Quasi-Stability of Partially Hyperbolic Diffeomorphisms

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Abstract

A partially hyperbolic diffeomorphism f is structurally quasi-stable if for any diffeomorphism g C^1-close to f, there is a homeomorphism π of M such that $\pi\circ g$ and $f\circ\pi$ differ only by a motion τ along center directions. f is topologically quasi-stable if for any homeomorphism g C^0-close to f, the above holds for a continuous map π instead of a homeomorphism. We show that any partially hyperbolic diffeomorphism f is topologically quasi-stable, and if f has C^1 center foliation W^c_f, then f is structurally quasi-stable. As applications we obtain continuity of topological entropy for certain partially hyperbolic diffeomorphisms with one or two dimensional center foliation.

0 Introduction

The motivation of this paper is to study topological properties of partially hyperbolic systems which are similar to those of uniformly hyperbolic systems.

Partial hyperbolicity theory was first studied in the work of Brin and Pesin ([5]) which emerged in attempts to extend the notion of complete hyperbolicity. A closely related notion of normal hyperbolicity was introduced earlier by Hirsh, Pugh and Shub [6]. For general theory of partial hyperbolicity and normal hyperbolicity, we refer to [12], [7], [2] and [3].

It is well known that Anosov diffeomorphisms are *structurally stable*([1]), that is, if f is an Anosov diffeomorphism on a compact manifold M then any diffeomorphism g C^1-close to f is topologically conjugate to f, i.e., there exists a homeomorphism π on M such that

$$\pi\circ g = f\circ\pi. \tag{0.1}$$

Moreover, f is also *topologically stable*([15]), that is, for any homeomorphism g C^0-close to f, there exists a continuous map π from M onto M such that equation (0.1) holds. For partially hyperbolic

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diffeomorphisms, we can not expect such stabilities because of the existence of the center direction. However, since the systems have both stable and unstable directions, we should be able to obtain some similar properties if we look at the behavior of the hyperbolic part, and "ignore" the motions along the center direction.

In this paper, we shall investigate the "stability" property of partially hyperbolic systems under C^0 and C^1 perturbations. Let f be a partially hyperbolic diffeomorphism. We show in Theorem A that for any homeomorphism g C^0-close to f, there exist a continuous map π from M to itself and a family of locally defined continuous maps $\{\tau_x : x \in M\}$, which send points along the center direction, such that

$$\pi \circ g(x) = \tau_{f(x)} \circ f \circ \pi(x) \quad \text{for all } x \in M. \tag{0.2}$$

In this case we say that f is *topologically quasi-stable.* Moreover, if center foliation $\mathcal{W}_f^c$ of f exists and is of C^1, then we can choose a new family $\{\tau_x : x \in M\}$, which map points along the center leaves such that an equation similar to (0.2) holds. The results are given in Theorem B. Theorem B$'$ deal with a particular case, i.e., one dimensional center foliation, in which the map τ can be determined by a flow along the foliation. In Theorem C we obtain *structural quasi-stability* property of f under C^1 perturbation. That is, if the center foliation $\mathcal{W}_f^c$ is C^1, then for any diffeomorphism g C^1-close to f, π is a leaf conjugacy between f and g.

As applications of the results, we obtain that if f is the time 1 map of an Anosov flow generated by a C^1 vector field, then any diffeomorphism g C^1-close to f is a time $1+\tau\circ f$ map of a flow, and the topological entropy of f and g are close (Theorem D). Also, if f has almost parallel center foliation (see next section for the precise definition), then so does any diffeomorphism g C^1-close to f, and the topological entropy function is locally constant at f in $\mathrm{Diff}^1(M)$ in the case of one dimensional center foliation, and continuous at f in $\mathrm{Diff}^\infty(M)$ in the case of two dimensional center foliation (Theorem E).

Our results concerning topological quasi-stability and structural quasi-stability can be regarded as generalizations of topological stability and structural stability for hyperbolic systems ([1] and [15]) to partially hyperbolic systems. They can also be regarded as generalizations of leaf conjugacy for the case that f has C^1 center foliation ([7] and [12]). However, for topologically quasi-stability, we do not require any additional assumption on f except for partial hyperbolicity. The methods we use for topological and structural quasi-stability are basically the same. We construct an operator in the Banach space consisting all continuous sections of the tangent bundle TM which is contracting in a neighborhood of the zero section such that under the inverse of the exponential map $\exp^{-1}$, π and τ are given by the fixed point of the operator. The methods are adopted from [15, 11] and are different from that used in [7]. We notice that there is another strategy to investigate the topological quasi-stability of partially hyperbolic diffeomorphism using the similar idea in [16], in which Walters obtained topological stability for expansive homeomorphisms with shadowing property. Actually, adapting the unified approach in this paper we will show in a forthcoming paper [8] that any partially hyperbolic diffeomorphism has the so-called "quasi-shadowing" property. Hence, we can obtain the similar results in Theorem A, Theorem B and Theorem B$'$.

With some additional condition Y. Hua showed that the topological entropy is continuous near the time one map f of an Anosov flow ([9]). Now the fact becomes a direct consequence of our result and a result in [14]. Our results for topological entropy of diffeomorphisms with almost parallel center foliation are similar to that in [10], which is under the assumption that the strongly unstable and stable foliations stably carry some unique nontrivial homologies.

This paper is organized as the following. The statements of results are given in Section 1. We also define some words and notations in the section. In Section 2 we deal with topological quasi-stability, including the proof of Theorem A, Theorem B and Theorem B$'$. The case of structural quasi-stability is discussed in Section 3, where we use the facts obtained in Section 2 to prove that the map π is a homeomorphism to obtain Theorem C. Section 4 is concerning the applications to topological entropy, where Theorem D and Theorem E are proved.

1 Definition, statement of results and notations

Let M be an m-dimensional C^∞ compact Riemannian manifold. We denote by $\|\cdot\|$ and $d(\cdot,\cdot)$ the norm on TM and the metric on M induced by the Riemannian metric respectively. Denote by $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphisms of M, $1 \le r \le \infty$.

A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is said to be *(uniformly) partially hyperbolic* if there exist numbers λ, λ', μ and μ' with $0 < \lambda < 1 < \mu$ and $\lambda < \lambda' \le \mu' < \mu$, and an invariant decomposition $T_xM = E^s_x \oplus E^c_x \oplus E^u_x \;\; \forall x \in M$, such that for any $n \ge 0$,

$$\begin{aligned} \|d_x f^n v\| &\le C\lambda^n \|v\| && \text{as } v \in E^s(x),\\ C^{-1}(\lambda')^n\|v\| \le \|d_x f^n v\| &\le C(\mu')^n\|v\| && \text{as } v \in E^c(x),\\ C^{-1}\mu^n\|v\| \le \|d_x f^n v\| & && \text{as } v \in E^u(x) \end{aligned}$$

hold for some number $C > 0$. E^s_x, E^c_x and E^u_x are called *stable, center* and *unstable* subspace, respectively. Via a change of Riemannian metric we always assume that $C = 1$. Moreover, for simplicity of notation, we assume that $\lambda = \dfrac{1}{\mu}$.

Since M is compact, we can take constant $\rho_0 > 0$ such that for any $x \in M$, the standard exponential mapping $\exp_x : \{v \in T_xM : \|v\| < \rho_0\} \to M$ is a C^∞ diffeomorphism to the image. Clearly, we have $d(x, \exp_x v) = \|v\|$ for $v \in T_xM$ with $\|v\| < \rho_0$. For any diffeomorphism $f : M \to M$, we take $\rho = \rho_f \in (0, \rho_0/2)$ such that for any $x, y \in M$ with $d(f^{-1}(x), y) \le \rho$, $v \in T_yM$ with $\|v\| \le \rho$,

$$d(x, f \circ \exp_y v) \le \rho_0/2.$$

Reduce ρ if necessary such that both sides in equation (2.3) and (2.20), in the proof of Theorem A and Theorem B respectively, are contained in the set $\{v \in T_xM : \|v\| < \rho_0\}$.

