Name: ______________________________ PID: ______________

1. Print your full name and your PID on your exam. Then finish reading these instructions and **sign the bottom of the page**.
2. Books, lecture notes, calculators or crib sheets (pre-compiled lists of formulas or other information) are not allowed.
3. All electronic equipment (such as cell phones, mp3 players, etc) must be turned off and stored away during the exam time.
4. Unless otherwise indicated, **SHOW ALL YOUR WORK**. If no work is shown, no partial credit can be awarded. Your work and answer need to be accurate and relevant to receive points.
5. You will be given exactly 50 minutes for this exam.
6. **Any student not following the above instructions nor behaving according to the above instructions during the exam may have their exam confiscated and points deducted.**

I have read and fully understood all of the above instructions:

Signature: ______________________

Problem	1	2	3	4	5	6	TOTAL
Points							
Maximum	20	20	25	15	20	10	110

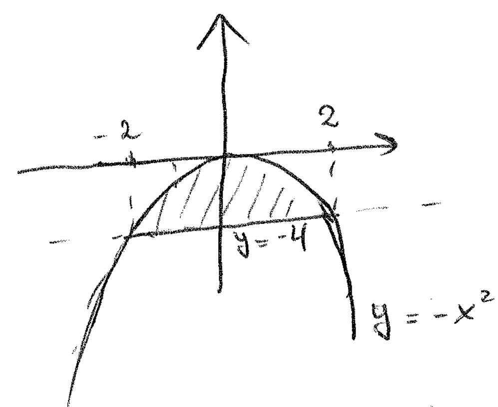
1. (20 points)
Let $\mathbf{F} = x^3y^3\mathbf{i} + y^3z^3\mathbf{j} + 6x^3z^3\mathbf{k}$. Find the work done by $\mathbf{F}$ for moving an object from $(0,0,0)$ to $(1,1,1)$ along the straight line segment connecting these two points.

$$r(t) = \vec{OA} + t\cdot\vec{AB} = \langle 0,0,0\rangle + t\langle 1,1,1\rangle = \langle t,t,t\rangle$$

$$\int_0^1 F(r(t))\cdot\frac{dr}{dt}\,dt = \int_0^1 \langle t^6, t^6, 6t^6\rangle\cdot\langle 1,1,1\rangle\,dt = \int_0^1 8t^6 = \boxed{\frac{8}{7}}$$

2. (20 points)
Let $\mathbf{F} = \overset{M}{(20x+30y)}\mathbf{i} + \overset{N}{(40x+50y)}\mathbf{j}$ be a planar vector field, and C be closed curve obtained by: $y = -x^2$ and $y = -4$ (oriented counterclockwise). Use Green's theorem to find:

- a. The circulation of $\mathbf{F}$ around C.
- b. The outward flux of $\mathbf{F}$ across C.



$$\text{circulation} = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy =$$

$$= \iint (40-30)\,dx\,dy = 10\iint dx\,dy = 10\iint dy\,dx$$

$$= 10\int_{-2}^{2}\int_{-x^2}^{x^2} dy\,dx = 10\int_{-2}^{2} 2x^2\,dx = 20\,\frac{x^3}{3}\Big|_{-2}^{2} = 20\left(\frac{8}{3}+\frac{8}{3}\right) = \frac{20\cdot 16}{3} = \boxed{\frac{320}{3}}$$

$$\text{Flux} = \iint\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dy\,dx = \iint (20+50)\,dx\,dy = 70\iint dx\,dy \quad \ominus$$

the same double integral

$$\ominus\ \frac{70\cdot 32}{3} = \frac{2240}{3}$$

f_x f_y f_z

3. (5+10+10=25 points) Given a vector field $\mathbf{F} = (4x + y + 3z)\mathbf{i} + (6y + x - 2z)\mathbf{j} + (2z - 2y + 3x)\mathbf{k}$.

- a.Use component test to show $\mathbf{F}$ is conservative.

$\frac{\partial}{\partial y}(4x+y+3x) = \frac{\partial}{\partial x}(6y+x-2z)$ ✓ $f_{xy} = f_{yx}$

$f_{xz} = f_{zx}$

$\frac{\partial}{\partial z}(4x+y+3x) = \frac{\partial}{\partial x}(2z-2y+3x)$ ✓ $f_{yz} = f_{zy}$

$\frac{\partial}{\partial z}(6y+x-2z) = \frac{\partial}{\partial y}(2z-2y+3x)$ ✓

- b. Find a potential function for $\mathbf{F}$.

