\mathcal{H}_\theta := \{\Phi : \ \exists H_\Phi > 0 \ \text{ s.t. } \ |\Phi(x) - \Phi(y)| \le H_\Phi |x-y|^\theta \text{ and } \operatorname{supp}(\Phi) \subset M_0\},$$

where $\theta \in (\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1]$, and $\theta^* \in (0,1)$ is specified in Lemma 7.1 of [4], which is dependent on the map f and the element P_0.

Set $\mathcal{P}_0 := \mathcal{P}$ and $\mathcal{P}_{k,n} := \bigvee_{i=k}^{n} f^{-i}(\mathcal{P}_0)$, and $\mathcal{P}_n = \mathcal{P}_{0,n}$.

For any $\Phi, \Psi \in \mathcal{H}_\theta$ and for any $k > 0$, we define $\overline{\Phi}_k$ by $\overline{\Phi}_k | B := \inf\{\Phi(x) : \ x \in f^k(B)\}$ for any $B \in \mathcal{P}_{0,2k}$, and define $\overline{\Psi}_k$ in the same way.

By Lemma 2.1 below, the direct calculation gives

$$\begin{aligned}
&|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\
\le& \Big| \int (\Psi \circ f^k - \overline{\Psi}_k) \circ (f^{n-k}) \cdot \Phi d\mu \Big| + \Big| \int (\Psi \circ f^k - \overline{\Psi}_k) d\mu \cdot \int \Phi d\mu \Big| \\
\le& (2 \max |\Phi|) \int |\Psi \circ f^k - \overline{\Psi}_k| d\mu \le (2 \max |\Phi|) \cdot \frac{C_A H_\Psi}{k^{\beta^*}},
\end{aligned} \tag{2.5}$$

where β^* is specified in Lemma 2.1.

For $\overline{\Phi}_k$ defined as above, let $\overline{\Phi}_k \mu$ be the signed measure whose density with respect to μ is $\overline{\Phi}_k$, and set $\Phi_k := \dfrac{d((f^k)_*(\overline{\Phi}_k \mu))}{d\mu}$.

Let $|\cdot|$ be the total variation of a signed measure, and note that $(f^k)_*(\Phi \circ (f^k)\mu) = \Phi\mu$, where $|\mu|(A) = \int_A d|\mu|$ for any Borel set $A \subset M$. Applying Lemma 2.1 for Φ we can get

$$\begin{aligned}
&\int |\Phi - \Phi_k| d\mu = |\Phi\mu - \Phi_k\mu|(M) = |(f^k)_*((\Phi \circ (f^k)\mu) - (f^k)_*(\overline{\Phi}_k \mu)|(M) \\
\le& |(\Phi \circ f^k - \overline{\Phi}_k)\mu|(M) = \int |\Phi \circ f^k - \overline{\Phi}_k| d\mu \le \frac{C_A H_\Phi}{k^{\beta^*}}.
\end{aligned}$$

Hence, by similar computation as previously, we have

$$
\begin{aligned}
&\left|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)\right| \\
\leq&\left| \int (\overline{\Psi}_k \circ (f^{n-k}))(\Phi - \Phi_k) d\mu \right| + \left| \int \overline{\Psi}_k d\mu \cdot \int (\Phi - \Phi_k) d\mu \right| \\
\leq&(2 \max |\Psi|) \int |\Phi - \Phi_k| d\mu \leq (2 \max |\Psi|) \frac{C_A H_\Phi}{k^{\beta^*}}.
\end{aligned} \tag{2.6}
$$

Now we show that $\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)$ can be expressed as functions only dependent on the unstable manifolds, which means that these functions are constant along stable manifolds on each element of P_i. Since $\overline{\Psi}_k$ is constant along stable manifolds on each rectangle $P_i \in \mathcal{P}$, we can regard it as a function on $\overline{M}$ as well. Also we have $\pi_*(\overline{\Phi}_k \mu) = \overline{\Phi}_k(\pi_* \mu) = \overline{\Phi}_k(\bar{\mu})$, and $\overline{f} \circ \pi = \pi \circ f$. So,

$$
\begin{aligned}
&\int (\overline{\Psi}_k \circ (f^{n-k})) \Phi_k d\mu = \int (\overline{\Psi}_k \circ (f^{n-k})) d((f^k)_*(\overline{\Phi}_k \mu)) \\
=&\int \overline{\Psi}_k d((f^{n-k})_*(f^k)_*(\overline{\Phi}_k \mu)) = \int \overline{\Psi}_k d((f^n)_*(\overline{\Phi}_k \mu)) \\
=&\int \overline{\Psi}_k d(\pi_*(f^n)_*(\overline{\Phi}_k \mu)) = \int \overline{\Psi}_k d((\overline{f}^n)_*(\overline{\Phi}_k \bar{\mu})) = \int \overline{\Psi}_k \circ \overline{f}^n \cdot \overline{\Phi}_k d\bar{\mu},
\end{aligned}
$$

and,

$$
\int \Phi_k d\mu \int \overline{\Psi}_k d\mu = \int d((f^k)_*(\overline{\Phi}_k \mu)) \cdot \int \overline{\Psi}_k d\bar{\mu} = \int \overline{\Phi}_k d\bar{\mu} \cdot \int \overline{\Psi}_k d\bar{\mu}.
$$

It means $|\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| = |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})|$. Hence, by (2.5) and (2.6), we have

$$
\begin{aligned}
&|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| = |\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu)| \\
\leq&|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\
+&|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| + |\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| \\
=&(2 \max |\Phi|) \cdot \frac{C_A H_\Psi}{k^{\beta^*}} + (2 \max |\Psi|) \cdot \frac{C_A H_\Phi}{k^{\beta^*}} + |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})|.
\end{aligned}
$$

Take $k = [n/2]$. Since $\beta^* > \dfrac{1}{\beta} - 1$, by (2.4), we obtain that there exist $A > 2^{1/\alpha - 1} A_\alpha$ and $A' < 2^{1/\beta - 1} A_\beta$ such that

$$
\frac{A'}{n^{\frac{1}{\beta} - 1}} \leq |\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| \leq \frac{A}{n^{\frac{1}{\alpha} - 1}}.
$$

This completes the whole proof of the Main Theorem.

Lemma 2.1. *Given any $\theta \in (\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1]$, there exist $C_A > 0$, $K > 0$ and $\beta^* = \beta^*(\theta) > \dfrac{1}{\beta} - 1$ such that for any $\Psi \in \mathcal{H}^\theta$ and $k \geq K$,*

$$
\int \left|\Psi \circ f^k - \overline{\Psi}_k\right| d\mu \leq \frac{C_A H_\Psi}{k^{\beta^*}}.
$$

Proof. Recall that by the definition, $\overline{\Psi}_k | B := \inf\{\Psi(x) : x \in f^k(B)\}$, where $B \in \mathcal{P}_{0,2k}$. So for any x, there is $y \in \mathcal{P}_{0,2k}(x)$ such that $\Psi \circ f^k(x) - \overline{\Psi}_k(x) =$

$\Psi \circ f^k(x) - \Psi \circ f^k(y)$. Since $\Psi \in \mathcal{H}_\theta$ and $f^k(\mathcal{P}_{0,2k}(x)) = \mathcal{P}_{-k,k}(f^k(x))$, we have that for $x \in B$ with $B \in \mathcal{P}_{0,2k}$,

$$\begin{aligned}&|\Psi \circ f^k(x) - \overline{\Psi}_k(x)| = |\Psi \circ f^k(x) - \Psi \circ f^k(y)| \\ \le& H_\Psi |f^k(x) - f^k(y)|^\theta \le H_\Psi \mathrm{diam}(f^k(B))^\theta = H_\Psi \mathrm{diam}(\mathcal{P}_{-k,k}(f^k(x)))^\theta.\end{aligned}$$

It means

$$|\Psi(x) - \overline{\Psi}_k(f^{-k}(x))| \le H_\Psi \mathrm{diam}(\mathcal{P}_{-k,k}(x))^\theta. \tag{2.7}$$

Hence, we need to estimate the diameter of the sets in $\mathcal{P}_{-k,k}$.

Let $\delta \in (0, \delta_0)$, where δ_0 is given in Proposition 6.2. Let

$$S_k = \{x \in M \setminus P : \mathrm{diam}(\mathcal{P}_{-k,k}(x)) \ge e^{-k\delta}\}.$$

By Remark 1.2, there is a uniform lower bound for the angle between E^u_x and E^s_x for all $x \in M \setminus P$. Hence, there exist $C_\ell > 0$ such that for any $x \in S_k$, either there exists an unstable manifold $\gamma^u_k(x^u) \subset \mathcal{P}_{-k,k}(x)$ with the length larger than $C_\ell e^{-k\delta}$, where $x^u \in \mathcal{P}_{-k,k}(x)$, or there exists a stable manifold $\gamma^s_k(x^s) \subset \mathcal{P}_{-k,k}(x)$ with the length larger than $C_\ell e^{-k\delta}$, where $x^s \in \mathcal{P}_{-k,k}(x)$.

In the former case, by the fact $f^k(\gamma^u_k(x^u)) = \gamma^s_0(f^k(x^u))$, there is $C_d > 0$ and $y^u \in \gamma^u_k(x^u)$ such that $|Df^k_{y^u}|_{E^u_{y^u}}| < C_d e^{k\delta}$. Hence, by distortion given in Lemma 3.1, for any $y \in \gamma^u_k(x^u)$, $|Df^k_y|_{E^u_y}| < C_d J_u e^{k\delta}$, and then for any $z \in \gamma^s_k(y)$, $y \in \gamma^u_k(x^u)$, $|Df^k_z|_{E^s_z}| < C_d J_u J_s e^{k\delta}$, that is, the inequality holds for all $z \in \mathcal{P}_{-k,k}(x)$. In particular, we have $|Df^k_x|_{E^s_x}| < C_d J_u J_s e^{k\delta}$. Similarly, in the latter case, we can get that $|Df^{-k}_x|_{E^s_x}| < C'_d J'_s J'_u e^{k\delta}$ for some $C'_D > 0$, where J'_s and J'_u are given in Lemma 3.2. So we can get

$$S_k \subset \Big\{x \in M :\ |Df^k_x|_{E^u_x}| < E e^{k\delta}\Big\} \bigcup \Big\{x \in M :\ |Df^{-k}_x|_{E^s_x}| < E' e^{k\delta}\Big\},$$

where $E = C_d J_u J_s$ and $E' = C'_d J'_u J'_s$. By applying Proposition 6.2, we get that there exist $C_D, C'_D > 0$ such that

$$\mu(S_k) \le \frac{C^*_D (\log k)^{2(1/\alpha - 1)}}{k^{1/\alpha - 1}},$$

where $C^*_D = C_D + C'_D$,

Let T_k be given in Proposition 5.1. By this proposition, $\mu(T_k) \le \dfrac{C_s \log k}{k^{1/\alpha}}$. For any $x \in T_k$, by Propositions 5.1 and 5.2, $\mathrm{diam}(\mathcal{P}_{-k,k}(x)) \le \dfrac{C_h}{k^{1/2+\alpha'}}$, where $\alpha' = \dfrac{b_0}{2a_0}$, and C_h is a constant larger than the constants C_s and C_u given by Proposition 5.1 and 5.2.