For any given continuous center section $u = \{u(x) \in E^c_x : x \in M\}$ with $\sup_{x\in M}\|u(x)\| < \rho$, we define a smooth map $\tau^{(1)}_x = \tau^{(1)}_x(\cdot, u)$ on $B(x,\rho)$ for any $x \in M$ by

$$\tau^{(1)}_x(y) = \exp_x(u(x) + \exp_x^{-1} y).$$

Theorem A. *Let $f \in \mathrm{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. Then there exists $\varepsilon_0 \in (0,\rho)$ satisfying the following conditions: For any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that for any* *homeomorphism g of M with $d(f,g) < \delta$ there exist a continuous center section u and a surjective continuous map $\pi : M \to M$ such that*

$$\pi \circ g(x) = \tau^{(1)}_{f(x)} \circ f \circ \pi(x), \quad x \in M. \tag{1.1}$$

Moreover, u and π can be chosen uniquely so as to satisfy the following conditions:

$$\begin{aligned} &d(\pi, \mathrm{id}_M) < \varepsilon, \\ &\exp_x^{-1}(\pi(x)) \in E^s_x \oplus E^u_x \quad \textit{for } x \in M. \end{aligned} \tag{1.2}$$

We mention again here that the theorem does not require any additional condition, provided f is a partially hyperbolic diffeomorphism, and g is a homeomorphisms C^0 close to f.

If f has C^1 center foliation $\mathcal{W}^c_f$, then we can require τ to move along the center foliation. In this case, we denote for any $\varepsilon > 0$, $\Sigma_\varepsilon(x) = \exp_x(H_x(\varepsilon))$, where $H_x(\varepsilon)$ is the $\varepsilon-$ball in $E^s_x \oplus E^u_x$. Obviously, $\Sigma_\varepsilon(x)$ is a smooth disk transversal to E^c_x at x. Since the center foliation $\mathcal{W}^c_f$ is C^1, we can conclude that if y is close enough to x, then there exist a locally defined smooth map $\tau^{(2)}_x$ on some neighborhood $U(x)$ of x and a constant $K_1 > 1$ independent of x such that for any $y \in U(x)$, we have

$$\tau^{(2)}_x(y) \in \Sigma_\varepsilon(x) \cap \mathcal{W}^c_f(y) \tag{1.3}$$

and

$$d(\tau^{(2)}_x(y), x) < K_1 d(y, x). \tag{1.4}$$

Theorem B. *Assume that $f \in \mathrm{Diff}^r(M)$ is a partially hyperbolic diffeomorphism with C^1 center foliation $\mathcal{W}^c_f$. Then there exists $\varepsilon_0 \in (0, \rho)$ satisfying the following conditions: For any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that for any homeomorphism g of M with $d(f,g) < \delta$ there exists a surjective continuous map $\pi : M \to M$ such that*

$$\pi \circ g(x) = \tau^{(2)}_{g(x)} \circ f \circ \pi(x), \quad x \in M. \tag{1.5}$$

Moreover, π can be chosen uniquely so as to satisfy the conditions in (1.2).

As a special case, if the center foliation $\mathcal{W}^c_f$ is C^1 and of dimension one, then we can define τ more directly. Let $u \in \mathfrak{X}^c$ with $\|u(x)\| = 1$ for any $x \in M$, and φ^t be the flow generated by u. For any continuous function $\tilde{\tau} : M \to \mathbb{R}$, define a smooth map $\tau^{(3)}_x = \tau^{(3)}_x(\cdot, \tilde{\tau})$ of $B(x, \rho)$ for any $x \in M$ by

$$\tau^{(3)}_x(y) = \varphi^{\tilde{\tau}(x)}(y).$$

Theorem B′. *Assume that $f \in \mathrm{Diff}^r(M)$ is a partially hyperbolic diffeomorphism with one dimensional C^1 center foliation $\mathcal{W}^c_f$. Then there exists $\varepsilon_0 \in (0, \rho)$ satisfying the following conditions: For any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that for any homeomorphism g of M with $d(f,g) < \delta$ there exists a continuous function $\widetilde{\tau} : M \to \mathbb{R}$ and a surjective continuous map $\pi : M \to M$ such that*

$$\pi \circ g(x) = \tau^{(3)}_{f(x)} \circ f \circ \pi(x), \quad x \in M. \tag{1.6}$$

Moreover, $\tilde{\tau}$ and π can be chosen uniquely so as to satisfy the conditions in (1.2).

Now we consider the structural quasi-stability as g is C^1-close to f.

A diffeomorphism f is called *dynamically coherent* if $E^{cu} := E^c \oplus E^u$, E^c, and $E^{cs} := E^c \oplus E^s$ are integrable, and everywhere tangent to $\mathcal{W}_f^{cu}$, $\mathcal{W}_f^c$ and $\mathcal{W}_f^{cs}$, the *center-unstable*, *center* and *center-stable foliations*, respectively; and $\mathcal{W}_f^c$ and $\mathcal{W}_f^u$ are subfoliations of $\mathcal{W}_f^{cu}$, while $\mathcal{W}_f^c$ and $\mathcal{W}_f^s$ are subfoliations of $\mathcal{W}_f^{cs}$. By Theorem 2.3 of [13], if f is as in Theorem B then it is dynamically coherent and this property is permanent under C^1 perturbation.

Theorem C. *Under the assumption of Theorem B (resp. Theorem B′), if g is a diffeomorphism C^1-close to f, then π can be chosen to be a homeomorphism and hence there exists a homeomorphism $\tau^{(2)}$ in Theorem B (resp. $\tau^{(3)}$ in Theorem B′) such that $\pi \circ g = \tau^{(2)} \circ f \circ \pi$ (resp. $\pi \circ g = \tau^{(3)} \circ f \circ \pi$). Also, π and $\tau^{(2)}$ (resp. $\tau^{(3)}$) can be chosen uniquely so as to satisfy the conditions in (1.2) if we replace E_x^s and E_x^u in* (1.2) *by their smooth approximation $\tilde{E}_x^s$ and $\tilde{E}_x^u$ respectively.*

Moreover, π sends $\mathcal{W}_g^{cu}$, $\mathcal{W}_g^c$ and $\mathcal{W}_g^{cs}$ to $\mathcal{W}_f^{cu}$, $\mathcal{W}_f^c$ and $\mathcal{W}_f^{cs}$ respectively. In particular, π is a leaf conjugacy from $(g, \mathcal{W}_g^c)$ to $(f, \mathcal{W}_f^c)$.

Remark 1.1. *In fact, if π is one to one, then for any $y \in M$, $x = \pi^{-1}(f^{-1}(y))$ is uniquely determined. Hence we can define $\tau^{(2)} : M \to M$ by $\tau^{(2)}(y) = \tau^{(2)}_{g(\pi^{-1} \circ f^{-1}(y))}(y)$ and obtain*

$$\pi \circ g(x) = \tau^{(2)} \circ f \circ \pi(x), \quad x \in M.$$

$\tau^{(3)}$ can be defined in a similar way.

The main result of Theorem C, in particular, leaf conjugacy, is well known ([7], [12]). Our proof provides a different approach.

Example. *Let N be a smooth compact Riemannian manifold and $h : N \longrightarrow N$ be an Anosov diffeomorphism. Then the diffeomorphism*

$$f = h \times \mathrm{id}_{S^1} : N \times S^1 \longrightarrow N \times S^1$$

is quasi-stable. In particular, if R is a rotation on S^1 close to the identity, and

$$g = h \times R : N \times S^1 \longrightarrow N \times S^1,$$

then we can take $\pi = \mathrm{id}_{N \times S^1}$ and $\tau^{(2)} = \mathrm{id}_N \times R$ in Theorem C.

Moreover, Theorem E below gives that the topological entropy is constant in a neighborhood of f in C^1 topology.

As applications of Theorem C, we have the following results about the continuity of the entropy.

We say that a diffeomorphism g is a time $1 + \tau$ map of a flow ψ for some real function τ on M if $g(x) = \psi^{1+\tau(x)}(x)$ for any $x \in M$.

Theorem D. *Let f be the time one map of an Anosov flow φ. Then for any diffeomorphism g C^1-close to f, there is a flow ψ and a continuous real function τ on M such that g is the time $1 + \tau \circ f$ map of the flow ψ. Hence the topological entropy function is continuous at f in $\mathrm{Diff}^1(M)$ with C^1 topology.*

Y. Hua proved the second part of the theorem under the condition that f is topologically transitive ([9]).

Let f be a partially hyperbolic diffeomorphism with center foliation $\mathcal{W}^c_f$. For a smooth surface Σ, we denote $\Sigma \perp \mathcal{W}^c_f$ if for any $x \in \Sigma \cap \mathcal{W}^c_f$, $T_x\Sigma \perp E^c_f(x)$. For any two surfaces Σ_1 and Σ_2 that are smooth or are images of homeomorphisms of some smooth surfaces, the holonomy map $\theta^c : \Sigma_1 \to \Sigma_2$ is a continuous map defined by sliding along the $\mathcal{W}^c_f$-leaves, i.e. for $x \in \Sigma_1$, $\theta^c(x) \in \Sigma_2 \cap \mathcal{W}^c_f(x)$.