$f_x = 4x+y+3z, \Rightarrow f = \int(4x+y+3z)\,dx = 2x^2+yx+3zx+C(y,z)$ (*)

$f_y = 6y+x-2z$ and (*) imply: $\left[2x^2+yx+3zx+C(y,z)\right]'_y = 6y+x-2z$

$\cancel{x} + C'_y(y,z) = 6y + x - 2z$ So $C'_y(y,z) = 6y-2z \Rightarrow C(y,z) = 3y^2-2yz+B(z)$

So $f = 2x^2+yx+3zX+3y^2-2yz+B(z)$. But $f_z = 2z-2y+3x$ and ~~z~~ imply

$\left(2x^2+yx+3zx+3y^2-2yz\right)+B(z)\Big]'_z = 2z-2y+3x$ So $3x-2y+B'(z) = 2z-2y+3x$

So $B'(z) = 2z\ \cancel{3x}$ Hence $B(z) = z^2$. Thus $\boxed{f = 2x^2+yx+3zx+3y^2-2yz+z^2}$

Yes

- c. Evaluate

$$\int_C \mathbf{F} \cdot dr$$

where C is any smooth curve from $(0, 0, 0)$ to $(1, 1, 1)$.

$$\int_C F\cdot dr = \int_{(0,0,0)}^{(1,1,1)} F\,dr = f(1,1,1) - f(0,0,0) = 2+1+3+3-2+1 = \boxed{8}$$

4. (5+10=15 points) Let S be the surface $z = (2xy)^{1/2}$ where $0 \le x \le 1$ and $0 \le x \le 1$

- a. Write down the formula for computing area of the surface S over the given domain (double integral with precise **limits**)
- b. Compute that integral.

$$(a)\ \iint d\sigma = \int_0^1\int_0^1 \sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy = \cancel{\int_0^1\int_0^1 \sqrt{1+\tfrac{1}{2}\left(\tfrac{2y}{x}\right)^{1/2}}}$$

$$(b)\quad f_x = \frac{1}{2}\left(\frac{2y}{x}\right)^{1/2},\quad f_y = \frac{1}{2}\left(\frac{2x}{y}\right)^{1/2}\quad \text{Hence}$$

$$\int_0^1\int_0^1 \sqrt{1+\frac{1}{4}\frac{2y}{x}+\frac{1}{4}\frac{2x}{y}}\,dx\,dy = \int_0^1\int_0^1 \sqrt{\frac{2xy+y^2+x^2}{2xy}}\,dx\,dy =$$

$$= \int_0^1\int_0^1 \sqrt{\frac{(x+y)^2}{2xy}}\,dx\,dy = \int_0^1\int_0^1 \frac{x+y}{\sqrt{2xy}}\,dx\,dy = \frac{1}{\sqrt{2}}\int_0^1\int_0^1 \left(\frac{x}{y}\right)^{1/2}+\left(\frac{y}{x}\right)^{1/2} dx\,dy =$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \left(\frac{x^{3/2}}{y^{1/2}}\cdot\frac{2}{3} + y^{1/2}x^{1/2}\cdot 2\right)\Bigg|_{x=0}^{x=1} dy =$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \left(\frac{2}{3}\cdot\frac{1}{y^{1/2}} + 2y^{1/2}\right)dy = \frac{1}{\sqrt{2}}\left(\frac{4}{3}\cdot y^{1/2} + 2\cdot\frac{2}{3}\cdot y^{3/2}\right)\Bigg|_{y=0}^{y=1} =$$

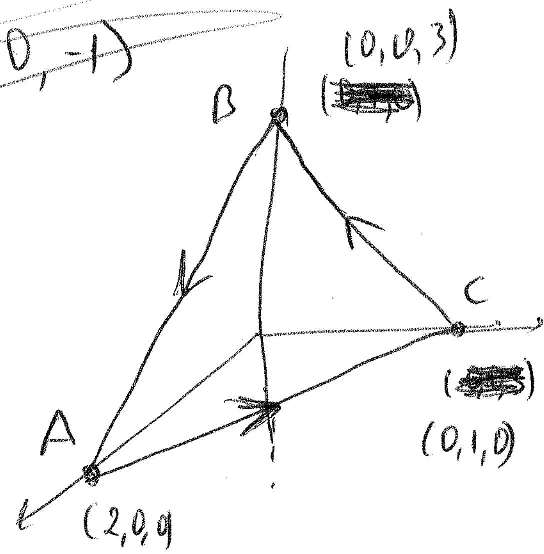
$$= \frac{1}{\sqrt{2}}\left(\frac{4}{3}+\frac{4}{3}\right) = \boxed{\frac{8}{3\sqrt{2}}}$$

