Name: ______________________________ PID: ______________

1. Print your full name and your PID on your exam. Then finish reading these instructions and **sign the bottom of the page**.
2. Books, lecture notes, calculators or crib sheets (pre-compiled lists of formulas or other information) are not allowed.
3. All electronic equipment (such as cell phones, mp3 players, etc) must be turned off and stored away during the exam time.
4. Unless otherwise indicated, **SHOW ALL YOUR WORK**. If no work is shown, no partial credit can be awarded. Your work and answer need to be accurate and relevant to receive points.
5. You will be given exactly 50 minutes for this exam.
6. **Any student not following the above instructions nor behaving according to the above instructions during the exam may have their exam confiscated and points deducted.**

I have read and fully understood all of the above instructions:

Signature: ______________________________

Problem	1	2	3	4	5	6	7	TOTAL
Points								
Maximum	16	16	20	12	20	16	10	110

1. (10+6=16 points)

- a. Evaluate

$$\int_0^1 \int_1^e \frac{2y}{x}\,dx\,dy$$

$$\int_0^1 2y \ln x \Big|_1^e dy = \int_0^1 2y(\underbrace{\ln e}_{1} - \underbrace{\ln 1}_{0})\,dy = \int_0^1 2y\,dy = y^2\Big|_0^1 = 1$$

- b. Find the average value of the function $f(x,y) = \frac{2y}{x}$ over the domain R given as $1 \le x \le e$, $0 \le y \le 1$.

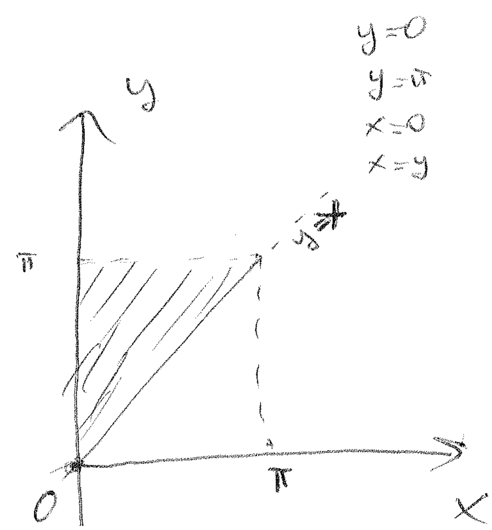
$$\text{Average} = \frac{1}{\underbrace{\int_0^1\int_1^e dx\,dy}_{=\text{Area}}} \int_0^1\int_1^e \frac{2y}{x}\,dx\,dy = \frac{1}{(e-1)} \cdot 1 = \frac{1}{e-1}$$

2. (16 points)
Evaluate

$$\int_0^\pi \int_0^y \frac{\sin x}{\pi - x}\,dx\,dy$$

$0 \le y \le \pi$

$0 \le x \le y$



$$\int_0^\pi \int_x^\pi \frac{\sin x}{\pi - x}\,dy\,dx =$$

$$= \int_0^\pi \frac{\sin x}{\pi - x} \cdot y\Big|_{y=x}^{y=\pi} dx = \int_0^\pi \frac{\sin x}{\pi - x} \cdot (\pi - x)\,dx = \int_0^\pi \sin x\,dx =$$

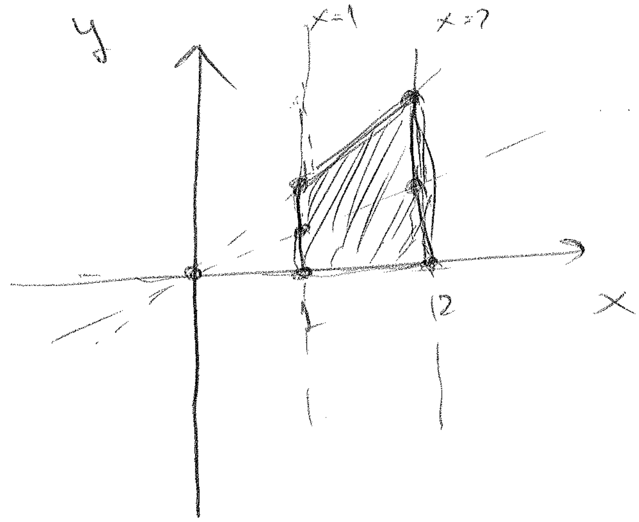
$$= -\cos x\Big|_{x=0}^{x=\pi} = -(-1-1) = 2$$