Definition 1.2. *A partially hyperbolic diffeomorphism f with integrable center foliation $\mathcal{W}^c_f$ is said to have* almost parallel center foliation *if for any $\alpha > 0$, there exists constant $\beta > 0$, such that for any smooth surfaces Σ_1, Σ_2 with $\Sigma_1 \perp \mathcal{W}^c_f$ and $\Sigma_2 \perp \mathcal{W}^c_f$, for any $x, y \in \Sigma_1$ with $d(x, y) \leq \beta$, we have $d(\theta^c(x), \theta^c(y)) \leq \alpha$ whenever they are defined.*

Remark 1.3. *In the definition we require $\Sigma_1 \perp \mathcal{W}^c_f$ and $\Sigma_2 \perp \mathcal{W}^c_f$ only for convenience. It is clear that we can change the definition by requiring the angles between Σ_1, Σ_2 and $\mathcal{W}^c_f$ uniformly bounded from below.*

Remark 1.4. *The requirements of the definition mean that the holonomy maps along the center foliation are equicontinuous for all possible holonomy maps whenever they are defined.*

It is clear that each of the maps f and g given by the above example has almost parallel center foliation, while Anosov flows do not.

Theorem E. *Let f be a partially hyperbolic diffeomorphism as in Theorem B. If the center foliation of f is almost parallel, then any diffeomorphism g that is C^1-close to f also has almost parallel center foliation.*

Moreover, if the center foliation of f is one dimensional, then the topological entropy function is locally constant in $\mathrm{Diff}^1(M)$; if $f \in \mathrm{Diff}^\infty(M)$ and the center foliation of f is two dimensional, then the topological entropy function is continuous near f in $\mathrm{Diff}^\infty(M)$ with C^1 topology.

For the case that the dimension of center subbundle of f is one or two, the same conclusions are obtained in [10] under the assumption that the strongly stable and unstable foliations of f stably carry some unique nontrivial homologies. Our proof uses some idea in the paper.

Denote by $\mathfrak{X}$ the Banach space of continuous vector fields on M with the norm

$$\|\omega\| = \sup_{x \in M} \|\omega(x)\|, \quad w \in \mathfrak{X}.$$

In other words, each element of $\mathfrak{X}$ is a continuous section of the tangent bundle TM. Similarly, we denote by $\mathfrak{X}^s, \mathfrak{X}^c$ and $\mathfrak{X}^u$ the space of continuous sections of the stable, center and unstable bundles E^s, E^c and E^u respectively. Also, we denote $\mathfrak{X}^{us} = \mathfrak{X}^u \oplus \mathfrak{X}^s$. Let $\Pi^s_x : T_xM \to E^s_x$ be the projection onto E^s_x along $E^c_x \oplus E^u_x$. Π^c_x and Π^u_x are defined in a similar way.

Recall that $\|\cdot\|$ is the norm on TM. We define the norm $\|\cdot\|_1$ on TM by $\|w\|_1 = \|u\| + \|v\|$ if $w = u + v \in T_xM$ with $u \in E^c_x$ and $v \in E^u_x \oplus E^s_x$. Similarly, if $w = u + v \in \mathfrak{X}$ with $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$, we also define $\|w\|_1 = \|u\| + \|v\|$. By triangle inequality and the fact that the angles between E^c and $E^u \oplus E^s$ are uniformly bounded away from zero, we know that there exists a constant L such that

$$\|w\| \leq \|w\|_1 \leq L\|w\|. \tag{1.7}$$

For any $\varepsilon > 0$, we denote

$$\mathfrak{B}(\varepsilon) = \{w \in \mathfrak{X} : \|w\| \le \varepsilon\}, \qquad \mathfrak{B}^{us}(\varepsilon) = \{w \in \mathfrak{X}^{us} : \|w\| \le \varepsilon\},$$

$$\mathfrak{B}_1(\varepsilon) = \{w \in \mathfrak{X} : \|w\|_1 \le \varepsilon\}.$$

2 Topological quasi-stability

2.1 The general case

Proof of Theorem A. We choose

$$\varepsilon_0 \in (0, \rho) \tag{2.1}$$

small enough such that any map π with $d(\pi, \mathrm{id}_M) < \varepsilon_0$ must be surjective (see e.g. Lemma 3 of [15] for existence of such ε_0).

To find a continuous center section $u \in \mathfrak{X}^c$ and a surjective continuous map $\pi : M \longrightarrow M$ satisfying (1.1) and the conditions in (1.2) of this theorem, we shall first try to solve the equation

$$\pi \circ g(x) = \tau^{(1)}_{f(x)} \circ f \circ \pi(x) \tag{2.2}$$

for unknown u and π. Putting $h = g \circ f^{-1}$ and $\pi(x) = \exp_x(v(x))$ for $v \in \mathfrak{B}^{us}(\rho)$, and replacing x by $f^{-1}(x)$, we see that (1.1) is equivalent to

$$\exp_x^{-1} \circ \exp_{h(x)}\big(v(h(x))\big) = \exp_x^{-1} \circ \tau_x^{(1)} \circ f \circ \exp_{f^{-1}(x)}\big(v(f^{-1}(x))\big). \tag{2.3}$$

By the definition of $\tau_x^{(1)}$, we have

$$\exp_x^{-1} \circ \tau_x^{(1)} \circ f \circ \exp_{f^{-1}(x)}\big(v(f^{-1}(x))\big) = u(x) + \exp_x^{-1} \circ f \circ \exp_{f^{-1}(x)}\big((v(f^{-1}(x))\big).$$

Define an operator $\beta : \mathfrak{B}(\rho) \to \mathfrak{X}$ and a linear operator $F : \mathfrak{X} \to \mathfrak{X}$ by

$$\beta(w)(x) = \exp_x^{-1} \circ f \circ \exp_{f^{-1}(x)}\big((w(f^{-1}(x))\big), \tag{2.4}$$

$$(Fw)(x) = d_{f^{-1}(x)} f w(f^{-1}(x)). \tag{2.5}$$

Clearly, $Fw = d_0\beta w$. Let

$$\eta(w)(x) = \beta(w)\big(x\big) - (d_0\beta w)\big(x\big). \tag{2.6}$$

Then we can write

$$\exp_x^{-1} \circ \tau_x^{(1)} \circ f \circ \exp_{f^{-1}(x)}(v(f^{-1}(x))) = (Fv)(x) + u(x) + \eta(v)(x). \tag{2.7}$$

Define a linear operator $J_h : \mathfrak{B}(\rho) \to \mathfrak{X}$ by

$$(J_h w)(x) = \sum_{i=s,c,u} \Pi_x^i \circ d_0(\exp_x^{-1} \circ \exp_{h(x)}) \circ \Pi_{h(x)}^i w(h(x)) \tag{2.8}$$

for any $w \in \mathfrak{B}(\rho)$. Set

$$\begin{aligned}\theta_h(w)(x) = {}& \exp_x^{-1} \circ \exp_{h(x)}(w(h(x))) - d_0(\exp_x^{-1} \circ \exp_{h(x)})w(h(x)) \\ &+ \sum_{i,j=s,c,u,i\ne j} \Pi_x^i \circ d_0(\exp_x^{-1} \circ \exp_{h(x)}) \circ \Pi_{h(x)}^j w(h(x)).\end{aligned}$$

Then we have

$$\exp_x^{-1} \circ \exp_{h(x)}(v(h(x))) = (J_h v)(x) + \theta_h(v)(x). \tag{2.9}$$

Also we mention that by definition,

$$\theta_h(0)(x) = \exp_x^{-1} \circ \exp_{h(x)} 0 = \exp_x^{-1} h(x). \tag{2.10}$$

Therefore, by (2.9) and (2.7), (2.3) is equivalent to

$$J_h v + \theta_h(v) = Fv + u + \eta(v),$$

further, is equivalent to

$$-J_h^{-1}u + (\mathrm{id}_{\mathfrak{X}} - J_h^{-1}F)v = J_h^{-1}(\eta(v) - \theta_h(v)).$$

Define a linear operator P_h from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$P_h \omega = -J_h^{-1}u + (\mathrm{id}_{\mathfrak{X}} - J_h^{-1}F)v \tag{2.11}$$

for $\omega = u + v \in \mathfrak{X}$, where $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$.