For any $x \notin T_k$, $\mathrm{diam}(\mathcal{P}_{-k,k}(x)) \le \dfrac{C_s}{k^{3/2+\alpha'}}$ by Proposition 5.1.

Hence, by invariance of μ and (2.7), the above estimates give

$$\begin{aligned}&\int \big|\Psi \circ f^k - \overline{\Psi}_k\big| d\mu = \int \big|\Psi - \overline{\Psi}_k \circ f^{-k}\big| d\mu \\ =& \int_{T^c_k \cap S^c_k} |\Psi \circ f^k - \overline{\Psi}_k| d\mu + \int_{T^c_k \cap S_k} |\Psi \circ f^k - \overline{\Psi}_k| d\mu + \int_{T_k} |\Psi \circ f^k - \overline{\Psi}_k| d\mu \\ \le& H_\Psi e^{-k\delta\theta} + \frac{H_\Psi C^\theta_s}{k^{(3/2+\alpha')\theta}} \cdot \frac{C^*_D (\log k)^{2(1/\alpha-1)}}{k^{1/\alpha-1}} + \frac{H_\Psi C^\theta_s}{k^{(1/2+\alpha')\theta}} \cdot \frac{C_s \log k}{k^{1/\alpha}} \le \frac{C_A H_\Psi}{k^{\beta^*}}\end{aligned}$$

for some $C_A > 0$ independent of Ψ, where $\beta^* > \left(\frac{3}{2} + \alpha'\right)\theta + \frac{1}{\alpha} - 1$. By the choice of θ, we have that $\beta^* > \dfrac{1}{\beta} - 1$. $\square$

3. Some distortion estimates

In this section we provide some distortion estimates which were used in Subsection 2.1 and will be used in Section 5 as well.

Lemma 3.1. *There are positive constants $J_s, J_u > 0$, and $\theta^* \in (0,1]$ such that for any $\gamma^s \in W^s(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^s$ and $n \geq 0$,*

$$\log \frac{|Df^n_y|_{E^u_y}|}{|Df^n_x|_{E^u_x}|} \leq J_s d^s(x,y)^{\theta^*}; \tag{3.1}$$

and for any $\gamma^u \in W^u(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^u$ and $n \geq 0$,

$$\log \frac{|Df^{-n}_y|_{E^u_y}|}{|Df^{-n}_x|_{E^u_x}|} \leq J_u d^u(x,y)^{\theta^*}. \tag{3.2}$$

Proof. Denote $P = P_0$. By the same method as in the proof of Lemma 7.4 in [4], we can get that there exists constant $I_s > 0$ such that if $\gamma^s \subset f^{-1}P \backslash P$ is a W^s-segment with $f^i\gamma^s \subset P$, $i = 1, \cdots n-1$, then for any $x, y \in \gamma^s$,

$$\log \frac{|Df^n_y|_{E^u_y}|}{|Df^n_x|_{E^u_x}|} \leq I_s d^u(x,y)^{\theta^*},$$

where $\theta^* = \theta$ is given in Lemma 7.1 of [4].

With this result we can get a proof of (3.1) using the same idea as in the proof of Proposition 7.5 in [4], whose details can be found in Proposition 3.1 in [3].

The second inequality (3.2) can be obtained similarly. $\square$

Similarly, we have the following result:

Lemma 3.2. *There are two positive constants J'_s and J'_u, and $\theta^* \in (0,1]$ such that for any $\gamma^s \in W^s(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^s$ and $n \geq 0$,*

$$\log \frac{|Df^n_y|_{E^s_y}|}{|Df^n_x|_{E^s_x}|} \leq J'_s d^s(x,y)^{\theta^*};$$

and for any $\gamma^u \in W^u(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^u$ and $n \geq 0$,

$$\log \frac{|Df^{-n}_y|_{E^s_y}|}{|Df^{-n}_x|_{E^s_x}|} \leq J'_u d^u(x,y)^{\theta^*}.$$

Lemma 3.3. (1) *Let $\gamma^u, \hat{\gamma}^u_i \in W^u(P_i)$. For the sliding map $\pi : \gamma^u \to \hat{\gamma}^u_i$, one has that $\pi_* \mu_\gamma = \mu_{\hat{\gamma}^u_i}$.*
(2) *$J(\overline{f})(x) = J(\overline{f})(y)$ for any $y \in \gamma^s(x)$.*

Proof. The statements and proof are the same as (1) and (2) of Lemma 1 in Subsection 3.1 in [20]. $\square$

Lemma 3.4. *There are $C > 0$, $\lambda \in (0,1)$, and $\theta^* \in (0,1)$ such that for any $\gamma^u \in W^u(P_i)$, $i = 1, ..., r$, $x, y \in \gamma^u$,*

$$\log \left| \frac{J(\widetilde{f})(x)}{J(\widetilde{f})(y)} \right| \le C \sqrt{\lambda}^{\theta^* s(x,y)},$$

where $s(x,y)$ is given in Subsection 2.2.

Proof. For any $x \in \gamma^u \cap P_i$, $i \neq 0$, one has

$$J(\widetilde{f})(x) = |D\widetilde{f}_x|_{E^u_x}| \cdot e^{u(\widetilde{f}(x))} \cdot e^{-u(x)}.$$

Denote $\phi(x) = \log |D\widetilde{f}_x|_{E^u_x}|$. We can write

$$\begin{aligned} |u(x) - u(y)| \le & \left| \sum_{i=0}^{k} [\phi(\widetilde{f}^i(x)) - \phi(\widetilde{f}^i(y))] \right| + \left| \sum_{i=0}^{k} [\phi(\widetilde{f}^i(\widehat{x})) - \phi(\widetilde{f}^i(\widehat{y}))] \right| \\ & + \left| \sum_{i=k+1}^{\infty} [\phi(\widetilde{f}^i(x)) - \phi(\widetilde{f}^i(\widehat{x}))] \right| + \left| \sum_{i=k+1}^{\infty} [\phi(\widetilde{f}^i(y)) - \phi(\widetilde{f}^i(\widehat{y}))] \right|. \end{aligned}$$

We take $k > 0$ such that $f^k = \widetilde{f}^{s^*(x,y)/2}$, where $s^*(x,y) = s(x,y)$ if $s(x,y)$ is even and $s^*(x,y) = s(x,y) + 1$, otherwise. Hence, $f^k(x), f^k(\hat{x}), f^k(y), f^k(\hat{y}) \notin P$, and (3.1) and (3.2) can be applied to the sums of the right hand side. So, we can get

$$\begin{aligned} |u(x) - u(y)| \le & J_u d^u(f^k(x), f^k(y))^{\theta^*} + J_u d^u(f^k(\hat{x}), f^k(\hat{y}))^{\theta^*} \\ & + J_s d^s(f^k(x), f^k(\hat{x}))^{\theta^*} + J_s d^s(f^k(y), f^k(\hat{y}))^{\theta^*}. \end{aligned}$$

Recall that λ is defined in (2.1). We can get that

$$\begin{aligned} & d^u(f^k(x), f^k(y))^{\theta^*} \\ = & d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*} \cdot \frac{d^u(\widetilde{f}^{s(x,y)^*/2}(x), \widetilde{f}^{s(x,y)^*/2}(y))^{\theta^*}}{d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*}} \le C_d \lambda^{\theta^* s(x,y)/2}, \end{aligned}$$

where C_d is determined by the maximum radius of each element in the Markov partition, we use the fact that $\widetilde{f}^{s(x,y)}(x)$ and $\widetilde{f}^{s(x,y)}(y)$ are in the same element of the Markov partition $\mathcal{P}$, and hence, $d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*}$ is uniformly bounded. Similarly, we have $d^u(f^k(\hat{x}), f^k(\hat{y}))^{\theta^*}$, $J_u d^s(f^k(x), f^k(\hat{x}))^{\theta^*}$, $J_u d^s(f^k(y), f^k(\hat{y}))^{\theta^*} \le C' \lambda^{\theta^* s(x,y)/2}$, where C' is a positive constant. Hence,

$$|u(x) - u(y)| \le 4C \lambda^{\theta^* s(x,y)/2},$$

where C is a positive constant.

Since $\log |D\widetilde{f}_x|_{E^u_x}| - \log |D\widetilde{f}_y|_{E^u_y}|$ and $u(\widetilde{f}(x)) - u(\widetilde{f}(y))$ can be estimated in a similar way, we get the inequality we need.

This competes the proof. □

4. Rates of convergence of the level sets

In this section, we prove Proposition 4.1 that is the key step to estimate the term $\mu[\tau > n]$.

Recall that $Q = Q_2 = f^{-1}P \setminus P$, and $Q_i = [\tau \geq i]$ for $i \geq 2$.

Note that the map f has a local product structure, that is, there exist positive constants ϵ and δ such that for any $x, y \in M$ with $d(x,y) \leq \delta$, $[x,y] := W^u_\epsilon(x) \cap W^s_\epsilon(y)$ contains exactly one point.

Take a coordinate system in a neighborhood U^* of p such that the map has the form given in (1.2) and (1.3). Hence, the y-axis and x-axis are the stable and unstable manifold of p, respectively. Recall that we assume $a_1 = 0 = b_1$.

Let $r > 0$ be small such that the ball centered at p of radius r is contained in U^*. We also assume that $P = P_0$ is small enough such that P, $f(P)$, and $f^{-1}(P)$ are contained in the ball.

