LECTURE 10

Section 14.5

Gradient

If you have a function of several variables then what is analog of a derivative?

Def: If $f(x,y,z)$ the gradient is denoted as ∇f and is defined to be the following vector

$$\nabla f = \langle f'_x , f'_y , f'_z \rangle$$

Example:

$$f(x,y,z) = x^2 + y^3 + z^4$$

Find ∇f - ?

Solution:

$$\nabla f = \langle f_x, f_y, f_z \rangle =$$
$$= \langle 2x, 3y^2, 4z^3 \rangle.$$

Q1: Find $\nabla f(1,2,3)$ if

$f = xyz$.

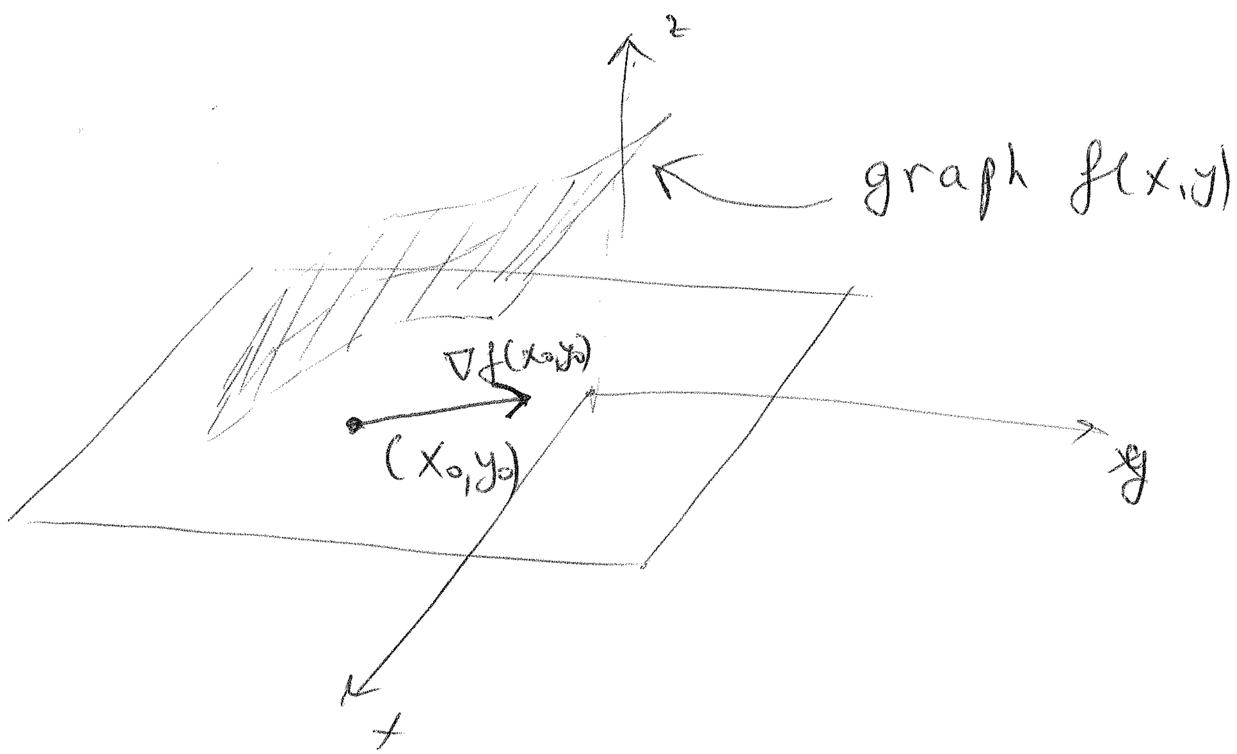
Solution:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle$$

$$\nabla f(1,2,3) = \langle 2\cdot 3, 1\cdot 3, 1\cdot 2 \rangle = \langle 6, 3, 2 \rangle$$

Remark: If $f(x,y)$ (2 variables) then $\nabla f = \langle f_x, f_y \rangle$ (Similarly for n- variables)

What does gradient do?