Define an operator Φ_h from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$\Phi_h(u+v) = P_h^{-1} J_h^{-1}(\eta(v) - \theta_h(v)).$$

Hence, equation (2.2) is equivalent to

$$\Phi_h(u+v) = u + v, \tag{2.12}$$

namely, $u+v$ is a fixed point of Φ_h.

By Lemma 2.1 below, we know that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon)$ such that for any homeomorphism h with $d(h, \mathrm{id}_M) \le \delta$, $\Phi_h : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ is a contracting map, and therefore has a fixed point in $\mathfrak{B}_1(\varepsilon)$. Hence, (2.2) has a unique solution. □

Lemma 2.1. *We can reduce ε_0 in (2.1) if necessary such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any homeomorphism h of M with $d(h, \mathrm{id}_M) \le \delta$, $\Phi_h(\mathfrak{B}_1(\varepsilon)) \subset \mathfrak{B}_1(\varepsilon)$ and for any $\omega, \omega' \in \mathfrak{B}_1(\varepsilon)$,*

$$\|\Phi_h(\omega) - \Phi_h(\omega')\|_1 \le \frac{1}{2}\|\omega - \omega'\|_1.$$

Proof. Recall that the constant L is given in (1.7).

Reduce ε_0 if necessary such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{4L}{1-\lambda} C(\varepsilon) < \frac{1}{4}, \tag{2.13}$$

where $C(\varepsilon)$ is the Lipschitz constant of $\eta(v)$ given in Sublemma 2.3. This is possible since by the sublemma, $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that ε only depends on f.

Then we take $\delta = \delta(\varepsilon)$ such that

$$\frac{4L}{1-\lambda}\delta < \frac{1}{4}\varepsilon; \tag{2.14}$$

and such that for any homeomorphism h with $d(h, \mathrm{id}_M) < \delta$,

$$\max\{\|J_h\|,\ \|J_h^{-1}\|\} \le \min\left\{2,\ \frac{1+\lambda^{-1}}{2}\right\}, \tag{2.15}$$

where J_h is defined in (2.8), and

$$\frac{4L}{1-\lambda} K(h) < \frac{1}{4}, \tag{2.16}$$

where $K(h)$ is the Lipschitz constant of $\theta_h(\cdot)$ given in Sublemma 2.4. This is possible since by the sublemma, $K(h) \to 0$ as $\varepsilon \to 0$.

By (2.15), Sublemma 2.2 below can be applied and therefore we get

$$\|P_h^{-1}\|_1 \le \frac{2}{1-\lambda}. \tag{2.17}$$

Note that $J_h(\mathfrak{X}^i) = \mathfrak{X}^i$ for $i = s, c, u$. Then it is easy to check that $\|J_h^{-1}\|_1 \le \|J_h^{-1}\|$. Hence by (2.15), we have $\|J_h^{-1}\|_1 \le 2$. Also, by (2.4), $\beta(0) = 0$ and therefore by (2.6), $\eta(0) = 0$; and by (2.10), $\|\theta_h(0)\| \le \delta$.

Take $\omega = u + v \in \mathfrak{B}_1(\varepsilon)$ with $u \in \mathfrak{X}^c$ and $v \in \mathfrak{X}^{us}$. By using the above estimates, Sublemma 2.3 and Sublemma 2.4, and then (2.13), (2.14) and (2.16), we can get

$$\begin{aligned}
\|\Phi_h(\omega)\|_1 &\le \|P_h^{-1}\|_1 \cdot \|J_h^{-1}\|_1 \cdot \|\eta(v) - \theta_h(v)\|_1 \\
&\le \frac{2}{1-\lambda} \cdot 2 \cdot L\|\eta(v) - \theta_h(v)\| \\
&\le \frac{4L}{1-\lambda}(\|\eta(v) - \eta(0)\| + \|\theta_h(v) - \theta_h(0)\| + \|\theta_h(0)\|) \\
&\le \frac{4L}{1-\lambda}(C(\varepsilon)\|\omega\|_1 + K(h)\|\omega\|_1 + \delta) \\
&\le \frac{1}{4}\|\omega\|_1 + \frac{1}{4}\|\omega\|_1 + \frac{1}{4}\varepsilon \le \frac{3\varepsilon}{4},
\end{aligned}$$

which implies that $\Phi_h(\mathfrak{B}_1(\varepsilon)) \subset \mathfrak{B}_1(\varepsilon)$.

Similarly, for two elements $\omega = u + v,\ \omega' = u' + v' \in \mathfrak{B}_1(\varepsilon)$ with $u, u' \in \mathfrak{X}^c$ and $v, v' \in \mathfrak{X}^{us}$, we have

$$\begin{aligned}
\|\Phi_h(\omega) - \Phi_h(\omega')\|_1 &\le \frac{4}{1-\lambda}(\|\eta(v) - \eta(v')\|_1 + \|\theta_h(v) - \theta_h(v')\|_1) \\
&\le \frac{4L}{1-\lambda}(\|\eta(v) - \eta(v')\| + \|\theta_h(v) - \theta_h(v')\|) \\
&\le \frac{4L}{1-\lambda}(C(\varepsilon_0)\|\omega - \omega'\|_1 + K(h)\|\omega - \omega'\|_1) \\
&\le \frac{1}{2}\|\omega - \omega'\|_1.
\end{aligned}$$

This proves that $\Phi_h : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ is a contraction. □

Sublemma 2.2. *For any homeomorphism h of M such that J_h satisfies* (2.15)*, P_h is invertible and*

$$\|P_h^{-1}\|_1 \le \frac{2}{1-\lambda}. \tag{2.18}$$

Proof. By the definitions of F and J_h, we have $F(\mathfrak{X}^i) = \mathfrak{X}^i$ and $J_h(\mathfrak{X}^i) = \mathfrak{X}^i$ for $i = u, s, c$. Let $F^i = F|_{\mathfrak{X}^i}$, $J_h^i = J_h|_{\mathfrak{X}^i}$ for $i = u, s, c$. By the definition of P_h we have $P_h|_{\mathfrak{X}^i} = id_{\mathfrak{X}^i} - (J_h^i)^{-1} \circ F^i$, $i = s, u$, and $P_h|_{\mathfrak{X}^c} = -(J_h^c)^{-1}$. So we also have $P_h(\mathfrak{X}^i) = \mathfrak{X}^i$ for $i = s, c, u$.

Since $\|F^s\|, \|(F^u)^{-1}\| \le \lambda$, by (2.15) we know that

$$\|(J_h^s)^{-1} \circ F^s\|,\ \|((F^{(u)})^{-1} \circ J_h^{(u)})\| \le \lambda \cdot (1 + \lambda^{-1})/2 = (1 + \lambda)/2 < 1.$$

Hence, both $P_h|_{\mathfrak{X}^s}$ and $P_h|_{\mathfrak{X}^u}$ are invertible and

$$(P_h|_{\mathfrak{X}^s})^{-1} = (id_{\mathfrak{X}^s} - (J_h^s)^{-1} \circ F^s)^{-1} = \sum_{k=0}^{\infty} ((J_h^s)^{-1} \circ F^s)^k,$$

$$(P_h|_{\mathfrak{X}^u})^{-1} = (id_{\mathfrak{X}^u} - (J_h^u)^{-1} \circ F^u)^{-1} = -\sum_{k=1}^{\infty} ((F^u)^{-1} \circ J_h^u)^k.$$

It follows that

$$\|(P_h|_{\mathfrak{X}^{us}})^{-1}\| \le \max\left\{\|(P_h|_{\mathfrak{X}^s})^{-1}\|, \|(P_h|_{\mathfrak{X}^u})^{-1}\|\right\} \le \frac{1}{1 - (1+\lambda)/2} = \frac{2}{1-\lambda}.$$

By (2.15) we also have

$$\|(P_h|_{\mathfrak{X}^c})^{-1}\| \le \|J_h\| \le 2.$$

So we obtain

$$\|P_h^{-1}\|_1 \le \max\left\{\|(P_h|_{\mathfrak{X}^{us}})^{-1}\|, \|(P_h|_{\mathfrak{X}^c})^{-1}\|\right\} \le \frac{2}{1-\lambda}.$$

