Name: ______________________________ PID: ____________________

1. Print your full name and your PID on your exam. Then finish reading these instructions and **sign the bottom of the page.**
2. Books, lecture notes, calculators or crib sheets (pre-compiled lists of formulas or other information) are not allowed.
3. All electronic equipment (such as cell phones, mp3 players, etc) must be turned off and stored away during the exam time.
4. Unless otherwise indicated, **SHOW ALL YOUR WORK**. If no work is shown, no partial credit can be awarded. Your work and answer need to be accurate and relevant to receive points.
5. You will be given exactly 50 minutes for this exam.
6. **Any student not following the above instructions nor behaving according to the above instructions during the exam may have their exam confiscated and points deducted.**

I have read and fully understood all of the above instructions:

Signature: ______________________________

Problem	1	2	3	4	5	6	7	TOTAL
Points								
Maximum	10	16	26	28	20	5	5	110

1. (10 points)
Find the center and radius of the sphere given by the following equation

$$x^2 + y^2 + z^2 = x + y + z$$

$$(x^2 - x) + (y^2 - y) + (z^2 - z) = 0$$

$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4} + \left(z^2 - z + \frac{1}{4}\right) - \frac{1}{4} =$$

$$= \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 - \frac{3}{4} = 0$$

So center is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $r = \sqrt{\frac{3}{4}}$.

2. (16=8+8 points)
The position vector of a particle moving in space is given by

$$\mathbf{r}(t) = (5\cos t)\mathbf{i} + (5\sin t)\mathbf{j} + (2t)\mathbf{k}$$

- (1)Find the velocity vector $\mathbf{v}(t)$ of the particle.
- (2)Find the length of the particle's path from $t = 11$ to $t = 2014$.

1) $V(t) = r'(t) = -5\sin t \cdot i + 5\cos t \cdot j + 2k$

2) $$\int_{11}^{2014} |r'(t)| dt = \int_{11}^{2014} \sqrt{25\sin^2 t + 25\cos^2 t + 4}\, dt =$$

$$= \int_{11}^{2014} \sqrt{29}\, dt = \sqrt{29}(2014 - 11) = 2003\sqrt{29}$$

3. (26=6+6+6+8 points) Let $\vec{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\vec{v} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

- (1)Find the dot product $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = \langle 1,2,3 \rangle \cdot \langle 3,2,1 \rangle = 3+4+3 = 10$$

- (2)Find the cross product $\vec{u} \times \vec{v}$.

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = i(2-6) - j(1-9) + k(2-6) =$$

$$= -4i + 8j - 4k$$

6

- (3)Find the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

$$\text{Area} = |u \times v| = \sqrt{16+64+16} = \sqrt{64+32} = \sqrt{96}$$

- (4)Find the projection vector $\text{Proj}_{\vec{v}}\vec{u}$ of $\vec{u}$ on $\vec{v}$.

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|v|^2}\right) \cdot v = \frac{10}{\left(\sqrt{1^2+2^2+3^2}\right)^2} \cdot \langle 3,2,1 \rangle =$$

$$= \frac{10}{1+4+9} \langle 3,2,1 \rangle = \frac{10}{14} \langle 3,2,1 \rangle = \frac{5}{7} \langle 3,2,1 \rangle = \left\langle \frac{15}{7}, \frac{10}{7}, \frac{5}{7} \right\rangle$$

4. (28=12+8+8 points)

- (1)Find an equation for the plane which contains the points $A(0,0,1)$, $B(1,0,0)$ and $C(0,0,2)$.

$$\overrightarrow{AB} = \langle 1, 0, -1 \rangle$$

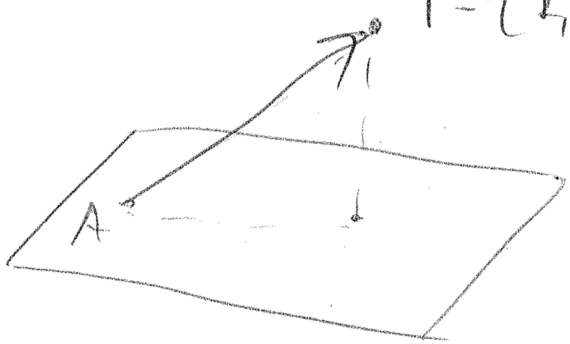
$$\overrightarrow{AC} = \langle 0, 0, 1 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = i(0) - j(1) + k(0) = -j = \langle 0, -1, 0 \rangle$$

Therefore $0(x-0) - 1(y-0) + 0(z-1) = 0$

$$-y = 0 \quad \text{or} \quad y = 0$$

- (2)Find the distance from the point $(1,1,1)$ to the plane in part (1).