Proposition 4.1. *Suppose $\alpha, \beta \in (0,1)$ satisfies $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then there exist D_α, $D_\beta > 0$ such that for any unstable curve $\gamma^u \in W^u(Q)$, for any $k > 0$, we have*

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \leq m^u_\gamma(\gamma^u_k) \leq \frac{D_\alpha}{k^{\frac{1}{\alpha}}}.$$

where $\gamma^u_k = \gamma^u \cap Q_k$ and m^u_γ is the Lebesgue measure restricted to γ^u.

Proof. Let $\gamma^u \in W^u(Q)$ be an unstable curve in Q. Denote $q = \gamma^u \cap W^s_\varepsilon(p)$.

For any $z = (x,y) \in \gamma^u$, denote $z_1 = (x_1, y_1) = f(z)$, and $\bar{z} = (\bar{x}, \bar{y}) = [z, fz] = W^u(z) \cap W^s(fz)$. Since both z_1 and $\bar{z}$ are in the same stable curve, $z \in Q_k$ if and only if $\bar{z} \in Q_{k-1}$. So if z is an endpoint of γ^u_k, then $\bar{z}$ is an endpoint of γ^u_{k-1}. In order to estimate the length of γ^u_k, we estimate the ratio $m^u_\gamma(\gamma^u_{k-1})/m^u_\gamma(\gamma^u_k)$ firstly. This is equivalent to estimate $\bar{x}/x$.

Denote by v^s_z a real number or ∞ such that $(v^s_z, 1)$ is a tangent vector of $W^s_r(z)$. Take the function $\hat{\rho}$ on $[0,r]$ as in Proposition 4.3. By Lemmas 4.4 and 4.6 below, we know that if $z = z_0$ is sufficiently close to q, then

$$-\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)(1 - x_0^\alpha)\frac{x_0}{y_0} \leq v^s_{z_0} \leq -\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)(1 - x_0^\beta)\frac{x_0}{y_0}.$$

With the estimates for v^s_z, we can get by Lemmas 4.5 and 4.7 that there exist $E_\alpha, E_\beta > 0$ such that

$$x_0 + E_\alpha x_0^{1+\alpha} \leq \bar{x}_0 \leq x_0 + E_\beta x_0^{1+\alpha}.$$

If we denote $s_k = m^u(\gamma^u_k)$, the inequalities mean

$$s_k + E_\alpha s_k^{1+\alpha} \leq s_{k-1} \leq s_k + E_\beta s_k^{1+\alpha}.$$

for all k sufficiently large. Hence, it follows (e.g. see Lemma 3.1 in [6]) that there exist D_α, $D_\beta > 0$ such that for all $k > 0$,

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \leq s_k \leq \frac{D_\alpha}{k^{\frac{1}{\alpha}}}.$$

This is what we need. □

To obtain Lemmas 4.4 and 4.6, we consider v^s_z, where z is near the y-axis. Assume that v^s_z has the form

$$v^s_z = -\rho\frac{x}{y},$$

where $\rho = \rho(x,y)$.

Since $(v_z^s, 1)$ is in the stable cone at z, without loss of generality, assume that

$$-1 \le v_z^s \le 1, \quad \forall z \in B(p,r). \tag{4.1}$$

Let ρ ba a function defined on U^*. Set $z_1 := f(z)$ and $\rho_1 := \rho(z_1)$. Define

$$\begin{aligned}\Delta_\rho(x,y) :&= (\rho - \rho_1)(1+\phi)(1-\psi) + \rho_1 y(1+\phi)\psi_y - y(1-\psi)\phi_y \\ &\quad - \rho_1 \rho x(1+\phi)\psi_x + \rho x(1-\psi)\phi_x,\end{aligned}$$

where $\phi = \phi(x,y)$ and $\psi = \psi(x,y)$. We need the following facts.

Lemma 4.2 ([4] Lemma 8.3)**.** *If $v_z^s \le -\rho(z)\frac{x}{y}$ and $0 \le \Delta_\rho(x,y)$, then $v_{z_1}^s \le -\rho(z_1)\frac{x_1}{y_1}$. The result also holds if all " $\leqslant$ " are replaced by " $\geqslant$ ".*

To get more precise form of ρ, we need the following results.

Proposition 4.3 ([4] Proposition 8.4)**.** *There exists a Lipschitz function $\hat\rho$ on $[0,r]$ with $\hat\rho(0) = 0$ satisfying the following two equations:*

$$\begin{aligned}\Delta_{\frac{a_2}{b_2}+\hat\rho}(0,y) &= (\hat\rho(y) - \hat\rho(y_1^{(0)}))(1+\phi)(1-\psi) \\ &\quad + \left(\frac{a_2}{b_2} + \hat\rho(y_1^{(0)})\right) y(1+\phi)\psi_y - y(1-\psi)\phi_y = 0,\end{aligned}$$

and

$$b_2 \log(1+\phi) + a_2 \log(1-\psi) - b_2 \int_{y_1^{(0)}}^{y} \frac{\hat\rho(t)}{t} dt = 0, \tag{4.2}$$

where $\phi = \phi(0,y)$, $\psi = \psi(0,y)$, and $y_1^{(0)} = y(1-\psi(0,y))$.

The upper bound estimates have been proved in [4]. We state the corresponding lemmas here for completion, which are Lemmas 9.1 and 9.2 in [4]

Lemma 4.4. *Suppose $\alpha a_2 > 2b_2$, $0 < \alpha < 1$, and $a_0 b_2 - a_2 b_0 > 0$. Then for any point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\epsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W_\epsilon^u(q)$ with $x_0 > 0$,*

$$v_{z_0}^s \ge -\left(\frac{a_2}{b_2} + \hat\rho(y_0)\right)(1 - x_0^\alpha)\frac{x_0}{y_0}.$$

Lemma 4.5. *Let $z_0 = (x_0, y_0)$ with $x_0 > 0$. If for all $z = (x,y)$ in the stable curve that joins $\bar z_0$ and z_1,*

$$v_z^s \ge -\left(\frac{a_2}{b_2} + \hat\rho(y)\right)(1 - x^\alpha)\frac{x}{y},$$

then

$$\bar x_0 \ge x_0 + E_\alpha x_0^{1+\alpha},$$

where E_α is a positive constant dependent on y_0.

The following lemma is the key step to get the lower bound estimates for $\bar x_0 / x_0$.

Lemma 4.6. *Given any $\alpha, \beta \in (0,1)$ with $\beta < \dfrac{2a_2 b_2}{a_2^2 + a_2 b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then for any point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W_\varepsilon^u(q)$ with $x_0 > 0$ small,*

$$v_{z_0}^s \le -\left(\frac{a_2}{b_2} + \hat\rho(y_0)\right)(1 - x_0^\beta)\frac{x_0}{y_0}. \tag{4.3}$$

Proof. For each $z_0 = (x_0, y_0) \in W_r^u(q)$, $z_i = (x_i, y_i) = f^i(z_0)$, define

$$c_0 := 0, \quad c_i := \frac{A_1 x_0^\beta y_0^2}{\prod_{j=0}^{i-1}\Big(1 - \theta_0 y_j \psi_y(0, y_j)\Big)} \qquad \forall i \geq 1,$$

where $A_1 = \dfrac{a_2}{2b_2}(2b_2 - \beta a_2)$ and θ_0 is specified in Lemma 4.8. It is evident that

$$c_{i+1} - c_i = c_{i+1}\theta_0 y_i \psi_y(0, y_i), \qquad \forall i > 0. \tag{4.4}$$

Set

$$\rho_i := \rho(z_i) = \left(\frac{a_2}{b_2} + \hat{\rho}(y_i)\right)(1 - x_i^\beta), \ i \geq 0, \tag{4.5}$$

and

$$\widetilde{\rho}_i := \rho_i - c_i, \ i \geq 0. \tag{4.6}$$

For any $z_i = (x_i, y_i)$, set

$$\begin{aligned}\Delta_{\widetilde{\rho}_i}(x_i, y_i) :=& (\widetilde{\rho}_i - \widetilde{\rho}_{i+1})(1 + \phi_i)(1 - \psi_i) \\ &+ \widetilde{\rho}_{i+1} y_i (1 + \phi_i)\psi_y(x_i, y_i) - y_i(1 - \psi_i)\phi_y(x_i, y_i) \\ &- \widetilde{\rho}_i \widetilde{\rho}_{i+1} x_i (1 + \phi_i)\psi_x(x_i, y_i) + \widetilde{\rho}_i x_i (1 - \psi_i)\phi_x(x_i, y_i),\end{aligned}$$

where $\phi_i = \phi(z_i) = \phi(x_i, y_i)$, $\psi_i = \psi(z_i) = \psi(x_i, y_i)$.

By contradiction, suppose that (4.3) is incorrect. It is to show that for $y_q > 0$ small enough, there is $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W_\varepsilon^u(q)$ with $q = (0, y_q)$, $x_0, y_0 > 0$,

$$v_{z_i}^s \geq -\widetilde{\rho}_i \frac{x_i}{y_i} \qquad \text{and} \qquad 0 \geq \Delta_{\widetilde{\rho}_i}(x_i, y_i),$$

this, together with Lemma 4.2, yields that

$$v_{z_{i+1}}^s \geq -\widetilde{\rho}_{i+1}\frac{x_{i+1}}{y_{i+1}}.$$

By Lemma 4.8 below, we can take $\varepsilon > 0$ small enough such that $c_{n_0} > 1 + \max\{a_2/b_2 + \hat{\rho}(y_i) : y \in [0, r]\}$ and hence, $\widetilde{\rho}_{n_0} < -1$ for some $n_0 = n(z_0)$. Since c_i increases with i, it follows that $\widetilde{\rho}_i < -1$ for any $i \geq n_0$. Note that x_i is increasing and y_i is decreasing when the orbit under the iteration of f is in the neighborhood of the origin. Then there exists $n_1 \geq n_0$ such that $v_{z_{n_1}}^s > -\widetilde{\rho}_{n_1}\frac{x_{n_1}}{y_{n_1}} > 1$. This contradicts (4.1).