See last page

5. (20 points) Find the circulation along the curve joining the points $A = (2, 0, 0)$, $B = (0, 0, 3)$ and $C = (0, 1, 0)$ (which is triangle) of the vector field $F(x, y, z) = \langle y, z^2, y^2\rangle$.

See ~~lecture~~ ~~back paper~~ Notes

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & y^2 \end{vmatrix} = i\left(2y-2z\right) - j\left(0\right) + k\left(-1\right) = \langle 2y-2z, 0, -1\rangle$$



$$\vec{AB} \times \vec{AC} = \langle -2, 1, 0\rangle \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ -2 & 0 & 3 \end{vmatrix} = i(3) - j(-6) + k(2) = \langle 3, 6, 2\rangle$$

Then ~~$\nabla \times F \cdot n$~~ unit will be

$$\frac{\langle 3, 6, 2\rangle}{\sqrt{9+36+4}} = \frac{\langle 3,6,2\rangle}{\sqrt{49}} = \frac{\langle 3,6,2\rangle}{7},$$

Hence $\nabla \times F \cdot n = \langle 2y-2z, 0, -1\rangle \cdot \frac{\langle 3,6,2\rangle}{7} =$

~~Hence~~ $= \boxed{\frac{6y-6z-2}{7}}$ ok

Equation of the plane is $3(x-2)+6(y-0)+2(z-0) = 3x-6+6y+2z = 0$

So z. Equation of the line will be $z=0$ or $3x-6+6y=0$ or $\boxed{x-2+2y=0}$

~~Hence~~ surface is $z = \frac{6-3x-6y}{2}$

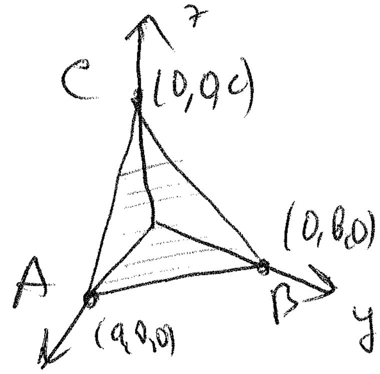
$$\text{So } \iint_D \nabla \times F \cdot n \, d\sigma = \iint_D \left(\frac{6y-6z-2}{7}\right) d\sigma = \iint_D \left(\frac{6y-6z-2}{7}\right)\sqrt{1+(f_x)^2+(f_y)^2}\, dx\, dy$$

shadow

See extra paper

6. (**Extra credit:** 10 points) Let **n** be the unit normal vector (with positive components) of the triangle Δ with vertices $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$ (where a, b, c are some given positive numbers) and let $F = \langle z, x, y\rangle$. Evaluate

$$\iint_{\Delta} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

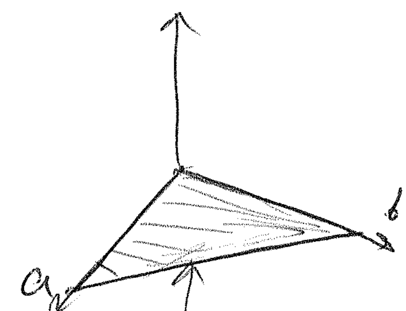


$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = i(1) - j(-1) + k(1) = \langle 1,1,1\rangle$$ good

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = i(bc) - j(-ac) + k(ab) = \langle bc, ac, ab\rangle$$

So $n = \dfrac{\langle bc, ac, ab\rangle}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}$, Hence $\nabla\times F \cdot n = \dfrac{bc+ac+ab}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}$ good.

$$\iint_{\Delta} \nabla\times F\cdot n\, d\sigma = \frac{bc+ac+ab}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} \iint_{\Delta} d\sigma$$



equation of the plane $bc(x-a) + ac\cdot y + ab\cdot z = 0$

$x\cdot bc + y\cdot ac + z\cdot ab = abc$

$z = c - \dfrac{xc}{a} - \dfrac{yc}{b}$ So $\sqrt{1+(f_x)^2+(f_y)^2} = \sqrt{1+\left(\frac{c}{a}\right)^2+\left(\frac{c}{b}\right)^2} = \sqrt{\dfrac{(ab)^2+(cb)^2+(ca)^2}{(ab)^2}} =$