3. (20 points) Write the following double integrals in polar coordinates (**Don't evaluate them.**)

- a. $\iint_D (x^2+y^2)dA$ where the domain D is bounded by the straight lines $x=1$, $x=2$, $y=0$ and $y=x$.
- b. $\int_0^\infty \int_0^\infty e^{x^2+y^2} dA$

a)

$\boxed{x^2+y^2=r^2}$



$$\iint_D (x^2+y^2)dA = \int_0^{\frac{\pi}{4}}\int r^2 \cdot r\,dr\,d\theta$$

$0 \leq \theta \leq \frac{\pi}{4}$

$r\cos\theta = x = 1$

so $r = \frac{1}{\cos\theta}$ and

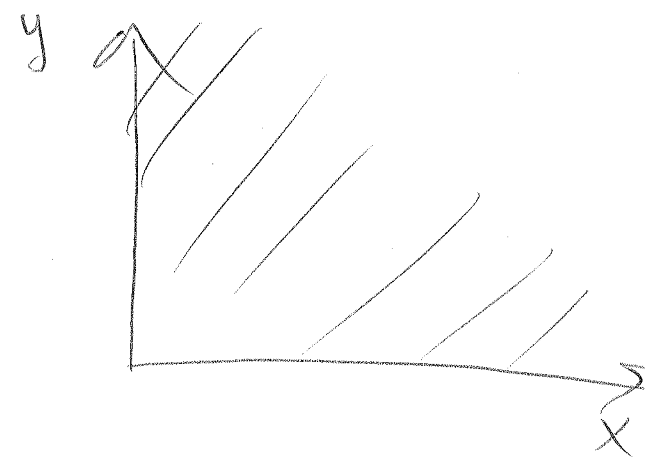
$r\cos\theta = x = 2$

$r = \frac{2}{\cos\theta}$

r

$$\boxed{\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} r^3\,dr\,d\theta}$$

b)