Now, we will show that for all $i \geq 0$ with $x_i < y_i$,

$$\Delta_{\widetilde{\rho}_i}(x_i, y_i) \leq 0.$$

Note that by (1.2) and (1.3)

$$\phi(x, y) = \phi(0, y) + O(x^2 + xy^2), \quad \psi(x, y) = \psi(0, y) + O(x^2 + xy^2), \tag{4.7}$$

$$\phi_y(x, y) = \phi_y(0, y) + O(x^2 + xy), \quad \psi_y(x, y) = \psi_y(0, y) + O(x^2 + xy). \tag{4.8}$$

Also,

$$\phi(x, y) = a_2 y^2 + O(x^2 + xy^2 + y^3) = a_2 y^2 + O(x^2 + y^3), \tag{4.9}$$

$$y\psi_y(x, y) = 2b_2 y^2 + O(x^2 y + xy^2 + y^3) = 2b_2 y^2 + O(x^2 y + y^3), \tag{4.10}$$

$$x\phi_x(x, y), x\psi_x(x, y) = O(x^2 + x^2 y + xy^2) = O(x^2 + xy^2). \tag{4.11}$$

Since $y_i - y_{i+1} = y_i - y_i(1-\psi(x_i,y_i)) = y_i\psi(x_i,y_i)$, and $\hat\rho$ is Lipschitz continuous,

$$\hat\rho(y_i) - \hat\rho(y_{i+1}) = O(y_i - y_{i+1}) = O(y_i\psi(x_i,y_i)) = O(y_i x_i^2 + y_i^3). \tag{4.12}$$

Denote $y_{i+1}^{(0)} := y_i(1-\psi(0,y_i))$. Then $y_{i+1} - y_{i+1}^{(0)} = O(y_i(\psi(0,y_i) - \psi(x_i,y_i)))$ and hence, by (4.8),

$$\hat\rho(y_{i+1}) - \hat\rho(y_{i+1}^{(0)}) = O(y_{i+1} - y_{i+1}^{(0)}) = O(x_i^2 y_i + x_i y_i^2). \tag{4.13}$$

Note $(1+a)^\beta - 1 = \beta a + O(a^2)$. By (4.9), we have

$$x_{i+1}^\beta - x_i^\beta = x_i^\beta((1+\phi(x_i,y_i))^\beta - 1) = \beta a_2 x_i^\beta y_i^2 + x_i^\beta O(x_i^2 + y_i^3). \tag{4.14}$$

First, using (4.4), (4.7), (4.12), (4.13), and (4.14), we get

$$\begin{aligned}
&(\widetilde\rho_i - \widetilde\rho_{i+1})(1+\phi_i)(1-\psi_i)\\
=&\Big(\hat\rho(y_i) - \hat\rho(y_{i+1})\Big)(1+\phi_i)(1-\psi_i)\\
&+\Big(\frac{a_2}{b_2}(x_{i+1}^\beta - x_i^\beta) + \big(\hat\rho(y_{i+1})x_{i+1}^\beta - \hat\rho(y_i)x_i^\beta\big)\Big)(1+\phi_i)(1-\psi_i)\\
&+(c_{i+1} - c_i)(1+\phi_i)(1-\psi_i)\\
=&\Big(\hat\rho(y_i) - \hat\rho(y_{i+1}^{(0)})\Big)(1+\phi(0,y_i))(1-\psi(0,y_i))\\
&+\frac{a_2}{b_2}\beta a_2 x_i^\beta y_i^2 + (c_{i+1} - c_i)(1+\phi_i)(1-\psi_i)\\
&+O(x_i^2 y_i + x_i y_i^2) + x_i^\beta O(x_i^2 + y_i^3).
\end{aligned} \tag{4.15}$$

Next, using (4.5) and (4.6), and then using (4.7), (4.8), (4.10), and (4.13), we get

$$\begin{aligned}
&\widetilde\rho_{i+1} y_i \psi_y(x_i,y_i)(1+\phi_i) - y_i(1-\psi_i)\phi_y(x_i,y_i)\\
=&\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1})\Big) y_i\psi_y(x_i,y_i)(1+\phi_i) - y_i(1-\psi_i)\phi_y(x_i,y_i)\\
-&\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1})\Big) x_{i+1}^\beta y_i \psi_y(x_i,y_i)(1+\phi_i) - c_{i+1} y_i \psi_y(x_i,y_i)(1+\phi_i)\\
=&\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1}^{(0)})\Big) y_i\psi_y(0,y_i)(1+\phi(0,y_i)) - y_i(1-\psi(0,y_i))\phi_y(0,y_i)\\
-&\frac{a_2}{b_2} 2b_2 x_i^\beta y_i^2 - c_{i+1} y_i \psi_y(0,y_i) + y_i O(x_i^2 + x_i y_i) + x_i^\beta y_i O(x_i^2 + y_i^2).
\end{aligned} \tag{4.16}$$

Also, denote

$$R_{\widetilde\rho}(x_i,y_i) := -\widetilde\rho_i \widetilde\rho_{i+1} x_i (1+\phi_i)\psi_x(x_i,y_i) + \widetilde\rho_i x_i (1-\psi_i)\phi_x(x_i,y_i). \tag{4.17}$$

The equations (4.15)-(4.17), (4.11), and Proposition 4.3 give

$$\begin{aligned}
\Delta_{\widetilde\rho_i}(x_i,y_i) = &-\frac{a_2}{b_2}(2b_2 - \beta a_2) x_i^\beta y_i^2 + (c_{i+1} - c_i)(1+\phi_i)(1-\psi_i)\\
&- c_{i+1} y_i \psi_y(0,y_i) + R_{\widetilde\rho}(x_i,y_i) + O(x_i^2 y_i + x_i y_i^2) + x_i^\beta O(x_i^2 + y_i^3).
\end{aligned} \tag{4.18}$$

Note that the choice of β implies $2b_2 - \beta a_2 > 0$. By (4.11), we have

$$R_{\widetilde\rho}(x_i,y_i) \begin{cases} = O(x_i^2 + x_i y_i^2) & \text{if } \widetilde\rho_i \ge -1;\\ < 0 & \text{if } \widetilde\rho_i < 0. \end{cases}$$

For $i=0$, $c_0=0$ and $c_1=\dfrac{a_2}{2b_2}(2b_2-\beta a_2)x_0^\beta y_0^2$ by the definition of c_i. Hence,

$$\Delta_{\widetilde{\rho}_0}(x_0,y_0)=-\frac{a_2}{2b_2}(2b_2-\beta a_2)x_0^\beta y_0^2-c_1y_i\psi_y(0,y_i)+O(x_0^2+x_0y_0^3+x_0^\beta y_0^3)<0,$$

since we assume x_0 is small compared with y_0.

For $0<i\le n_0(z_0)$, where n_0 is given in Lemma 4.8, by (4.4), we have

$$(c_{i+1}-c_i)(1+\phi_i)(1-\psi_i)-c_{i+1}y_i\psi_y(0,y_i)=-c_{i+1}(1-\theta_0)y_i\psi_y(0,y_i)+y_i^2O(x_i^2+y_i^2).$$

So, we have

$$\Delta_{\widetilde{\rho}_i}(x_i,y_i)=-\frac{a_2}{2b_2}(2b_2-\beta a_2)x_i^\beta y_i^2-c_{i+1}(1-\theta_0)y_i\psi_y(0,y_i)+O(x_i^2+x_iy_i^3+x_0^\beta y_i^3)<0,$$

since we have $Kx_i<y^{1+\beta/(2-\beta)}$, or $x_i^2\le K^{-(2-\beta)}x_i^\beta y_i^2$, for some $K>0$ sufficiently large.

If $i\ge n_0$, then $\widetilde{\rho}_i<0$. Hence, $R_{\widetilde{\rho}}(x_i,y_i)<0$. Then by (4.18),

$$\begin{aligned}\Delta_{\widetilde{\rho}_i}(x_i,y_i)=&-\frac{a_2}{2b_2}(2b_2-\beta a_2)x_i^\beta y_i^2-c_{i+1}(1-\theta_0)y_i\psi_y(0,y_i)\\&-|R_{\widetilde{\rho}}(x_i,y_i)|+O(x_i^2y_i+x_iy_i^2)+x_i^\beta O(x_i^2+y_i^3))<0.\end{aligned}$$

This completes the proof. □

Lemma 4.7. *Let $z_0=(x_0,y_0)$ with $x_0,y_0>0$. If for all $z=(x,y)$ in the stable curve that joins $\bar z_0$ and z_1,*

$$v_z^s\le-\Big(\frac{a_2}{b_2}+\hat\rho(y_0)\Big)(1-x_0^\beta)\frac{x_0}{y_0},\tag{4.19}$$

then

$$\bar x_0\le x_0+E_\beta x_0^{1+\beta},$$

where E_β is a positive constant dependent on y_0.

Proof. Since $(v_z^s,1)$ forms a tangent line of the stable manifold $W_r^s(z)$, (4.19) gives

$$\frac{dx}{dy}\le-\Big(\frac{a_2}{b_2}+\hat\rho(y)\Big)(1-x^\beta)\frac{x}{y},$$

which implies that

$$\frac{dx}{x(1-x^\beta)}+\Big(\frac{a_2}{b_2}+\hat\rho(y)\Big)\frac{dy}{y}\le 0.$$

Integrating the function from $z_1=(x_1,y_1)$ to $\bar z_0=(\bar x_0,\bar y_0)$, we have

$$\log\frac{\bar x_0}{x_1}-\frac{1}{\beta}\log\frac{1-\bar x_0^\beta}{1-x_1^\beta}+\frac{a_2}{b_2}\log\frac{\bar y_0}{y_1}+\int_{y_1}^{\bar y_0}\frac{\hat\rho(y)}{y}dy\le 0.$$

In the following discussions, we omit the subscript 0. The above inequality gives

$$\frac{\bar x}{x_1}\le\Big(\frac{1-\bar x^\beta}{1-x_1^\beta}\Big)^{\frac1\beta}\Big(\frac{y_1}{\bar y}\Big)^{\frac{a_2}{b_2}}\exp\Big(-\int_{y_1}^{\bar y}\frac{\hat\rho(y)}{y}dy\Big).$$

This, together with $x_1=x(1+\phi(x,y))$ and $y_1=y(1-\psi(x,y))$, yields that

$$\frac{\bar x}{x}\le(1+\phi(x,y))(1-\psi(x,y))^{\frac{a_2}{b_2}}\Big(\frac{1-\bar x^\beta}{1-x_1^\beta}\Big)^{\frac1\beta}\Big(\frac{y}{\bar y}\Big)^{\frac{a_2}{b_2}}\exp\Big(-\int_{y(1-\psi(x,y))}^{\bar y}\frac{\hat\rho(y)}{y}dy\Big).$$