This is what we need. ◻

Sublemma 2.3. *For any $0 < \varepsilon \le \rho$, there exists constant $C(\varepsilon) > 0$ such that for any $w, w' \in \mathfrak{B}(\varepsilon)$,*

$$\|\eta(w') - \eta(w)\| \le C(\varepsilon)(\|w' - w\|).$$

Moreover, $C(\varepsilon)$ can be chosen in such a way that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Let $\beta_x : T_{f^{-1}(x)}M \longrightarrow T_xM$ be the map defined by $\beta_x(\xi) = \exp_x^{-1} \circ f \circ \exp_{f^{-1}(x)} \xi$. Therefore $\beta(w)(f^{-1}(x)) = \beta_x(w(f^{-1}(x)))$ for any $w \in \mathfrak{X}$. Then by (2.6), for any $w, w' \in \mathfrak{X}$ with $\|w\|, \|w'\| < \varepsilon$,

$$\begin{aligned}
&\|\eta(w')(x) - \eta(w)(x)\| \\
=&\|\beta_x(w'(f^{-1}(x))) - \beta_x(w(f^{-1}(x))) - d_0\beta_x(w'(f^{-1}(x)) - w(f^{-1}(x)))\| \\
=&\Big\|\int_0^1 d_{w(f^{-1}(x)) + t(w'(f^{-1}(x)) - w(f^{-1}(x)))}\beta_x(w'(f^{-1}(x)) - w(f^{-1}(x)))dt \\
&- d_0\beta_x(w'(f^{-1}(x)) - w(f^{-1}(x)))\Big\| \\
\le& \sup_{w^* \in I_x} \|d_{w^*}\beta_x - d_0\beta_x\| \cdot \|w'(f^{-1}(x)) - w(f^{-1}(x))\|,
\end{aligned}$$

where $I_x = \{w(f^{-1}(x)) + t(w'(f^{-1}(x)) - w(f^{-1}(x))) : t \in [0,1]\}$. Since $d_{w^*}\beta_x$ is continuous with w^* and the continuity is uniform with respect to x, we can take

$$C(\varepsilon) = \sup\left\{\|d_{w(f^{-1}(x))}\beta_x - d_0\beta_x\| :\ w \in \mathfrak{B}_1(\varepsilon), x \in M\right\}.$$

Now the results of the lemma are clear. ◻

Sublemma 2.4. *For any h with $d(h, \mathrm{id}_M) \le \rho$, there exists a constant $K = K(h) > 0$ such that for any $w, w' \in \mathfrak{B}(\varepsilon)$,*

$$\|\theta_h(w') - \theta_h(w)\| \le K(h)\|w' - w\|.$$

Moreover, $K(h)$ can be chosen in such a way that $K(h) \to 0$ as $d(h, \mathrm{id}_M) \to 0$.

Proof. Since the map θ is C^∞ with respect to w and h, we can use the same method in the proof of the previous lemma to get the inequality.

Note that if $h = \mathrm{id}_M$, then the partial derivative of θ with respect to w is zero. So we get $K(h) \to 0$ as $d(h, \mathrm{id}_M) \to 0$. □

2.2 The center foliation $\mathcal{W}_f^c$ is C^1

2.2.1 The general case

Proof of Theorem B. The proof is similar to that of Theorem A.

To find π satisfying (1.5) and the conditions in (1.2) of this theorem, we shall try to solve the equation

$$\pi \circ g(x) = \tau^{(2)}_{g(x)} \circ f \circ \pi(x) \tag{2.19}$$

for unknown π. Putting $\pi(x) = \exp_x(v(x))$ with $v \in \mathfrak{B}^{us}(\rho)$, we see that (2.19) is equivalent to

$$v(x) = \exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \pi(v(g^{-1}(x))). \tag{2.20}$$

Define an operator $\beta : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ and a linear operator $A : \mathfrak{B}^{us}(\rho) \to \mathfrak{X}^{us}$ by

$$(\beta(v))(x) = \exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \pi(v(g^{-1}(x))), \tag{2.21}$$

$$(Av)(x) = (A^s_{g^{-1}(x)} + A^u_{g^{-1}(x)})v(g^{-1}(x)), \tag{2.22}$$

where

$$A^s_{g^{-1}(x)} = \Pi^s_x \circ d_0(\exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{g^{-1}(x)}) \circ \Pi^s_{g^{-1}(x)}$$

and

$$A^u_{g^{-1}(x)} = \Pi^u_x \circ d_0(\exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{g^{-1}(x)}) \circ \Pi^u_{g^{-1}(x)}.$$

Let $\eta = \beta - A$. By (2.21) and (2.22), (2.20) is equivalent to

$$v = Av + \eta(v),$$

further, is equivalent to

$$v - Av = \eta(v).$$

Define a linear operator P from a neighborhood of $0 \in \mathfrak{X}^{us}$ to $\mathfrak{X}^{us}$ by

$$Pv = (\mathrm{id}_{\mathfrak{X}^{us}} - A)v \tag{2.23}$$

for $v \in \mathfrak{X}^{us}$.

Define an operator Φ from a neighborhood of $0 \in \mathfrak{X}^{us}$ to $\mathfrak{X}^{us}$ by

$$\Phi(v) = P^{-1}\eta(v).$$

Hence, the equation (2.19) is equivalent to

$$\Phi(v) = v, \tag{2.24}$$

namely, v is a fixed point of Φ.

The remaining work is to show that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon)$ such that for any homeomorphism g with $d(g, f) \le \delta$, $\Phi : \mathfrak{B}(\varepsilon) \to \mathfrak{B}(\varepsilon)$ is a contracting map, and therefore has a fixed point in $\mathfrak{B}(\varepsilon)$. Hence, (2.19) has a unique solution. To this end we only need to modify the proof of Lemma 2.1 to a easer version since in this case we need not to concern with the center direction. We leave the details to the reader. □

2.2.2 $\mathcal{W}_f^c$ is of one dimensional

Proof of Theorem B′. The proof is also similar to that of Theorem A.

To find π and $\tilde{\tau}$ satisfying (1.6) and the conditions in (1.2) of this theorem, we shall try to solve the equation

$$\pi \circ g(x) = \tau^{(3)}_{f(x)} \circ f \circ \pi(x) \tag{2.25}$$

for unknown $\tilde{\tau}$ and π. Putting $h = g \circ f^{-1}$ and $\pi(x) = \exp_x(v(x))$ with $v \in \mathfrak{X}^{us}$, we see that (2.25) is equivalent to

$$\exp_x^{-1} \circ \exp_{h(x)}(v(h(x))) = \exp_x^{-1} \circ \varphi^{\tilde{\tau}(x)} \circ f \circ \exp_{f^{-1}(x)}\big(v(f^{-1}(x))\big). \tag{2.26}$$

Define $\beta : \mathfrak{B}(\rho) \times \mathfrak{C}(\rho) \to \mathfrak{X}$, where $\mathfrak{C}(\rho) = \{\tilde{\tau} \in C^0(M) : \|\tilde{\tau}\| \le \rho\}$, by

$$\beta(\omega, \tilde{\tau})(x) = \exp_x^{-1} \circ \varphi^{\tilde{\tau}(x)} \circ f \circ \exp_{f^{-1}(x)}\big(\omega(f^{-1}(x))\big). \tag{2.27}$$

It is easy to see that

$$\big(d_{(0,0)}\beta(\omega, \tilde{\tau})\big)(x) = (F\omega)(x) + \tilde{\tau}(x) \cdot u(x),$$

where $F : \mathfrak{X} \to \mathfrak{X}$ is defined in (2.5) (also recall that in this case u is a unit center vector field). Let

$$\eta(\omega, \tilde{\tau})(x) = \beta\big(\omega, \tilde{\tau}\big)(x) - \big(d_{(0,0)}\beta(\omega, \tilde{\tau})\big)(x). \tag{2.28}$$

Then we can write

$$\exp_x^{-1} \circ \varphi^{\tilde{\tau}(x)} \circ f \circ \exp_{f^{-1}(x)}(v(f^{-1}(x))) = (Fv)(x) + \tilde{\tau}(x) \cdot u(x) + \eta(v, \tilde{\tau})(x). \tag{2.29}$$