$0 \leq \theta \leq \frac{\pi}{2}$

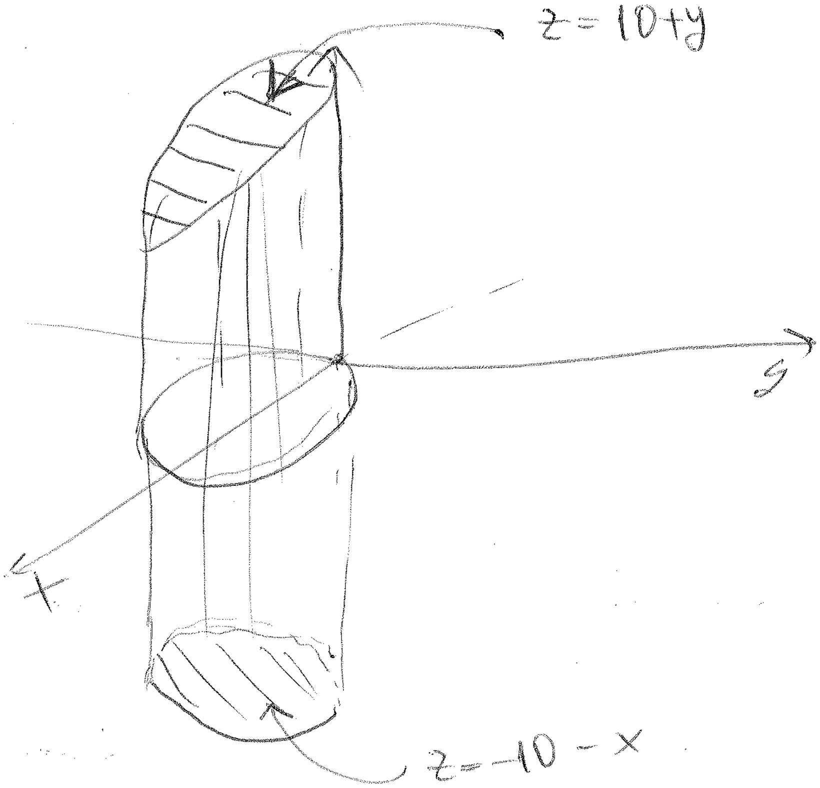
$0 \leq r \leq \infty$

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{r^2}\,r\,dr\,d\theta$$

4. (12 points) Write the following triple integral in cylindrical coordinate system:

$$\iiint_D \left(\ln(x^2+y^2) - z\right) dV.$$

where D is the cylindrical domain which has a circle in the base $(x-1)^2+y^2=1$ (in XY plane) and it is bounded by the **planes** $z=-10-x$ and $z=10+y$ (**Don't evaluate the integral.**)



$(x-1)^2+y^2=1$

$x^2-2x+1+y^2=1$

$x^2-2x+y^2=0$

$x=r\cos\theta$

$y=r\sin\theta$

$r^2-2r\cos\theta=0$

$r-2\cos\theta=0$

$r=2\cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

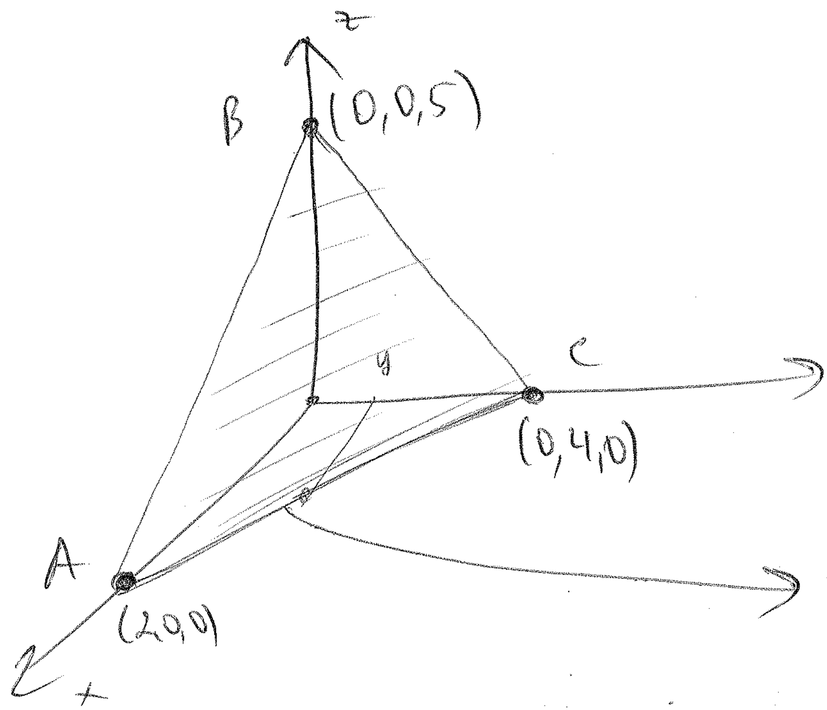
$0 \le r \le 2\cos\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_{-10-r\cos\theta}^{10+r\sin\theta} (\ln r^2 - z)\, d\cancel{z}\, r\, dr\, d\theta$$

So

$$\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta}\int_{-10-r\cos\theta}^{10+r\sin\theta} (\ln r^2 - z)\, dz \cdot r\, dr\, d\theta$$

5. (20 points) By using triple integral, find the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 5)$.