By (4.7), $\phi(x,y) = \phi(0,y) + O(x^2 + xy^2)$ and $\psi(x,y) = \psi(0,y) + O(x^2 + xy^2)$. Hence, $\displaystyle\int_{y(1-\psi(x,y))}^{y(1-\bar{\psi}(0,y))} \frac{\hat{\rho}(y)}{y} dy = O(x)$, where we treat y as a constant. By (4.2), one has

$$\begin{aligned}&(1+\phi(x,y))(1-\psi(x,y))^{\frac{a_2}{b_2}} \exp\Big(- \int_{y(1-\psi(x,y))}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big)\\ =&(1+O(x^2)) \exp\Big(\int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy\Big).\end{aligned}$$

Since $\bar{z} = (\bar{x}, \bar{y})$ and $z = (x,y)$ are in the same local unstable manifold, one has that

$$|\bar{y} - y| \le N(\bar{x} - x) \le N(x_1 - x) = Nx\phi,$$

where N is a positive constant. So,

$$\Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \le \Big(1 + \frac{Nx\phi}{\bar{y}}\Big)^{\frac{a_2}{b_2}} = 1 + O(x) \quad \text{and} \quad \exp\Big(\int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy\Big) = 1 + O(x).$$

Now we get

$$\frac{\bar{x}}{x} \le \Big(\frac{1 - \bar{x}^\beta}{1 - x_1^\beta}\Big)^{\frac{1}{\beta}} \big(1 + O(x)\big).$$

Using the facts $x_1^\beta = x^\beta(1+\phi)^\beta = x^\beta + \beta x^\beta \phi + x^\beta O(\phi^2)$ and $x < \bar{x}$, we have

$$\frac{1-\bar{x}^\beta}{1-x_1^\beta} = 1 + \frac{x_1^\beta - \bar{x}^\beta}{1 - x_1^\beta} = 1 + \frac{x^\beta + \beta x^\beta\phi - \bar{x}^\beta + x^\beta O(\phi^2)}{1-x_1^\beta} \le 1 + \frac{\beta x^\beta \phi + x^\beta O(\phi^2)}{1 - x_1^\beta}.$$

Therefore,

$$\frac{\bar{x}}{x} \le 1 + E_\beta x^\beta,$$

where E_β is a positive constant dependent on y_0.

This completes the proof. □

Lemma 4.8. *Suppose $\alpha, \beta \in (0,1)$ satisfies $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then there exist $\theta_0 \in (0,1)$ and $\eta > \dfrac{a_2b_2}{a_2^2+b_2^2}$ such that for any positive constants K and N, a point $q = (0, y_q)$ with $y_q > 0$ small, there is $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W_\varepsilon^u(q)$ with $x_0 > 0$, the following inequalities hold simultaneously for some positive integer $n = n(z_0)$:*

$$x_0^\beta y_0^2 \prod_{j=0}^{n} \Big(1 - \theta_0 y_j \psi_y(0, y_j)\Big)^{-1} \ge N, \qquad Kx_n < y^{1+\eta}.$$

Proof. Since $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} = \dfrac{2a_2b_2}{(a_2^2+b_2^2)\Big(1 + \dfrac{a_2b_2}{a_2^2+b_2^2}\Big)}$, there is $\gamma > 1 + \dfrac{a_2b_2}{a_2^2+b_2^2}$ such that $\beta = \dfrac{2a_2b_2}{\gamma(a_2^2+b_2^2)}$. Take $\dfrac{a_2b_2}{a_2^2+b_2^2} < \eta < \gamma - 1$ and then take $\theta_0 > 0$ such that

$$1 > \theta_0 > \max\Big\{\frac{\gamma\beta}{2}, \ \frac{2 - (\gamma-1)\beta + \eta\beta}{2}\Big\}.$$

Clearly we have $\dfrac{\beta}{2-\beta} < \dfrac{2a_2b_2}{2(a_2^2+a_2b_2+b_2^2)-2a_2b_2} = \dfrac{a_2b_2}{a_2^2+b_2^2} < \eta.$

By the choices of θ_0 and γ, we could assume that K is large enough such that if $Kx \le y$, then

$$1-\theta_0 y\psi_y(0,y) \le 1-\theta_1 2b_2y^2 \le (1-\psi)^{2\theta_2} \tag{4.20}$$

and

$$(1+\phi)^{\beta}(1-\psi)^{2-\gamma\beta} \le 1, \tag{4.21}$$

where θ_1 and θ_2 satisfy

$$\max\left\{\frac{\gamma\beta}{2},\ \frac{2-(\gamma-1)\beta+\beta\eta}{2}\right\} < \theta_2 < \theta_1 < \theta_0. \tag{4.22}$$

Hence, for any $z_0=(x_0,y_0)$ with $Kx_0 < y_0$, by (4.21), we have

$$x_1^{\beta}y_1^{2-\gamma\beta} \le x_0^{\beta}(1+\phi_0)^{\beta}y_0^{2-\gamma\beta}(1-\psi_0)^{2-\gamma\beta} \le x_0^{\beta}y_0^{2-\gamma\beta}. \tag{4.23}$$

Set $n := n(z_0)$ as the largest positive integer such that $Kx_n \le y_n^{1+\eta}$ and $Kx_{n+1} > y_{n+1}^{1+\eta}$. Since $0<y<1$, we have that if $Kx < y^{1+\eta}$, then $Kx<y$. So,

$$x_0^{\beta}y_0^{2-\gamma\beta} \ge x_{n+1}^{\beta}y_{n+1}^{2-\gamma\beta} \ge K^{-\beta}y_{n+1}^{\beta(1+\eta)}y_{n+1}^{2-\gamma\beta} = K^{-\beta}y_{n+1}^{2+(1-\gamma)\beta+\eta\beta}.$$

By (4.20) and (4.23), we get

$$\begin{aligned}\frac{x_0^{\beta}y_0^2}{\prod_{j=0}^{n}\Big(1-\theta_0y_j\psi_y(0,y_j)\Big)} &\ge \frac{y_0^{2\theta_2}x_0^{\beta}y_0^2}{y_0^{2\theta_2}\prod_{j=0}^{n}(1-\psi_j)^{2\theta_2}}\\ \ge \frac{x_0^{\beta}y_0^{2+2\theta_2}}{y_{n+1}^{2\theta_2}} &\ge \frac{y_0^{2\theta_2-\gamma\beta}}{K^{\beta}y_{n+1}^{2\theta_2-(2-(\gamma-1)\beta+\eta\beta)}}.\end{aligned}$$

By (4.22), $2\theta_2-(2-(\gamma-1)\beta+\eta\beta) > 0$. Hence, if z_0 is sufficiently close to q, then y_{n+1} can be arbitrarily small and the right hand side of the inequality can be arbitrarily large. This lemma is thus proved. □

5. Estimates of the size of elements of $\mathcal{P}_{-k,k}$

Recall that $\mathcal{P}$ is a Markov partition. Denote $\mathcal{P}_{k,n} = \vee_{i=k}^{n} f^i(\mathcal{P})$ and $\mathcal{P}_n = \mathcal{P}_{0,n}$. Denote by $\mathcal{P}_{k,n}(x)$ the element of $\mathcal{P}_{k,n}$ that contains x.

Also, denote by $\gamma_n^s(x)$ the connected stable curves that contains x and is contained in $\mathcal{P}_n(x)$, and by $\gamma_n^u(x)$ the connected unstable curves that contains x and is contained in $\mathcal{P}_{-n,0}(x)$.

Recall that m^s is the Lebesgue measure restricted to stable curves. Recall also that $Q=Q_2=f^{-1}P\backslash P$, and $Q_k=[\tau \ge k]$, $k\ge 2$, are introduced in Subsection 2.2. Denote $R_k=[\tau=k]=Q_k\setminus Q_{k+1}$ for $k\ge 2$. Then we denote $Q_k^+=f^{\tau}(Q_k)$ and $R_k^+=f^{\tau}(R_k)=f^k(R_k)$, where f^{τ} is the first return map of f with respect to $M_0=M\setminus P_0$. Clearly $Q_k=\cup_{i=k}^{\infty}R_i$ and $Q_k^+=\cup_{i=k}^{\infty}R_i^+$.

Proposition 5.1. *There exist $K_s>0$ and $C_s>0$ such that for any $k\ge K_s$, we can find a set T_k with the following properties:*

(i) $\mu(T_k) \le \dfrac{C_s\log k}{k^{1/\alpha}}$;

(ii) $m^s(\gamma_k^s(x)) \le \dfrac{C_s}{k^{1/2+\alpha'}}$ *for any $x\in T_k$;*

(iii) $m^s(\gamma_k^s(x)) \le \dfrac{C_s}{k^{3/2+\alpha'}}$ *for any* $x \notin T_k \cup P$,

where $\alpha' = b_0/2a_0$.

Proof. Take $K_s \ge 2K_1$, where K_1 is given in Corollary 5.6.

Recall that λ is defined in (2.1). For each $k > 0$, take $\ell = \ell_k = -\left\lfloor \dfrac{\log k}{\log \lambda} \right\rfloor$. Then for any $j \ge \ell_k$, $\lambda^j < \dfrac{1}{k}$.

Define

$$T_k = \bigcup_{i=0}^{\ell} (f^\tau)^i (Q^+_{\lfloor k/2 \rfloor}),$$

where τ is the first return time with respect to $M \backslash P$. By (2.3), $\mu(Q_{\lfloor k/2 \rfloor}) \le \dfrac{2^{1/\alpha} B_\alpha}{k^{1/\alpha}}$ for some $B_\alpha > 0$. Since μ is preserved under the map f^τ, we can get

$$\mu(T_k) \le \frac{2^{1/\alpha} B_\alpha}{k^{1/\alpha}} \cdot \ell \le \frac{C' \log k}{k^{1/\alpha}}$$

for some $C' > 0$. Hence, we get part (i) if $C_s \ge C'$.

For any $x \in M$, denote $x_k := f^{-k}(x)$. If $x_k \in P$, we define $\tau(x_k) = \min\{i > 0 : f^i(x_k) \in M \setminus P\}$, the first time the orbit of x_k enter $M \setminus P$.

We now prove a claim stronger than the requirements in (ii) and (iii): For any $x \notin P$, the inequality in (ii) holds for any $x \in T_k$ with $x_k \in P$ and $\tau(x_k) > k/2$; and that in (iii) holds otherwise.

If $x_k \notin P$, then by Corollary 5.7(i), $m^s(\gamma_k^s(x)) \le \dfrac{C_2}{k^{3/2+\alpha'}}$.