By (2.9) and (2.29), (2.26) is equivalent to

$$J_h v + \theta_h(v) = Fv + \tilde{\tau} \cdot u + \eta(v, \tilde{\tau}).$$

where J_h is a linear operator defined in (2.8). Further, the equation is equivalent to

$$-J_h^{-1}(\tilde{\tau} \cdot u) + (\mathrm{id}_{\mathfrak{X}} - J_h^{-1}F)v = J_h^{-1}(\eta(v, \tilde{\tau}) - \theta_h(v)).$$

Similarly we define a linear map P_h by

$$P_h\omega = -J_h^{-1}(\tilde{\tau} \cdot u) + (\mathrm{id}_{\mathfrak{X}} - J_h^{-1}F)v \tag{2.30}$$

for $\omega = \tilde{\tau} \cdot u + v \in \mathfrak{X}$ with $v \in \mathfrak{B}^{us}(\rho)$ and $\tilde{\tau} \in \mathfrak{C}(\rho)$. Hence, the above equation becomes

$$P_h(\tilde{\tau} \cdot u + v) = J_h^{-1}(\eta(v, \tilde{\tau}) - \theta_h(v)).$$

Define a map Φ_h from a neighborhood of $0 \in \mathfrak{X}$ to $\mathfrak{X}$ by

$$\Phi_h(\tilde{\tau} \cdot u + v) = P_h^{-1} J_h^{-1}(\eta(v, \tilde{\tau}) - \theta_h(v)).$$

Hence, the equation (2.25) is equivalent to

$$\Phi_h(\tilde{\tau} \cdot u + v) = \tilde{\tau} \cdot u + v, \tag{2.31}$$

namely, $\tilde{\tau} \cdot u + v$ is a fixed point of Φ_h.

Also similar to what we have done in the proof of Theorem A, there exists $\varepsilon_0 \in (0, \rho)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon)$ such that for any homeomorphism h with $d(h, \mathrm{id}_M) \le \delta$, $\Phi_h : \mathfrak{B}_1(\varepsilon) \to \mathfrak{B}_1(\varepsilon)$ is a contracting map, and therefore has a fixed point in $\mathfrak{B}_1(\varepsilon)$. Hence, (2.25) has a unique solution. We leave the details to the reader. □

3 Structural quasi-stability

Proof of Theorem C. We only prove this theorem under the assumption of Theorem B. We shall find π and u using the similar strategy in the proof of Theorem B. Furthermore, in order to obtain a leaf conjugacy π we shall give some necessary modification in techniques.

Since the center foliation of f is C^1, hence from Theorem 5.10 of [12] and Section 6 of [6], we know that if a diffeomorphism g is sufficiently close to f in C^1 topology, then it is also partially hyperbolic, the corresponding splitting $E_g^s \oplus E_g^c \oplus E_g^u$ is near that of f and the center distribution E_g^c is integrable. Now choose C^1 bundle $\tilde{E}^s \oplus \tilde{E}^u$ sufficiently close to $E_g^s \oplus E_g^u$, and hence close to $E_f^s \oplus E_f^u$. We want to find a continuous center section $u \in \mathfrak{X}^c$ and a homeomorphism $\pi : M \longrightarrow M$ ε close to id_M such that (1.5) holds and

$$\exp_x^{-1}(\pi(x)) \in \tilde{E}_x^s \oplus \tilde{E}_x^u \tag{3.1}$$

for $x \in M$. Put $h = g \circ f^{-1}$ and $\pi(x) = \exp_x(v(x))$ for $v \in \tilde{\mathfrak{X}}^s \oplus \tilde{\mathfrak{X}}^u$, where $\tilde{\mathfrak{X}}^s$ and $\tilde{\mathfrak{X}}^u$ denote the spaces of continuous sections of $\tilde{E}^s$ and $\tilde{E}^u$ respectively. We see that (1.5) is equivalent to

$$\exp_x^{-1} \circ \exp_{h(x)}(v(h(x))) = \exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{f^{-1}(x)}\big(v(f^{-1}(x))\big). \tag{3.2}$$

Then we can write

$$\exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{f^{-1}(x)}(v(f^{-1}(x))) = (Fv)(x) + u(x) + \eta(w)(x), \tag{3.3}$$

where $\omega = u + v \in \mathfrak{X}$ with $u \in \mathfrak{X}^c$ and $v \in \tilde{\mathfrak{X}}^{us}$,

$$(Fv)(x) = \sum_{i=s,u} \tilde{\Pi}_x^i \circ d_{f^{-1}(x)} f \circ \tilde{\Pi}_{f^{-1}(x)}^i v(f^{-1}(x))$$

and

$$\begin{aligned} \eta(w)(x) \quad = \quad & \exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{f^{-1}(x)}\big(v(f^{-1}(x))\big) \\ & - d_0\big(\exp_x^{-1} \circ \tau_x^{(2)} \circ f \circ \exp_{f^{-1}(x)}\big) v(f^{-1}(x)) \\ & + \sum_{i=s,c,u,j=s,u,i\neq j} \tilde{\Pi}_x^i \circ d_{f^{-1}(x)} f \circ \tilde{\Pi}_{f^{-1}(x)}^j v(f^{-1}(x)), \end{aligned} \tag{3.4}$$

in which $\tilde{\Pi}^s_x$ is the projection from T_xM onto $\tilde{E}^s_x$ along $\tilde{E}^c_x \oplus \tilde{E}^u_x$, where $\tilde{\Pi}^c_x = \Pi^c_x$. $\tilde{\Pi}^c_x$ and $\tilde{\Pi}^u_x$ are defined in the similar manner. It is clear that $F(\tilde{\mathfrak{X}}^s) = \tilde{\mathfrak{X}}^s, F(\tilde{\mathfrak{X}}^u) = \tilde{\mathfrak{X}}^u$. Moreover, for any $\lambda < \tilde{\lambda} < 1$, we can choose g sufficiently close to f, $\tilde{E}^s \oplus \tilde{E}^u$ sufficiently close to $E^s_f \oplus E^u_f$ such that

$$\|F|_{\mathfrak{X}^s}\|, \ \ \|F^{-1}|_{\mathfrak{X}^u}\|^{-1} \leq \tilde{\lambda}.$$

Similar to (2.9), we have

$$\exp_x^{-1} \circ \exp_{h(x)}(v(h(x))) = (J_h v + \theta_h(v))(x), \tag{3.5}$$

where J_h and θ_h is redefined with respect to $\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}^s \oplus \tilde{\mathfrak{X}}^c \oplus \tilde{\mathfrak{X}}^u$. By (3.3) and (3.5), (3.2) is equivalent to

$$J_h v + \theta_h(v) = Fv + u + \eta(w).$$

From now, we can find π and u in a similar way as we have done in the proof of Theorem B. We omit the details.

In the following, we prove that π obtained above is a leaf conjugacy from $(g, \mathcal{W}^c_g)$ to $(f, \mathcal{W}^c_f)$. Since the bundle $\tilde{E}^s \oplus \tilde{E}^u$ is C^1, (3.1) implies that the restriction of π to each center leaf of g is one-to-one. If we can get that π sends center leaves of g to that of f then by the same arguments of Pesin in Lemma 5.11 of [12] we can conclude that π is a leaf conjugacy. Therefore, the remaining work is to prove that π sends center leaves of g to that of f.

Now we show that for any $x \in M$, $\pi(W^c_g(x)) \subset W^c_f(\pi x)$. It is enough to show that the set $\pi(W^c_g(x))$ is tangent to $E^c_f(\pi x)$ for any $x \in M$. Suppose not, then there exist a small number $c_1 > 0$ and a sequence of points $\{y_k\} \subset W^c_g(x)$ with $y_k \to x$ as $k \to \infty$ such that

$$d(\pi z'_k, \pi z_k) + d(\pi z_k, \pi y_k) \geq c_1 d(\pi x, \pi y_k), \tag{3.6}$$

where z'_k and z_k are the unique points such that $\pi z_k \in W^u_f(\pi y_k)$ and $\pi z'_k \in W^s_f(\pi z_k) \cap W^c_f(\pi x)$. By taking a subsequence we may assume that $d(\pi z_k, \pi y_k) \geq d(\pi z'_k, \pi z_k)$ for all $k > 0$, and the other case can be discussed similarly by using f^{-1}.