If $f(x,y)$ is a function of 2 variables then roughly speaking $\nabla f(x_0,y_0) =$

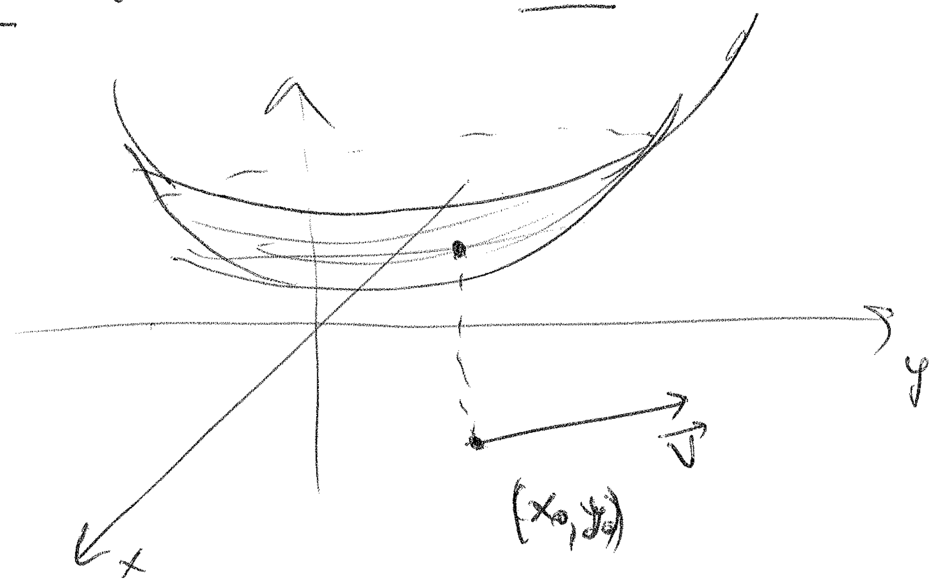
$= \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle$ at point (x_0,y_0)

shows a direction wher the function $f(x,y)$ is going to increase most rapidly. And the magnitude of the gradient $|\nabla f|$ - shows how quickly

It is going to increase.

directional derivative:

directional derivative shows how does the function changes in specific direction $\vec{v}$?



~~Example~~

And it can be computed by the following formula:

$$\boxed{\nabla f(x_0, y_0) \cdot \frac{\vec{V}}{|\vec{V}|}}$$

Example.

Let $f = \cos x + \cos y + \cos z$, and consider a vector $\vec{v} = \langle 1, 2, 3 \rangle$. Find directional derivative at point $(0, 0, 1)$.

Solution:

1 step: $\nabla f = \langle f_x, f_y, f_z \rangle = \langle -\sin x, -\sin y, -\sin z \rangle$

2 step:

$$\nabla f(0,0,1) = \langle -\sin 0, -\sin 0, -\sin 1 \rangle = \langle 0, 0, -\sin 1 \rangle$$

3 step:

$$\nabla f(0,0,1) \cdot \frac{\vec{v}}{|v|} = \langle 0, 0, -\sin 1 \rangle \cdot \frac{\langle 1,2,3 \rangle}{\sqrt{1^2+2^2+3^2}} =$$

$$= \frac{0 \cdot 1 + 0 \cdot 2 - \sin 1 \cdot 3}{\sqrt{14}} = \frac{-3 \cdot \sin 1}{\sqrt{14}}$$

Done

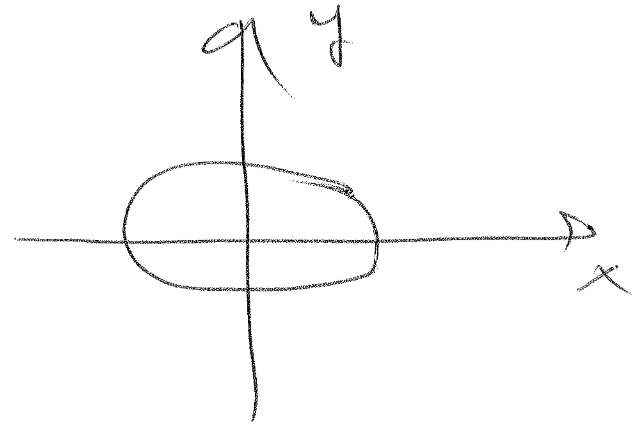
Gradients and tangent to ~~the~~ level curves.