If $x_k \in P$ and $\tau(x_k) \le k/2$, then we have $f^\tau(x_k) \notin P$ and $k - \tau(x_k) \ge \max\{K_1, k/2\}$. Using Corollary 5.7(i) with $f^{\tau(x_k)}(x_k)$ and $x = f^{k-\tau(x_k)}(f^{\tau(x_k)}(x_k))$ we get

$$m^s(\gamma_k^s(x)) \le \frac{C_2}{(k - \tau(x_k))^{3/2+\alpha'}} \le \frac{2^{3/2+\alpha'} C_2}{k^{3/2+\alpha'}}.$$

If $x_k \in P$, $\tau(x_k) > k/2$ and $x \notin T_k$, then we have $\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k)) \subset Q^+_{\lfloor k/2 \rfloor}$. By Corollary 5.7(ii) we have $m^s(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))) \le \dfrac{C_2}{\lfloor k/2 \rfloor^{1/2+\alpha'}} \le \dfrac{2^{1/2+\alpha'} C_2}{k^{1/2+\alpha'}}$. On the other hand, $x \notin T_k$ implies $k - \tau(x_k) \ge \tau(f^\tau(x_k)) + \tau((f^\tau)^2(x_k)) + \cdots + \tau((f^\tau)^\ell(x_k))$. Hence $\|Df_y^{k-\tau(x_k)}|_{E_y^s}\| \le \lambda^\ell \le \dfrac{1}{k}$ for any $y \in \gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))$ by the choice of ℓ. Note that $f^{k-\tau(x_k)}\left(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\right) = \tau_k^s(x)$. We get

$$m^s(\gamma_k^s(x)) \le \frac{1}{k} \cdot m^s\left(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\right) \le \frac{1}{k} \cdot \frac{2^{1/2+\alpha'} C_2}{k^{1/2+\alpha'}} = \frac{2^{1/2+\alpha'} C_2}{k^{3/2+\alpha'}}.$$

On the other hand, if $x_k \in P$, $\tau(x_k) > k/2$ and $x \in T_k$, then we can only get

$$m^s(\gamma_k^s(x)) \le m^s\left(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\right) \le \frac{C_2}{\lfloor k/2 \rfloor^{1/2+\alpha'}} \le \frac{2^{1/2+\alpha'} C_2}{k^{1/2+\alpha'}}.$$

Now we get what we claimed if we take $C_s = 2^{1/2+\alpha'} C_2$. $\square$

Proposition 5.2. *There exist* $K_u > 0$ *and* $C_u > 0$ *such that for any* $k \ge K_u$, $m^u(\gamma_k^u(x)) \le \dfrac{C_u}{k^{1/\alpha}}$ *for any* $x \notin P$.

Proof. The proof is similar to that for Proposition 5.1 by using the estimates given in Proposition 4.1 for $\gamma_k^u \in W^u(Q_k)$, instead of Corollary 5.7 for $\gamma_k^s \in W^s(Q_k^+)$. $\square$

To prove Lemma 5.5 below, we need the following facts.

Lemma 5.3 ([6] Lemmas 3.1 and 3.2). *If*

$$t_{n-1} \geq t_n + Ct_n^{1+\varrho} + O(t_n^{1+\varrho'}) \qquad \forall n > 0, \tag{5.1}$$

where $\varrho' > \varrho$, then for all large n,

$$t_n \leq \frac{1}{(\varrho C(n+k))^{1/\varrho}} + O\Big(\frac{1}{(n+k)^{\delta'}}\Big), \tag{5.2}$$

for some $\delta' > 1/\varrho$ and $k \in \mathbb{Z}$.

Moreover, if (5.2) *holds and for all $n > 0$,*

$$r(t_n) \leq 1 - C't_n^{\varrho} + O(t_n^{1+\varrho'}),$$

where $C' > 0$, then there exists $D > 0$ such that for all $k_0 > k$,

$$\prod_{i=k_0-k}^{n+k_0-k} r(t_i) \leq D\Big(\frac{k}{n+k}\Big)^{C'/\varrho C}.$$

The results remain true if we interchange "$\leq$" and "$\geq$". Therefore, if (5.1) becomes an equality, then so does (5.2).

Lemma 5.4 ([4] Propositions 2.6 and 2.8). *For any $\varepsilon > 0$, there exists a constant $0 < r_* \leq r_0$ such that for any $r \in (0, r_*)$ and $x \in B(p, r)$, $t \in (0, 1]$, $j = 1, \cdots, \left\lfloor \frac{2}{t^2} \right\rfloor$, we have*

$$(1-\epsilon)|tx| \leq \left|f^j(tx)\right| \leq (1+\epsilon)|tx|;$$

and for any $x, y \in B(p, r)$ with $|\Theta(x, y)| \leq |\Theta(x, f(x))|$ and $|y| = t|x|$, $t \in (0, 1]$, we have

$$|\Theta(y, f^j(y))| \leq |\Theta(x, f(x))| + \varepsilon|x|^2 \qquad \forall\ 0 \leq j \leq \left\lfloor \frac{1}{t^2} \right\rfloor;$$

$$|\Theta(y, f^j(y))| \geq |\Theta(x, f(x))| - \varepsilon|x|^2 \qquad \forall \left\lfloor \frac{1}{t^2} \right\rfloor \leq j \leq \left\lfloor \frac{2}{t^2} \right\rfloor,$$

where r_0 is specified in Definition 1.3, and $\Theta(x, y)$ denotes the angle from x to y counterclockwise in $\mathbb{R}^2$.

Lemma 5.5. *There exists $C_1 > 0$ such that for any $x \in Q$ with $n = \tau(x)$, $\|Df_x^n|_{E_x^s}\| \leq \dfrac{C_1}{n^{3/2+\alpha'}}$, where $\alpha' = b_0/2a_0$.*

Proof. Choose $\theta^u, \theta^s > 0$ small. Then take sectors $\mathcal{S}^u = \{z \in U : |\angle(z, E_p^u)| \leq \theta^u\}$ and $\mathcal{S}^s = \{z \in U : |\angle(z, E_p^s)| \leq \theta^s\}$, where $\angle(z, E_p^u)$ is the angle between the vector from p to z and the line E_p^u. Then let $\mathcal{S}^c = P \setminus (\mathcal{S}^s \cup \mathcal{S}^s)$.

If $N_0 > 0$ is large enough, then for any $x \in Q_{N_0}$, the orbit of x passes through $\mathcal{S}^s$, $\mathcal{S}^c$, and $\mathcal{S}^u$ consecutively before it leaves P. Note that if $x \in R_n \subset Q_{N_0}$, then $n = n_x = \tau(x) \geq N_0$. We take n^s,n^c, $n^u > 0$ such that $n^s = \max\{j > 0 : f^i(x) \in \mathcal{S}^s,\ \forall 1 \leq i \leq j\}$, $n^c = \max\{j > 0 : f^{n^s+i}(x) \in \mathcal{S}^c,\ \forall 1 \leq i \leq j\}$, and $n^u = n_x - n^s - n^c$. That is, $x, f(x), \ldots, f^{n^s}(x) \in \mathcal{S}^s$, $f^{n^s+1}(x), \ldots, f^{n^s+n^c}(x) \in \mathcal{S}^c$, and $f^{n^s+n^c+1}(x), \ldots, f^{n_x}(x) \in \mathcal{S}^u$.

Note that (1.3) implies that f has the form $f(r)=r(1-b_2r^2+O(r^3))$ restricted to $W^s_\varepsilon(p)$, and Df has the form $Df|_{E^s}=1-3b_2r^2+O(r^3)$ restricted to E^s_x for $x=(0,r)\in W^s_\varepsilon(p)$. Hence, by Lemma 5.3, for any point $\hat{x}\in W^s_\varepsilon(p)\cap Q$, $|f^n(\hat{x})|\approx \dfrac{1}{\sqrt{2b_2n}}$ and $\|Df^n_{\hat{x}}|_{E^s_{\hat{x}}}\|\sim \dfrac{\hat{d}_s}{\sqrt{n^3}}$ for some constant $\hat{d}_s>0$, where $a_k\approx b_k$ means $\lim\limits_{k\to\infty}\dfrac{a_k}{b_k}=1$, and $a_k\sim b_k$ means a_k/b_k is bounded away from 0 and infinity. Since the points in $\mathcal{S}^s$ are close to $W^s_\varepsilon(p)$, we can get that there exist $c_s>c'_s>0$ and $d_s>d'_s>0$ such that

$$\frac{c'_s}{\sqrt{n^s}}\le |f^{n^s}(x)|\le \frac{c_s}{\sqrt{n^s}} \qquad \text{and} \qquad \frac{d'_s}{\sqrt{(n^s)^3}}\le \|Df^{n^s}_x|_{E^s_x}\|\le \frac{d_s}{\sqrt{(n^s)^3}}. \tag{5.3}$$

Now we consider the part of the orbit in $\mathcal{S}^c$. Take $z\in\mathcal{S}^s$ such that $f^k(z)\in\mathcal{S}^u\cap Q^+_{N_0}$ with some $k>0$. Define k^s and k^c in a way similar with that of n^s and n^c as above, that is, k^s is the largest positive integer such that $f^1(z),\dots,f^{k^s}(z)\in\mathcal{S}^s$, and k^c is the largest positive integer such that $f^{k^s+1}(z),\dots,f^{k^s+k^c}(z)\in\mathcal{S}^c$. Consider Lemma 5.4 with ε small. If N_0 is sufficiently large, then for $x\in Q_{N_0}$, $|f^{n^s}(x)|=t|f^{k^s}(z)|$ is small. Hence, by Lemma 5.4, for $i=0,1,\dots,n^c$,

$$(1-\varepsilon)^{k^c}|f^{n^s}(x)|\le|f^{n^s+i}(x)|\le(1+\varepsilon)^{k^c}|f^{n^s}(x)| \quad \text{and} \quad n^c\sim\frac{k^c}{t^2}=\frac{k^c|f^{k^s}(z)|^2}{|f^{n^s}(x)|^2}.$$

So, there exist $c_n>c'_n>0$ and $c_c>c'_c>0$ such that for $i=0,1,\dots,n^c$,

$$\frac{c'_n}{|f^{n^s}(x)|^2}\le n^c\le\frac{c_n}{|f^{n^s}(x)|^2}, \quad \text{and} \quad c'_c|f^{n^s}(x)|\le|f^{n^s+i}(x)|\le c_c|f^{n^s}(x)|. \tag{5.4}$$