$= \dfrac{1}{ab}\sqrt{(ab)^2+(cb)^2+(ca)^2}$. So

$$\iint_{\Delta} \nabla\times F\cdot n\, d\sigma = \frac{bc+ac+ab}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} \cdot \frac{1}{ab}\cdot\sqrt{(ab)^2+(cb)^2+(ca)^2}\ \overbrace{\underbrace{\iint dx\,dy}_{\text{shadow} \,=\, \frac{a\cdot b}{2}}}^{\text{area of}} = \boxed{\frac{bc+ac+ab}{2}}$$

$n^o = 5$

Solution:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & y^2 \end{vmatrix} = i(2y-2z) - j(0) + k(-1) =$$

$$= \langle 2y-2z, 0, -1 \rangle$$

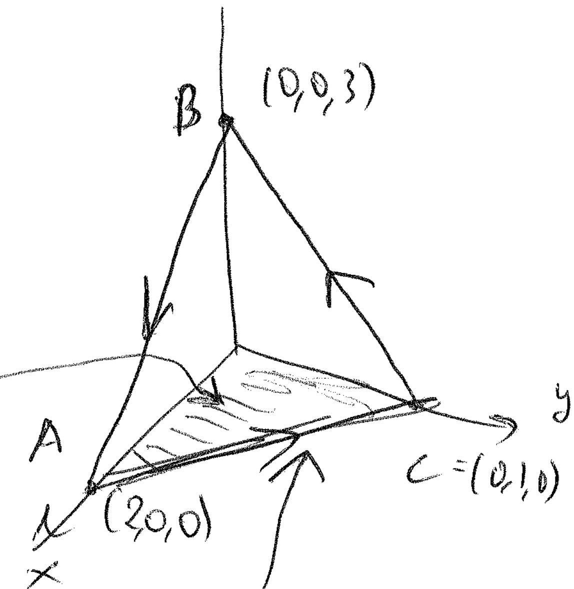
$\overline{AB} \times \overline{AC} =$

$$= \begin{vmatrix} i & j & k \\ -2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = i(-3) - j(6) + k(-2) =$$

$= \langle -3, -6, -2 \rangle$

change the sign

$$n = \frac{\langle 3,6,2 \rangle}{\sqrt{49}} = \frac{\langle 3,6,2\rangle}{7}$$



So $\nabla \times F \cdot n = \langle 2y-2z, 0, -1 \rangle \cdot \frac{\langle 3,6,2\rangle}{7} = \frac{6y-6z-2}{7}$

$3y - 6$

~~$d\sigma$~~ = Let's find equation of the plane $3(x-2) + 6\cdot y + 2\cdot z = 0$

$3x + 6y + 2z = 6$, so $z = 3 - 3y - \frac{3}{2}x$. equation of the line will be $z = 0$ or $3x + 6y = 6 \Rightarrow \boxed{x + 2y = 2}$

$d\sigma = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dx\, dy$

$= \sqrt{1 + 9 + \frac{9}{4}} = \frac{7}{2}$

$$\iint_{\Delta} \nabla \times F \cdot n \boxed{d\sigma} = \iint_{\Delta} \left(\frac{6y-6z-2}{7}\right) \cdot \frac{7}{2}\, dx\, dy = \iint_{\Delta} (3y - 3z - 1)\, dx\, dy$$

$$\int_0^1 \int_0^{2-2y} 3y - 1 - 3\left(3 - 3y - \frac{3}{2}x\right) dx\, dy = \int_0^1 \int_0^{2-2y} \left(12y + \frac{9}{2}x - 10\right) dx\, dy = \int_0^1 12y(2-2y) + \frac{9}{4}(2-2y)^2 - 10(2-2y)\, dy =$$

$$= \int_0^1 24y - 24y^2 + 9(1 - 2y + y^2) - 20 + 20y\, dy = \int_0^1 -15y^2 + 26y - 11\, dy = -15\frac{y^3}{3} + 13y^2 - 11y \Big|_{y=0}^{1} = \boxed{-3}$$

$-5 + 13 - 11$