$20x + 10y = 40$ or
$2x + y = 4$

Lets find the equation of the plane

$\overrightarrow{AB} = \langle -2, 0, 5 \rangle$ $\quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -2 & 0 & 5 \\ -2 & 4 & 0 \end{vmatrix} =$

$\overrightarrow{AC} = \langle -2, 4, 0 \rangle$

$= i(-20) - j(10) + k(-8) = \langle -20, -10, -8 \rangle$

$-20(x-2) - 10y - 8z = 0$ $\quad 20x + 10y + 8z - 40 = 0$

$\boxed{20x + 10y + 8z = 40}$

$\boxed{10x + 5y + 4z = 20}$

So $\cancel{\int_0^4 \int_0^{\frac{4-y}{2}} \int_0^{\frac{40-20x-10y}{8}} dz\, dx\, dy}$ $\quad \cancel{\frac{1}{8} \int_0^4 \int_0^{\frac{4-y}{2}} (40 - 20x - 10y)\, dx\, dy} = \cancel{\frac{1}{8} \int_0^4 \left(40\left(\frac{4-y}{2}\right) - 10y\left(\frac{4-y}{2}\right) - 10\left(\frac{4-y}{2}\right)^2 \right) dy} = \cancel{\frac{10}{8} \int_0^4 \left(8 - 2y - \frac{y^2}{2}\right)}$

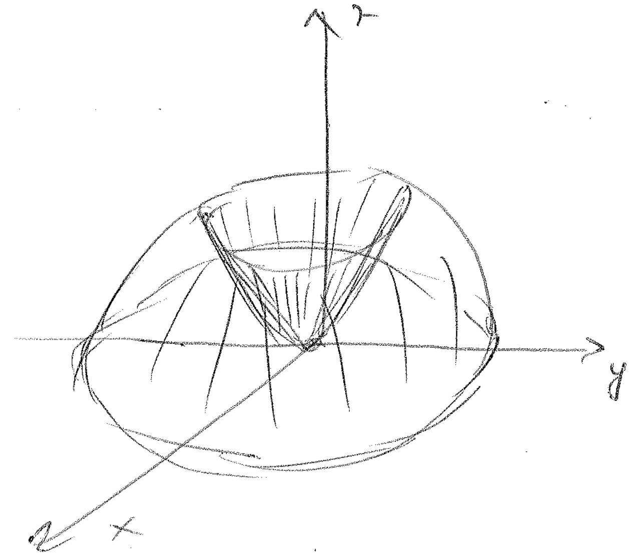
see the end of the page

6. (16 points) Consider half of a ball of radius 2 (i.e. $x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$). Let's remove the following cone $z^2 \geq x^2 + y^2$. What is the volume of the obtained domain? (You should use spherical coordinate system)

$0 \leq \theta \leq 2\pi$

$\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$

$0 \leq \rho \leq 2$



$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

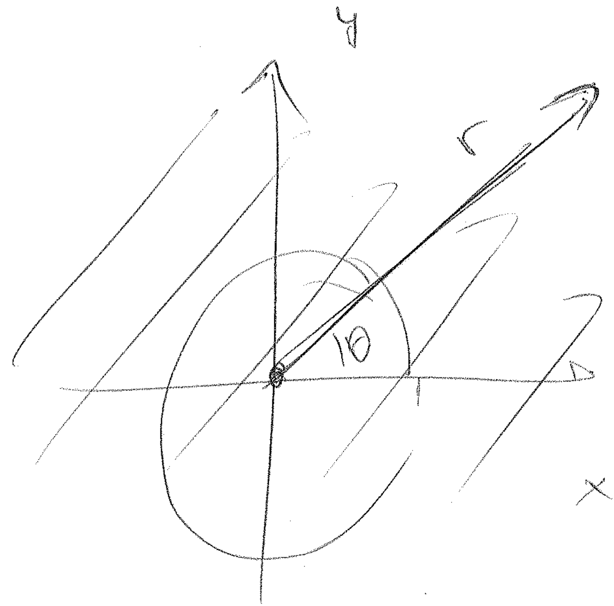
$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\rho^3}{3}\sin\varphi \Big|_{\rho=0}^{\rho=2} d\varphi \, d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{8}{3}\sin\varphi \, d\varphi \, d\theta =$$