Note that (1.2) and (1.3) imply that there exist $c>c'>0$ such that $1-c|y|^2\le\|Df_y|_{E^s_y}\|\le 1-c'|y|^2$ for any y with $|y|$ small. Hence, by taking $y=f^{n^s+i}(x)$, $i=0,1,\dots,n^c$, we obtain that there exist $0<d'_c\le d_c<1$ such that

$$d'_c\le\|Df^{n^c}_{f^{n^s}(x)}|_{E^s_{f^{n^s}(x)}}\|\le d_c. \tag{5.5}$$

For the part of the orbit in $\mathcal{S}^u$, we note that (1.3) implies that f has the form $f(r)=r(1+a_0r^2+O(r^3))$ restricted to $W^u_\varepsilon(p)$, and Df has the form $Df|_{E^s}=1-b_0r^2+O(r^3)$ restricted to E^s_x for $x=(r,0)\in W^u_\varepsilon(p)$. Hence, by Lemma 5.3, for any point $\hat{x}\in W^u_\varepsilon(p)$, $|f^{-n}(\hat{x})|\approx\dfrac{1}{\sqrt{2a_0n}}$ and $\|Df^n_{\hat{x}}|_{E^s_{\hat{x}}}\|\sim\dfrac{1}{n^{b_0/2a_0}}$. Since points in $\mathcal{S}^u$ are close to $W^u_\varepsilon(p)$, we can get that there exist $c_u>c'_u>0$ and $d_u>d'_u>0$ such that

$$\begin{aligned}\frac{c'_u}{\sqrt{n^u}}\le&|f^{n^s+n^c}(x)|\le\frac{c_u}{\sqrt{n^u}},\\ \frac{d'_u}{(n^u)^{b_0/2a_0}}\le&\|Df^{n^u}_{f^{n^s+n^c}(x)}|_{E^s_{f^{n^s+n^c}(x)}}\|\le\frac{d_u}{(n^u)^{b_0/2a_0}}.\end{aligned} \tag{5.6}$$

By the second inequality of (5.4), $|f^{n^s+n^c}(x)|\sim|f^{n^s}(x)|$. Hence, by (5.3), (5.4), and (5.6), all n^s, n^c and n^u are roughly proportional. Since $n^s+n^c+n^u=n=n_x$, we know that there exist $\rho^s,\rho^u\in(0,1)$ such that $n^s\ge\rho^sn$ and $n^u\ge\rho^un$. So by (5.3), (5.5) and (5.6), we get

$$\|Df^n_x|_{E^s_x}\|\le\frac{C_1}{n^{3/2+b_0/2a_0}}$$

for some $C_1>0$.

The proof is completed. $\square$

Corollary 5.6. *There exists $K_1 > 0$ such that for any $n > K_1$, if $x, f^n(x) \notin P$, then $\|Df^n_x|_{E^s_x}\| \le \dfrac{C_1}{n^{3/2+\alpha'}}$, where C_1 and α' are as in Lemma 5.5.*

Proof. Take $K_1' > 0$ such that $\dfrac{C_1}{k^{3/2+\alpha'}} \cdot \dfrac{C_1}{n^{3/2+\alpha'}} \le \dfrac{C_1}{\big(2(k+n)\big)^{3/2+\alpha'}}$, whenever $k, n \ge K_1'$.

Let $S = S_{K_1'} = \{f^i(x) \in P : x \in Q_{K_1'}, i = 1, \dots, n_x - 1\}$, where $n_x = \tau(x)$. Since f is uniformly hyperbolic on $M \setminus \overline{S}$, there exists $\rho = \rho_S \in (0,1)$ such that $\|Df_z|_{E^s_x}\| \le \rho$ for any $x \in M \setminus \overline{S}$. Take $K_1'' > 0$ such that for any $n \ge K_1''$, $\rho^n \le \dfrac{C_1}{(2n)^{3/2+\alpha'}}$.

Take $K_1 = \max\{2K_1', 2K_1''\}$. For $x, f^n x \notin P$ with $n \ge K_1$, we denote $I = \{i \in (1,n) : f^i(x) \notin S\}$, and let k_x be the cardinality of I. If $k_x \ge n/2 > K_1''$, then

$$\|Df^n_x|_{E^s_x}\| \le \prod_{i\in I} \|Df_{f_i(x)}|_{E^s_{f_i(x)}}\| \le \rho^{k_x} \le \frac{C_1}{(2k_x)^{3/2+\alpha'}} \le \frac{C_1}{n^{3/2+\alpha'}}.$$

If $k_x \le n/2$, then we may assume that the orbit $\{x, \dots, f^{n-1}(x)\}$ passes through $Q_{K_1'}$ ℓ times. Let $k_1 < k_2 < \cdots < k_\ell < n$ such that $f^{k_j}(x) \in Q_{K_1'}$, $j = 1, \dots, \ell$. Denote $n_j = \tau(f^{k_j}(x))$. So, we have $n_j \ge K_1'$ for all j. Now we get

$$\begin{aligned} \|Df^n_x|_{E^s_x}\| \le & \prod_{1\le j\le \ell} \|Df^{n_j}_{f^{k_j}(x)}|_{E^s_{f^{k_j}(x)}}\| \le \prod_{1\le j \le \ell} \frac{C_1}{n_j^{3/2+\alpha'}} \\ \le & \frac{C_1}{\big(2(n_1 + \cdots + n_\ell)\big)^{3/2+\alpha'}} = \frac{C_1}{\big(2(n-k_x)\big)^{3/2+\alpha'}} \le \frac{C_1}{n^{3/2+\alpha'}}, \end{aligned}$$

where we use the fact $n_1 + \cdots + n_\ell = n - k_x > n/2$.

This completes the proof. $\square$

Recall that Q_n, R_n, Q_n^+, R_n^+ and $\gamma^s_n(x)$ are given at the beginning of this section. Also, we have $Q_n^+ \in \mathcal{P}_n$.

Corollary 5.7. *There exists $C_2 > 0$ such that for any $k > 0$,*

(i) $m^s\big(\gamma^s_k(f^k(x))\big) \le \dfrac{C_2}{k^{3/2+\alpha'}}$ *if $x, f^k(x) \notin P$;*

(ii) $m^s\big(\gamma^s_k(x)\big) \le \dfrac{C_2}{k^{1/2+\alpha'}}$ *if $x \in Q_k^+$.*

Proof. (i) Note that $f^n(\gamma^s_0(x)) = \gamma^s_k(f^k(x))$. By Corollary 5.6, and distortion estimates given in Lemma 3.2, we can get that $m^s\big(\gamma^s_k(f^k(x))\big) \le \dfrac{C_1'}{k^{3/2+\alpha'}} \cdot m^s\big(\gamma^s_0(x)\big)$ for some $C_1' > 0$. Then we use the fact that $m^s\big(\gamma^s_0(x)\big)$ are bounded above for all $x \in M$.

(ii) Note that for $y \in R_i$, $f^i(y) \in R_i^+$ and $f^i(\gamma^s_0(y_i)) = \gamma^s_i(f^i(y_i))$. By using the same arguments as above, and using Lemma 5.5 to replace Corollary 5.6, we can get $m^s\big(\gamma^s_i(f^i(y))\big) \le \dfrac{C_2}{i^{3/2+\alpha'}}$ for all $y \in R_i$. Since for any $x \in Q_k^+$, $\gamma^s_k(x)$ is the union of the stable curves $\gamma_i^s(z_i)$, $z_i \in R_i^+ \cap \gamma_k^s(x)$, $i = k, k+1, \dots,$ we get that $m^s(\gamma_k^s(x)) \le \sum_{i=k}^{\infty} \frac{C_2}{i^{3/2+\alpha'}}$. Now we can increase C_2 to get the result of part (ii). $\square$

6. Some large deviation estimation

In this section, we study the large deviation estimates for the observable function $\Psi \in \mathcal{L}$ with respect to the quotient map $\bar{f}$. We adopt the discussions used in [14].

Recall that $(\overline{f}, \overline{M})$ is the one-dimensional system induced from (f, M), and $(\widetilde{f}, \widetilde{M})$ is the first return maps of $\overline{f}$ with respect to $\widetilde{M} = \overline{M} \setminus \overline{P}_0$.

Lemma 6.1. *Let $0 < \alpha < \frac{1}{2}$. Given any $\epsilon > 0$, for any function $\Psi \in \mathcal{L}$ satisfying $|\int \Psi d\bar{\mu}| \ge \epsilon$, one has that*

$$\bar{\mu}\Big\{\overline{x} \in \overline{M} : \ \Big|\sum_{i=0}^{n-1}\Big(\Psi(\bar{f}^i(\overline{x})) - \int \Psi d\bar{\mu}\Big)\Big| > n\epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}). \tag{6.1}$$

The transfer operator of the Markov map $\bar{f}$ is defined as follows:

$$\mathcal{T}\Psi(\overline{x}) = \sum_{\bar{f}\overline{y} = \overline{x}} g_{\bar{\mu}}(\overline{y})\Psi(\overline{y}),$$

where $g_{\bar{\mu}} = d\bar{\mu}/d\bar{\mu} \circ \bar{f}$ and $\Psi \in L^1(\overline{M})$. Since $\bar{\mu}$ is invariant with respect to the quotient map $\bar{f}$, $g_{\bar{\mu}}$ is said to be the g-function of $\bar{\mu}$.

Define the following operators:

$$T_n\Psi := 1_{\overline{Q}}\mathcal{T}^n(\Psi \cdot 1_{\overline{Q}}), \quad R_n\Psi := 1_{\overline{Q}}\mathcal{T}(\Psi \cdot 1_{[R_{\overline{Q}}=n]}).$$

By Proposition 1 of [17], one has the renewal equation:

$$T(z) = (I - R(z))^{-1}, \quad z \in \mathbb{D},$$

where $\mathbb{D}$ is the unit disk in the complex plane, and

$$R(z) = \sum_{n=1}^{\infty} z^n R_n, \quad T(z) = I + \sum_{n=1}^{\infty} z^n T_n, \quad z \in \mathbb{D}.$$

Proof of Lemma 6.1. For convenience, set $\Phi := \Psi - \int \Psi d\bar{\mu}$.