$P = (1,1,1)$ $A = (0,0,0)$

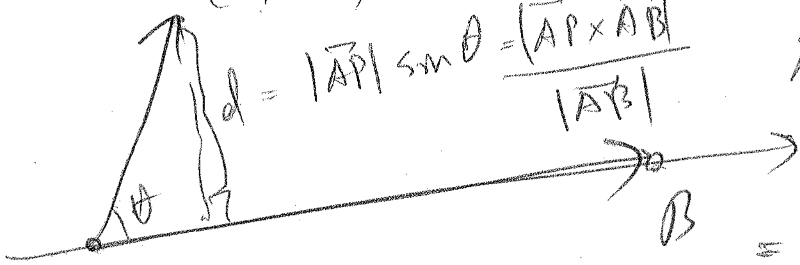
$\overrightarrow{AP} = \langle 1, 1, 1 \rangle$

$\vec{n} = \langle 0, -1, 0 \rangle$

distance $= |\mathrm{Proj}_{\vec{n}} \overrightarrow{AP}| = \dfrac{|\overrightarrow{AP} \cdot \vec{n}|}{|\vec{n}|} = \dfrac{|-1|}{1} = 1$

- (3)Find the distance from the point $(0,1,1)$ to the line determined by $A(0,0,1)$ and $B(1,0,0)$.

$P\ (0,1,1)$



$d = |\vec{AP}| \sin\theta = \dfrac{|\vec{AP} \times \vec{AB}|}{|\vec{AB}|}$

$A = (0,0,1)$

$d = \dfrac{\sqrt{1^2+1^2}}{\sqrt{1^2+1^2}} = 1$

$\vec{AB} = \langle 1, 0, -1 \rangle$

$\vec{AP} = \langle 0, 1, 0 \rangle$

$\vec{AP} \times \vec{AB} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} =$

$= i(1) - j(0) + k =$

$= \langle 1, 0, 1 \rangle$

5. (20=6+14) A bird is flying straightly from $(0,0,0)$ to $(2,0,2)$ at a constant speed $2 units/sec$.

- (1)How long does it take for the bird to fly from $(0,0,0)$ to $(2,0,2)$?

$\text{time} = \dfrac{\text{distance}}{\text{speed}} = \dfrac{\sqrt{4+0+4}}{2} = \dfrac{\sqrt{8}}{2}$

- (2)Find the parametric equation of the path with parameter t as time in *sec*.

$r(t) = \langle 0,0,0 \rangle + t \cdot \langle 2,0,2 \rangle \cdot \lambda$

$|r'(t)| = \lambda |\langle 2,0,2 \rangle| = \lambda \cdot \sqrt{8} = 2$

$\lambda = \dfrac{2}{\sqrt{8}}$ Therefore $r(t) = t \cdot \langle 2,0,2 \rangle \cdot \dfrac{2}{\sqrt{8}} =$

$= \left\langle \dfrac{4t}{\sqrt{8}}, 0, \dfrac{4t}{\sqrt{8}} \right\rangle$

6. (5 points, extra credit; no partial credit for this problem)
   Let $r(t) = \langle e^{t^2}, e^{-t}, \ln(1+t)\rangle$ where $t > -1$. A plane which passes through the point $r(t_0)$ and which is perpendicular to the tangent vector of the curve $r(t)$ at time t_0 is called a **normal plane** at point $r(t_0)$. Find the equation of a normal plane at point $r(0)$.

$r'(t) = \langle 2te^{t^2}, -e^{-t}, \frac{1}{1+t}\rangle$, $r'(0) = \langle 0, -1, 1\rangle$

normal vector

$r(0) = \langle 1, 1, 0\rangle$ - point which belongs to a plane

$\cancel{r(1) = \langle 1, 1, 0\rangle}$

So $0(x-1) - 1(y-1) + 1\cdot(z-0) = 0$ or $\boxed{z - y = -1}$

7. (5 points, extra credit; no partial credit for this problem)
   Consider any two vectors $\vec{u}, \vec{v}$ in $3D$. Show that the following vectors $\vec{v}$ and $\vec{u} - \mathrm{Proj}_{\vec{v}}\vec{u}$ are perpendicular. (Hint: dot product)

$\vec{u} - \mathrm{Proj}_{\vec{v}}\vec{u} = \vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{|v|^2}\right)\cdot\vec{v}$

Therefore

$\vec{v}\cdot(\vec{u} - \mathrm{Proj}_{\vec{v}}\vec{u}) = \vec{v}\cdot\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)(\vec{v}\cdot\vec{v}) =$

$= \vec{v}\cdot\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{|v|^2}|v|^2\right) =$

$= \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} = 0$.