For each $k > 0$, there exists $n = n(k) > 0$ such that

$$d(g^i(x), g^i(y_k)) \leq (\mu'_g)^i d(x, y_k) \leq \varepsilon \qquad \forall\, 0 \leq i \leq n,$$

where μ'_g is the upper bounds of $\|Dg|_{E^c_g}\|$ given in the definition of partially hyperbolic diffeomorphism. We can see that $n(k) \sim -\log d(x, y_k)/\log \mu'_g$ if $\mu'_g > 1$, and we regard $n(k) = \infty$ if otherwise. Since $d(\pi, \mathrm{id}_M) < \varepsilon$, we have

$$d\big(\pi(g^i x), \pi(g^i y_k)\big) \leq 3\varepsilon \qquad \forall\, 0 \leq i \leq n. \tag{3.7}$$

Since the foliation $\mathcal{W}^c_f$ is smooth, for any $x \in M$ there is a coordinate chart U_x at x of size $r > 0$ such that the local leaves of the center foliation can be viewed as parallel disks. For any $z \in U_x$, denote such a local center disk passing through z by $B^c_f(z)$. We consider the coordinate charts $U_{\pi(g^i x)}$ at $\pi(g^i x)$ of size r, $i = 0, 1, \cdots, n$. We assume that $r > 0$ is small and the coordinates are taken in a way such that the metrics on the charts are close to the metric on the manifold. Also, we assume that ε and δ are small such that as $d(f, g) < \delta$, all the points $\pi(g^i w)$ and $f(\pi(g^{i-1} w))$ are in the chart $U_{\pi(g^i x)}$, where $w = x, y_k, z_k, z'_k$. Since τ is a motion along leaves, we have $B^c_f(\pi(gx)) = B^c_f(f(\pi x))$. So by the fact that $d(f(\pi z_k), f(\pi y_k)) \geq \mu_f d(\pi z_k, \pi y_k)$, we get

$$d(B^c_f(\pi(gz_k)), B^c_f(\pi(gy_k))) \geq \tilde{\mu} d(B^c_f(\pi z_k), B^c_f(\pi y_k))$$

for some $\max\{1, \mu'_g\} < \tilde{\mu} < \mu_f$. Inductively, we have

$$d\big(B^c_f(\pi(g^n z_k)), B^c_f(\pi(g^n y_k)\big) \geq \tilde{\mu}^n d\big(B^c_f(\pi z_k), B^c_f(\pi y_k)\big).$$

Since $\pi z'_k \in W^s_f(\pi z_k)$ and $B^c_f(\pi(g^n z'_k)) = B^c_f(\pi(g^n x)$, we have

$$d\big(B^c_f(\pi(g^i z_k)), B^c_f(\pi(g^i x))\big) \to 0 \text{ as } i \to \infty.$$

Hence

$$d\big(B^c_f(\pi(g^n x)), B^c_f(\pi(g^n y_k))\big) \geq \tilde{\mu}^n d\big(B^c_f(\pi z_k), B^c_f(\pi y_k)\big).$$

Since $\pi z_k \in W^u_f(\pi y_k)$, it is easy to see that $d\big(B^c_f(\pi z_k), B^c_f(\pi y_k)\big) \geq c_2 d(\pi z_k, \pi y_k)$ for some constant $c_2 > 0$ only depends on the system. Also by (3.6) and the fact that $d(\pi z_k, \pi y_k) \geq d(\pi z'_k, \pi z_k)$, we have $d(\pi z_k, \pi y_k) \geq 0.5 c_1 d(\pi x, \pi y_k)$. Since $y_k \in W^c_g(x)$ and the map $\pi(x)$ is along $\tilde{E}^u_x \oplus \tilde{E}^s_x$, which is a smooth tangent subbundle, we have $d(\pi x, \pi y_k) \geq c_3 d(x, y_k)$ for some $c_3 > 0$ independent of x and y_k. Therefore, we have

$$d\big(B^c_f(\pi(g^n x)), B^c_f(\pi(g^n y_k))\big) \geq C\tilde{\mu}^n d(x, y_k),$$

where C is a constant independent of x, y_k and n. Since $n = n(k)$ increases like $-\log d(x, y_k)/\log \mu'_g$ and $\min\{1, \mu'_g\} < \tilde{\mu}$, we have $\tilde{\mu}^n d(x, y_k) \to \infty$ as $d(x, y_k) \to 0$. This contrdicts to (3.7) which implies that $d\big(B^c_f(\pi(g^n x)), B^c_f(\pi(g^n y_k))\big)$ is bounded.

Replacing the center leaves by the center-stable leaves or the center-unstable leaves and using similar arguments in the above paragraph, we can prove that π sends $\mathcal{W}^{cs}_g$ and $\mathcal{W}^{cu}_g$ to $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$ respectively. $\square$

4 Applications in the entropy theory

In this section, we apply our results to continuity of entropy. It is well known that continuity properties of entropy are very delicate. Obviously, the topological entropy of Anosov diffeomorphisms is locally a constant since it is structurally stable. For partially hyperbolic systems, Hua, Saghin and Xia ([10]) proved that for the case that the unstable and stable foliations stably carry some unique nontrivial homologies, the topological entropy is locally constant if the center foliation is one dimensional, and continuous if the center foliation is two dimensional. Hua ([9]) showed that the topological entropy is continuous at the time one map of transitive Anosov flows. In this section, we will use our results on the structural quasi-stability to investigate continuity of entropy for some partially hyperbolic diffeomorphisms.

4.1 Time one map of Anosov flow

Proof of Theorem D. Let g be a diffeomorphism sufficiently close to f. By Theorem B′ and Theorem C, there exist a homeomorphism $\pi : M \to M$ with $d(\pi, \mathrm{id}_M)$ sufficiently small and a small

$\tilde{\tau} : M \to \mathbb{R}$ such that

$$\pi \circ g(x) = \varphi^{\tilde{\tau}\circ f(x)}(f \circ \pi(x)) \ \ \forall x \in M.$$

Now we can define a flow ψ by $\psi^t(x) = \pi^{-1}\varphi^t(\pi(x))$ for $x \in M$ and $t \in \mathbb{R}$. Obviously, φ and ψ are conjugate and

$$g(x) = \psi^{1+\tilde{\tau}\circ f(x)}(x) \tag{4.1}$$

for any $x \in M$. By Theorem B of [14], we have that

$$(1 + \min_{x\in M} \tilde{\tau}(x))h(\varphi) \leq h(\psi) \leq (1 + \max_{x\in M} \tilde{\tau}(x))h(\varphi), \tag{4.2}$$

where $h(\varphi)$ and $h(\psi)$ are the topological entropies of φ and ψ respectively. From Proposition 21 of [4], we have that for any $t \in \mathbb{R}$,

$$h(\varphi^t) = |t|h(\varphi^1) = |t|h(f) \quad \text{and} \quad h(\psi^t) = |t|h(\psi^1). \tag{4.3}$$

By (4.1),

$$h(\psi^{1+\min_{x\in M}\tilde{\tau}(x)}) \leq h(g) \leq h(\psi^{1+\max_{x\in M}\tilde{\tau}(x)}). \tag{4.4}$$

Therefore, by (4.2), (4.3) and (4.4), we have

$$(1 + \min_{x\in M} \tilde{\tau}(x))^2 h(f) \leq h(g) \leq (1 + \max_{x\in M} \tilde{\tau}(x))^2 h(f).$$

Note that $|\tilde{\tau}| \to 0$ as $g \to f$. Hence we conclude that the topological entropy function is continuous at f. □

4.2 Systems with almost parallel center foliation

For a smooth surface Σ, $y \in \Sigma$ and $r > 0$, we denote

$$\Sigma(y, \alpha) = \{z \in \Sigma : d(z, y) < r\}.$$

The volume growth rate of the unstable foliation of f is defined by

$$\chi^u(x, r) = \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}(f^n \mathcal{W}^u_f(x, r)),$$

and

$$\chi^u(f) = \sup_{x\in M} \chi^u(x, r).$$

(See [10].) Clearly, $\chi^u(f)$ is independent of r. $\chi^s(f)$ is defined similarly by using stable manifolds $\mathcal{W}^s_f$.

