Name: ______________________________ PID: ______________

1. Print your full name and your PID on your exam. Then finish reading these instructions and **sign the bottom of the page**.
2. Books, lecture notes, calculators or crib sheets (pre-compiled lists of formulas or other information) are not allowed.
3. All electronic equipment (such as cell phones, mp3 players, etc) must be turned off and stored away during the exam time.
4. Unless otherwise indicated, **SHOW ALL YOUR WORK**. If no work is shown, no partial credit can be awarded. Your work and answer need to be accurate and relevant to receive points.
5. You will be given exactly 50 minutes for this exam.
6. **Any student not following the above instructions nor behaving according to the above instructions during the exam may have their exam confiscated and points deducted.**

I have read and fully understood all of the above instructions:

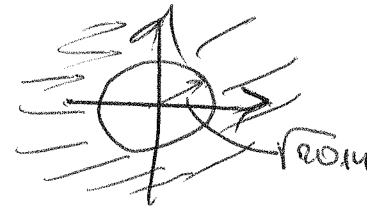
Signature: ______________________________

Problem	1	2	3	4	5	6(1)	6(2)	TOTAL
Points								
Maximum	14	16	14	16	24	16	10	110

1. (5+3+6=14 points)
   Let $f(x,y) = \ln(x^2 + y^2 - 2014)$.

   - (1) What is the domain of f? (describe it by an inequality and draw a picture)

$x^2+y^2-2014 > 0$



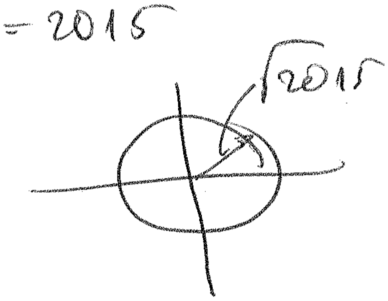
   - (2) What is the range of the function.

$(-\infty, \infty)$

   - (2) Sketch the level curve $f(x,y) = 0$.

$\ln(x^2+y^2-2014)=0$ or $x^2+y^2-2014=1$

$x^2+y^2=2015$



2. (16 points)
   Find the following limit or show it doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{(3x+y)^2 - xy}$$

$y = kx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+yx+y^2}{(3x+y)^2-xy} = \frac{x^2+kx^2+k^2x^2}{(3x+kx)^2-kx^2} =$$

$$= \frac{1+k+k^2}{(3+k)^2-k}$$

does not exist

3. (14 points) Find a number c such that the following function $f(x, y, t) = (\cos 10x + \sin 10y)e^{-ct}$ satisfies the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -100\cos 10x\, e^{-ct}, \quad \frac{\partial^2 f}{\partial y^2} = -100 \sin 10x\, e^{-ct}$$

$$\frac{\partial f}{\partial t} = -ce^{-ct} \cancel{\text{Hence}} (\cos 10x + \sin 10y)$$

Hence

$$-100\underline{(\cos 10x + \sin 10x)}e^{-ct} = -ce^{-ct}\underline{(\cos 10x + \sin 10y)}$$

$$-c = -100 \quad \text{or} \quad \boxed{c = 100}$$

4. (16 points) Let $w = \ln \frac{x^2e^y+z}{x^2e^y-z}$, $x = u$, $y = \ln v$ and $z = p^2$. Use **chain rule** to find $\frac{\partial w}{\partial p}$ at point $(u, v, p) = (1, 1, 2)$ (No credit for other methods.)

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\underbrace{\frac{\partial x}{\partial p}}_{0} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial p}}_{0} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p} =$$

$$= \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial p} = \left(\frac{1}{x^2e^y+z} + \frac{1}{x^2e^y-z}\right) \cdot 2p =$$

$$= \left(\frac{1}{u^2v+p^2} + \frac{1}{u^2v-p^2}\right) \cdot 2p = \frac{2u^2v \cdot 2p}{(u^2v+p^2)(u^2v-p^2)}$$

$$\frac{\partial w}{\partial p}(1,1,2) = \frac{8}{(1\cdot 1+4)(1\cdot 1-4)} = \frac{8}{5\cdot(-3)} = \frac{8}{-15} = -\frac{8}{15} \quad \boxed{-\frac{8}{15}}$$

5. (6+6+6+6=24 points) Let $f(x,y) = 3x^3 - 2y^3 + xy$ and $P(1,1)$.