It follows from (2.4) and the fact that $\bar{\mu}$ is an invariant measure of $\bar{f}$ that

$$\begin{aligned}\Big|\int \Phi \circ \bar{f}^k \cdot \Phi d\bar{\mu}\Big| &= \Big|\int \Big(\Psi \circ \bar{f}^k - \int \Psi d\bar{\mu}\Big)\Big(\Psi - \int \Psi d\bar{\mu}\Big) d\bar{\mu}\Big| \\ &= \Big|\int \Psi \circ \bar{f}^k \cdot \Psi d\bar{\mu} - \int \Psi \circ \bar{f}^k d\bar{\mu} \int \Psi d\bar{\mu}\Big| = |\mathrm{Cor}_n(\Psi, \Psi; \bar{f}, \bar{\mu})| \le \frac{C(\Psi)}{k^{\frac{1}{\alpha}-1}}.\end{aligned}$$

By the renewal theory, Theorem 1 in [17] or Theorem 1.1 in [2],

$$T_n = \frac{1}{r}\mathsf{Pr} + \frac{1}{r^2}\sum_{k=n+1}^{\infty} \mathsf{P}_k + E_n,$$

where Pr is the eigenprojection of $R(1)$ at 1, r is given by $\mathsf{Pr}R'(1)\mathsf{Pr} = r\mathsf{Pr}$, $\mathsf{P}_n = \sum_{l>n} \mathsf{Pr}R_l\mathsf{Pr}$, $E_n \in \mathrm{Hom}(\mathcal{L}, \mathcal{L})$. By using Lemma 6.5 in [2] and (2.3), we have that $\|R_n\| = O(\frac{1}{n^\alpha})$. So, we have $\|E_n\| = o(1/n^{\frac{1}{\alpha}-1})$.

By the fact that $\mathsf{Pr}\Phi = \int_{\overline{Q}} \Phi d\bar{\mu}$ (see the proof of Theorem 2 in [17]), $\int \Phi d\bar{\mu} = 0$, and Theorem 1.2 in [2], one has

$$\int \|\mathcal{T}^n \Phi\| d\bar{\mu} = \int \|T_n \Phi\| d\bar{\mu} = O\Big(\frac{1}{n^{\frac{1}{\alpha}-1}}\Big).$$

Next, it is to apply the method of the proof of Proposition 2.3 in [14] to prove (6.1).

By Proposition 1.2 in [18] and the fact that $\bar{f}$ is measure preserving with respect to the measure $\bar{\mu}$, $\mathbb{E}_{\bar{\mu}}(\Phi|\bar{f}^{-k}\overline{\mathcal{B}}) = (\mathcal{T}^k \Phi) \circ \bar{f}^k$ for any positive integer k and $\Phi \in L^1(\overline{M})$. By direct computation,

$$\begin{aligned}
&\bar{\mu}\Big\{\overline{x} \in \overline{M} : \ \Big|\sum_{i=0}^{n-1} \Phi(\bar{f}^i(\overline{x})) \cdot \Big| > n\epsilon\Big\} \leq \frac{1}{(n\epsilon)^{2\vartheta}} \int \Big|\sum_{i=0}^{n-1} \Phi(\bar{f}^i(\overline{x}))\Big|^{2\vartheta} d\bar{\mu}(\overline{x}) \\
\leq &\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}} \Big(\|\Phi\|_{2\vartheta} + 240 \sum_{k=1}^{n} k^{-1/2} \|\mathbb{E}_{\bar{\mu}}(\Phi \circ \bar{f}^k|\overline{\mathcal{B}})\|_{2\vartheta}\Big)^{2\vartheta} \\
= &\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}} \Big(\|\Phi\|_{2\vartheta} + 240 \sum_{k=1}^{n} k^{-1/2} \|\mathbb{E}_{\bar{\mu}}(\Phi|\bar{f}^{-k}\overline{\mathcal{B}})\|_{2\vartheta}\Big)^{2\vartheta} \\
= &\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}} \Big(\|\Phi\|_{2\vartheta} + 240 \sum_{k=1}^{n} k^{-1/2} \|\mathcal{T}^k \Phi\|_{2\vartheta}\Big)^{2\vartheta} \\
\leq &\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}} \Big(\|\Phi\|_{2\vartheta} + 240 \|\Phi\|_{\infty}^{(2\vartheta-1)/(2\vartheta)} \sum_{k=1}^{n} k^{-1/2} \Big(\int |\mathcal{T}^k \Phi| d\bar{\mu}\Big)^{\frac{1}{2\vartheta}}\Big)^{2\vartheta} \\
\leq &\frac{C}{n^{\vartheta}\epsilon^{2\vartheta}} \Big(\|\Phi\|_{2\vartheta} + 240 \|\Phi\|_{\infty}^{(2\vartheta-1)/(2\vartheta)} \sum_{k=1}^{n} \frac{1}{k}\Big)^{2\vartheta},
\end{aligned}$$

where $\vartheta = \frac{1}{\alpha} - 1 > 1$ and Corollary 1 from [12] is used in the second inequality. This shows (6.1). $\square$

Finally we show a proposition which is used in Subsection 2.3.

Proposition 6.2. *There exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, $E, E' > 0$, we can find C_D, $C'_D > 0$ respectively and $N_d > 0$ satisfying*

$$\mu\Big\{x \in M : \ |Df_x^n|_{E_x^u}| < Ee^{n\delta}\Big\} \leq \frac{C_D(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}}; \tag{6.2}$$

$$\mu\Big\{x \in M : \ |Df_x^{-n}|_{E_x^s}| < E'e^{n\delta}\Big\} \leq \frac{C'_D(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}} \tag{6.3}$$

for all $n \geq N_d$.

Proof. Without loss of generality, we can assume that $E = E' = 1$. This is because we can always take N_d sufficiently large and incease δ to some $\delta^* > \delta$ such that $Ee^{n\delta} \leq e^{n\delta^*}$ for all $n > N_d$.

Now let us prove (6.2).

For the finite Markov partition $\mathcal{P} = \{P_0, P_1, \cdots, P_r\}$ and fixed $\hat{\gamma}_i^u \in W^u(P_i)$, $0 \leq i \leq r$, consider the following function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in P_0; \\ \log |Df_{\pi(x)}|_{E^u_{\pi(x)}}| & \text{if } x \notin P_0, \end{cases}$$

where π is the sliding map defined in Subsection 2.1. Clearly ψ is constant along the stable manifolds in P_i, $0 \le i \le r$. It can be regarded as an element in $\mathcal{L}$ as well. It is evident that $\int \psi d\bar{\mu} > 0$.

Since f is uniformly hyperbolic on $M \setminus P$, there exist two positive constants C_u and C_u' such that

$$C_u \le \log |Df_x|_{E^u_x}| \le C_u' \qquad \forall x \in M \setminus P.$$

Hence, if we let $C_L = \dfrac{C_u}{C_u'}$ and $C_L' = \dfrac{C_u'}{C_u}$, then

$$C_L \le \frac{\log |Df_x|_{E^u_x}|}{\log |Df_{\pi(x)}|_{E^u_{\pi(x)}}|} \le C_L' \qquad \forall x \in P_i,\ i \ne 0.$$

So,

$$\begin{aligned} \log |Df^n_x|_{E^u_x}| &= \sum_{i=0}^{n-1} \log |Df_{f^i(x)}|_{E^u_{f^i(x)}}| \\ &\ge \sum_{i=0}^{n-1} \mathbb{1}_{M\setminus P_0} \log |Df_{f^i(x)}|_{E^u_{f^i(x)}}| \ge C_L \sum_{i=0}^{n-1} \psi(f^i(x)), \end{aligned}$$

where $\mathbb{1}_{M\setminus P_0}$ is the indicator function. Hence,

$$\Big\{x \in M :\ \frac{1}{n} \log |Df^n_x|_{E^u_x}| < \delta\Big\} \subset \Big\{x \in M :\ \frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i(x)) < \frac{\delta}{C_L}\Big\} \tag{6.4}$$

for any $\delta > 0$.

Take $\delta_0 = C_L \int \psi d\mu$, and let $0 < \delta < \delta_0$. Set $\epsilon := \int \psi d\mu - \delta/C_L$. Clearly $\epsilon > 0$. Recall that we mentioned that ψ can be regarded as functions in $\mathcal{L}$. So by Lemma 6.1, one has that

$$\bar{\mu}\Big\{\overline{x} \in \overline{M} :\ \Big|\sum_{i=0}^{n-1} \Big(\psi(\bar{f}^i(\overline{x})) - \int \psi d\bar{\mu}\Big)\Big| > n\epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}),$$

and therefore,

$$\bar{\mu}\Big\{\overline{x} \in \overline{M} :\ \frac{1}{n}\sum_{i=0}^{n-1} \psi(\bar{f}^i(\overline{x})) < \int \psi d\bar{\mu} - \epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}). \tag{6.5}$$

By (6.4) and (6.5), and the fact that $\bar{\mu}$ is the quotient measure of μ, we have that

$$\begin{aligned} &\mu\Big\{x \in M :\ |Df^n_x|_{E^u_x}| < e^{n\delta}\Big\} \le \mu\Big\{x \in M :\ \frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i(x)) < \frac{\delta}{C_L}\Big\} \\ =&\bar{\mu}\Big\{x \in M :\ \frac{1}{n}\sum_{i=0}^{n-1} \psi(\bar{f}^i(x)) < \int \psi d\bar{\mu} - \epsilon\Big\} \le \frac{C_D (\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}} \end{aligned}$$

for some $C_D > 0$. This is (6.2).

To get (6.3), we introduce the following function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in P_0; \\ -\log |Df_{\pi(x)}|_{E^s_{\pi(x)}}| & \text{if } x \notin P_0. \end{cases}$$

Hence ψ is constant along the stable manifolds and can be regarded as a function in $\mathcal{L}$. It is also obvious that $\int \psi d\bar{\mu} > 0$. By using similar methods as above, we can obtain

$$\mu\Big\{x \in M : \ |Df_x^n|_{E_x^s}| > e^{-n\delta}\Big\} \leq \frac{C_D'(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}}$$

for some $C_D' > 0$. Note that E^s is one-dimensional. So $|Df_{f^n(x)}^{-n}|_{E_{f^n(x)}^s}| < e^{n\delta}$ if and only if $|Df_x^n|_{E_x^s}| > e^{-n\delta}$. Since μ is an invariant measure, we get (6.3). $\square$

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