Proof of Theorem E. The first part of the theorem follows from Lemma 4.1 below. By Lemma 4.2 below, the volume growth satisfies $\chi^u(f) = \chi^u(g)$. So, following the same arguments in the proof of Theorem 1.1 in [10], we can obtain the second part of the theorem. □

Recall that θ^c is a holonomy map defined by sliding the center leaves. When we use the map, we will allow the domain to be a nonsmooth surface or even an arbitrary set. Also, we use θ^c_f and θ^c_g to denote the maps along the center leaves of f and g respectively.

Lemma 4.1. *Let f be a partially hyperbolic diffeomorphism as in Theorem B and g be a diffeomorphism that is C^1-close to f. If f has almost parallel center foliation, then so does g.*

Moreover, for any $x, y \in M$ with $y \in \mathcal{W}^c_f(\pi(x))$, and any smooth surfaces $\Sigma_f(x)$ and $\Sigma_g(y)$ with $\Sigma_f(x) \perp \mathcal{W}^c_f$ and $\Sigma_g(y) \perp \mathcal{W}^c_g$, the map $\theta^c_g \circ \pi : \Sigma_f(x) \to \Sigma_g(y)$ is uniformly continuous with respect to x and y.

Proof. Take $\beta > 0$. Take $x \in M$ and $y \in \mathcal{W}^c_g(x)$. Let $\Sigma_g(x)$ and $\Sigma_g(y)$ denote the smooth surfaces with $\Sigma_g(x) \perp \mathcal{W}^c_g$ and $\Sigma_g(y) \perp \mathcal{W}^c_g$. Denote $\Sigma_g(x, \beta) = \{z \in \Sigma_g(x) : d(x, z) \le \alpha\}$. We need to show that there exists $\alpha > 0$ independent of x and y such that $\theta^c_g(\Sigma_g(x, \alpha)) \subset \Sigma_g(y, \beta)$.

Denote $\mathcal{R}_g(y, \beta) = \cup_{z \in \Sigma_g(y,\beta)} \mathcal{W}^c_g(z, \beta)$, where $\mathcal{W}^c_g(z, \beta)$ is the local center leaf of g at z of size β. Clearly, $\mathcal{R}_g(y, \beta)$ contains a ball about y of radius β. Since π is a homeomorphism, π^{-1} is uniformly continuous on M. So there is $\beta' > 0$ independent of y such that $\pi\big(\mathcal{R}_g(y, \beta)\big)$ contains a ball of radius β' about $\pi(y)$. In particular, $\pi(\mathcal{R}_g(y, \beta)) \supset \Sigma_f(\pi(y), \beta')$, where $\Sigma_f(\pi(y), \beta')$ is the part of a smooth surface $\Sigma_f(\pi(y))$ that is contained in a ball of radius β', and $\Sigma_f(\pi(y)) \perp \mathcal{W}^c_f$.

Note that by Theorem C, $\pi(\mathcal{W}^c_g(y)) = \mathcal{W}^c_f(\pi(y))$. Since f has almost parallel center foliation, there is $\alpha' > 0$ independent of $\pi(y)$ such that $\theta^c_f(\Sigma_f(\pi(x), \alpha')) \subset \Sigma_f(\pi(y), \beta')$.

Consider the set $\mathcal{R}_f(\pi(x), \alpha') = \cup_{z \in \Sigma_f(\pi(x),\beta')} \mathcal{W}^c_f(z, \alpha')$. It contains a ball of radius α' about $\pi(x)$. Since π is uniformly continuous, there exists $\alpha > 0$ only depending on α', such that $\pi^{-1}\big(\mathcal{R}_f(\pi(x), \alpha')\big)$ contains a ball of radius α about x. In particular, it contains $\Sigma_g(x, \alpha)$.

Now it is easy to check that $\theta^c_g\big(\Sigma_g(x, \alpha)\big) \subset \Sigma_g(y, \beta)$.

The proof of the second part of the lemma can be obtained in a similar way. $\square$

Lemma 4.2. *Let f and g be as in Lemma 4.1. Then we have*

$$\chi^u(g) = \chi^u(f).$$

Proof. Take $x \in M$ and $r > 0$.

By the last lemma we know that there exists $r^* \ge r' > 0$ such that

$$\mathcal{W}^u_f(\pi(x), r') \subset (\theta^c \circ \pi)(\mathcal{W}^u_g(x, r)) \subset \mathcal{W}^u_f(\pi(x), r^*), \tag{4.5}$$

where θ^c is the holonomy map into $\mathcal{W}^u_f(\pi(x), r^*)$.

Define $\psi_n = \theta^c_n \circ \pi$, where θ^c_n is the holonomy map into $f^n \mathcal{W}^u_f(\pi(x), r^*)$. It is easy to check that (4.5) implies

$$f^n \mathcal{W}^u_f(\pi(x), r') \subset \psi_n(g^n \mathcal{W}^u_g(x, r)) \subset f^n \mathcal{W}^u_f(\pi(x), r^*)$$

for any $n > 0$. Moreover, by the second part of Lemma 4.1, we know that for any $\alpha > 0$, there exists $\beta > 0$ such that for any $x \in M$, $n > 0$, and $y \in g^n \mathcal{W}^u_g(x, r)$,

$$\psi_n \mathcal{W}^u_g(y, \beta) \subset \mathcal{W}^u_f(\psi_n(y), \alpha).$$

The inclusions mean that Condition (b) of Sublemma 4.3 below is satisfied with $\psi_n = \psi$, $W = g^n \mathcal{W}^u_g(x, r)$ and $W' = f^n \mathcal{W}^u_f(\pi(x), r')$ for all $n \ge 0$. Since $\mathcal{W}^u_f$ and $\mathcal{W}^u_g$ are smooth submanifolds with bounded curvature, Condition (a) of the sublemma holds.

Now we apply Sublemma 4.3 to get that there is $C > 0$ independent of n such that for all $n \geq 0$,

$$\mathrm{Vol}(f^n \mathcal{W}^u_f(x, r')) \leq C\, \mathrm{Vol}(g^n \mathcal{W}^u_g(x, r)).$$

Since this is true for any x, we get $\chi^u(f) \leq \chi^u(g)$.

We can apply similar arguments, by using the inverse of π and the fact that g has almost parallel center foliation, to get $\chi^u(g) \leq \chi^u(f)$. $\square$

Sublemma 4.3. *Let $W, W', W^* \subset M$ be k dimensional manifolds with $W' \subset W^*$ and $\psi : W \to W^*$ be a one to one map such that $W' \subset \psi(W)$. Suppose that for all $n \geq 0$,*

(a) *there are constants $\overline{C}$ and $\underline{C}$ such that for any small $\alpha > 0$, $y' \in W'$ and $y \in W$,*

$$\mathrm{Vol}\, W'(y', \alpha) \leq \overline{C}\alpha^k, \qquad \underline{C}\alpha^k \leq \mathrm{Vol}\, W(y, \alpha);$$

(b) *for any $\alpha > 0$, there is a constant $\beta > 0$ such that for any $y \in W$ with $\psi(y) \in W'$,*

$$\psi(W(y, \beta)) \subset W'(\psi(y), \alpha).$$

Then there exists $C > 0$, only depending on $\underline{C}$, $\overline{C}$, α and β, such that

$$\mathrm{Vol}(W') \leq C\, \mathrm{Vol}(W).$$

Proof. Fix $\alpha > 0$. Take a 2α separated set $y'_1, \cdots, y'_{\ell_n} \in W'$, that is, $d(y'_i, y'_j) \geq 2\alpha$ for any $1 \leq i, j \leq \ell_n$. We also require that the set has maximal cardinality. Hence, $\{B_{W'}(y'_i, 2\alpha)\}$ form a cover of W'. So by part (a) we have

$$\mathrm{Vol}(W') \leq \ell_n \cdot \overline{C}(2\alpha)^k.$$

Take $\beta > 0$ as in Condition (b) of the sublemma. Since the balls in $\{W'(y'_i, \alpha)\}$ are pairwise disjoint, and $\psi W(y_i, \beta) \subset B_{W'}(y'_i, \alpha)$, where $y_i = \psi^{-1} y'_i \in W$, we see that $\{W(y_i, \beta)\}$ are pairwise disjoint. Hence,

$$\mathrm{Vol}(W) \geq \ell_n \cdot \underline{C}\beta^k.$$

So we have

$$\mathrm{Vol}(W') \leq C\, \mathrm{Vol}(W),$$

where $C = \overline{C}(2\alpha)^k / \underline{C}\beta^k$. $\square$

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