$$= \frac{8}{3}\int_0^{2\pi} -\cos\varphi \Big|_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}} d\theta = \frac{8}{3}\int_0^{2\pi} -(\underbrace{\cos\tfrac{\pi}{2}}_{0} - \underbrace{\cos\tfrac{\pi}{4}}_{\frac{\sqrt{2}}{2}}) \, d\theta =$$

$$= \frac{8}{3}\int_0^{2\pi} \frac{\sqrt{2}}{2} \, d\theta = \frac{8}{3} \cdot \frac{\sqrt{2}}{2} \cdot 2\pi = \boxed{\frac{8\pi\sqrt{2}}{3}}$$

7. (**Extra credit** – 10 points) The following function $f(x) = e^{-x^2/2}$ plays crucial role in statistics and probability theory. It is known as Gaussian function and wikipedia says that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. You need to show this by using polar coordinate system and here is an important hint which you can use as a fact

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$$



$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy =$$

Polar coordinate system

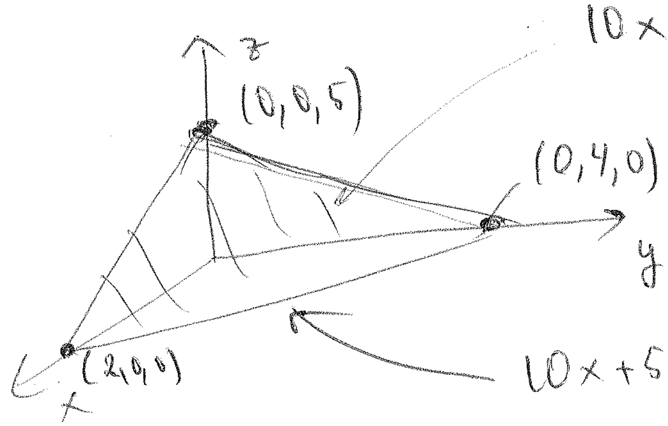
$$= 0 \le \theta \le 2\pi, \quad 0 \le r \le \infty \qquad =$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta = 2\pi \cdot \int_{0}^{\infty} e^{-r^2} r\,dr =$$

$$= 2\pi\left(-\frac{e^{-r^2}}{2}\right)\Bigg|_{r=0}^{r=\infty} = \pi \quad \text{So}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$

$10x+5y+4z=20$



$10x+5y=20$ or

$2x+y=4$

So $\displaystyle\int_0^4\int_0^{\frac{4-y}{2}}\int_0^{\frac{20-10x-5y}{4}} dz\,dx\,dy =$

$\displaystyle= \int_0^4\int_0^{\frac{4-y}{2}} \frac{20-10x-5y}{4}\,dx\,dy = \frac{5}{4}\int_0^4\int_0^{\frac{4-y}{2}}(4-2x-y)\,dx\,dy =$

$\displaystyle= \frac{5}{4}\int_0^4 4\cdot\left(\frac{4-y}{2}\right) - \left(\frac{4-y}{2}\right)^2 - y\left(\frac{4-y}{2}\right)dy = \frac{5}{4}\int_0^4 8-2y-\left(\frac{16-8y+y^2}{4}\right)-2y+\frac{y^2}{2}\,dy$

$\displaystyle= \frac{5}{4}\int_0^4 8-2y-\left(4-2y+\frac{y^2}{4}\right)-2y+\frac{y^2}{2}\,dy = \frac{5}{4}\int_0^4 4-2y+\frac{y^2}{4}\,dy =$

$\displaystyle= \frac{5}{4}\left(4y-y^2+\frac{y^3}{12}\right)\Bigg|_{y=0}^{y=4} = \frac{5}{4}\left(16-16+\frac{4^3}{12}\right) = \frac{5}{4}\cdot\frac{4^2}{3} = \boxed{\frac{20}{3}}$