- (1)Find ∇f at P.

$\nabla f = \langle 9x^2 + y, -6y^2 + x \rangle$.

$\nabla f(1,1) = \langle 9+1, -6+1 \rangle = \langle 10, -5 \rangle$

- (2)Find the derivative of f at P in the direction of $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$.

$\nabla f(1,1) \cdot \frac{\langle 1, 2 \rangle}{\sqrt{1^2+2^2}} = \langle 10, -5 \rangle \cdot \frac{\langle 1, 2 \rangle}{\sqrt{5}} = 0.$

- (3)Find the direction in which the function **decreases** most rapidly. (Write the direction as a unit vector.)

$-\frac{\nabla f}{|\nabla f|} = -\frac{\langle 10, -5 \rangle}{\sqrt{100+25}} = \frac{\langle -10, 5 \rangle}{\sqrt{125}}$

- (4)Find the equation of the tangent line to the level curve of $f(x,y)$ at point P.

$\cancel{f(1,1)} = \cancel{f_x = 9x^2}$ $\nabla f(1,1) = \langle 10, -5 \rangle$

$10(x-1) \cancel{+} 5(y-1) = 0$

$10x - 5y - 5 = 0$

$\boxed{10x - 5y = 5}$ or $\boxed{2x - y = 1}$

6. Let $f(x,y) = y^5 - x^5 - 5xy$

- (16 points)(1)View $f(x,y)$ as a function defined on the whole plane for the moment and find all critical points and label them as local minimum, local maximum or saddle points.
- (10 points, **extra credit**)(2) Find a function $f(x,y)$ which has infinitely many **saddle points**. Hint: Try to consider $f(x,y) = y \cdot g(x)$.

(a)

$\nabla f = 0, \quad \langle -5x^4 - 5y,\ 5y^4 - 5x \rangle = 0$

$\begin{cases} -5x^4 - 5y = 0 \\ 5y^4 - 5x = 0 \end{cases}$ or $\begin{cases} y = -x^4 \\ (-x^4)^4 - x = 0 \end{cases}$ or $x^{16} - x = 0$

$x(x^{15} - 1) = 0$ So two solutions

$(0,0)$ and $(1,-1)$

$x = 0$ then $y = 0$

or

$x = 1$ then $y = -1$.

First pick $(0,0)$

$f_{xx} f_{yy} - (f_{xy})^2 = -20x^3 \cdot 20y^3 - (-5)^2 =$

$= -400x^3y^3 - 25$

at $(0,0)$ $f_{xx} f_{yy} - (f_{xy})^2 = 0 - 25 < 0$ saddle point

at $(1,-1)$ $f_{xx} f_{yy} - (f_{xy})^2 = 400 - 25 = 375 > 0$ (min or max)

$f_{xx} = -20x^3$. and at $(1,-1)$ $f_{xx} = -20 \cdot 1 < 0$

So it is local max

(6)

$f(x,y) = y \cdot g(x)$ Then

$\nabla f = 0$

$\nabla f = \langle y \cdot g'(x), g(x) \rangle = 0$ So

$\begin{cases} y \cdot g'(x) = 0 \\ g(x) = 0 \end{cases}$ ok

$f_{xx} f_{yy} - (f_{xy})^2 =$

$= y \cdot g''(x) \cdot 0 - (g'(x))^2 =$

$= -(g'(x))^2 < 0$

we want it

Therefore if $g(x) = 0$

For infinitely many points but $g'(x) \neq 0$ They we are done. Therefore let's take

$g(x) = \sin x$, then $\sin x = 0$ for $x = \pi k$

k is ~~and~~ an integer

but $(\sin x)' = \cos x$ $\cos(\pi k) \neq 0$

So saddle points are $(\pi k, 0)$ where k is an integer.

and $\boxed{f(x,y) = y \cdot \sin x}$