Let $f(x,y) = c$ (for example

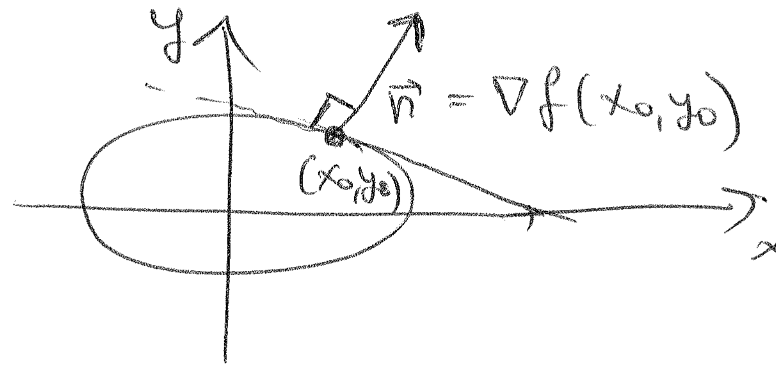
$f(x,y) = x^2+y^2$, and $c = 1$ i.e. $x^2+y^2 = 1$)

~~Usually~~ Usually it gives you a curve on the plane

(i.e. Solution is a curve)



Then $\nabla f(x_0, y_0)$ is a normal ~~v~~ector (perpendicular to the curve at point (x_0, y_0))



Then the equatio of the line passing through that point is

$$\boxed{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0}$$

Example: Let $f(x,y)=x^2+y^2$.

Find the equation of a tangent line to the level curve $f(x,y)=1$ at point $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

Solution:

1 step: $\nabla f=\langle 2x, 2y\rangle$

2 step: $\nabla f(x_0,y_0)=\left\langle \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right\rangle=\langle\sqrt{2},\sqrt{2}\rangle$

3 step.

Equation of a tangent line is:

$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=$

$=0$

or

$$\sqrt{2}\left(x-\frac{1}{\sqrt{2}}\right)+\sqrt{2}\left(y-\frac{1}{\sqrt{2}}\right)=0$$

or

$\sqrt{2}\cdot x - 1 + \sqrt{2}\cdot y - 1 = 0$

or $\boxed{x\sqrt{2}+y\cdot\sqrt{2}=2.}$

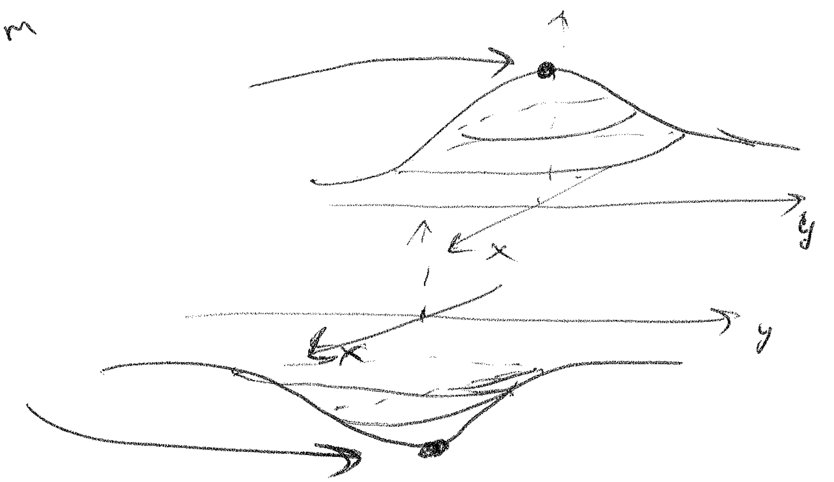
Remark: Remember that the function increases most rapidly in the direction $\frac{\nabla f}{|\nabla f|}$ and decreases most rapidly in the direction $-\frac{\nabla f}{|\nabla f|}$.

Section 14.7

Extreme values and suddle points

local maximum

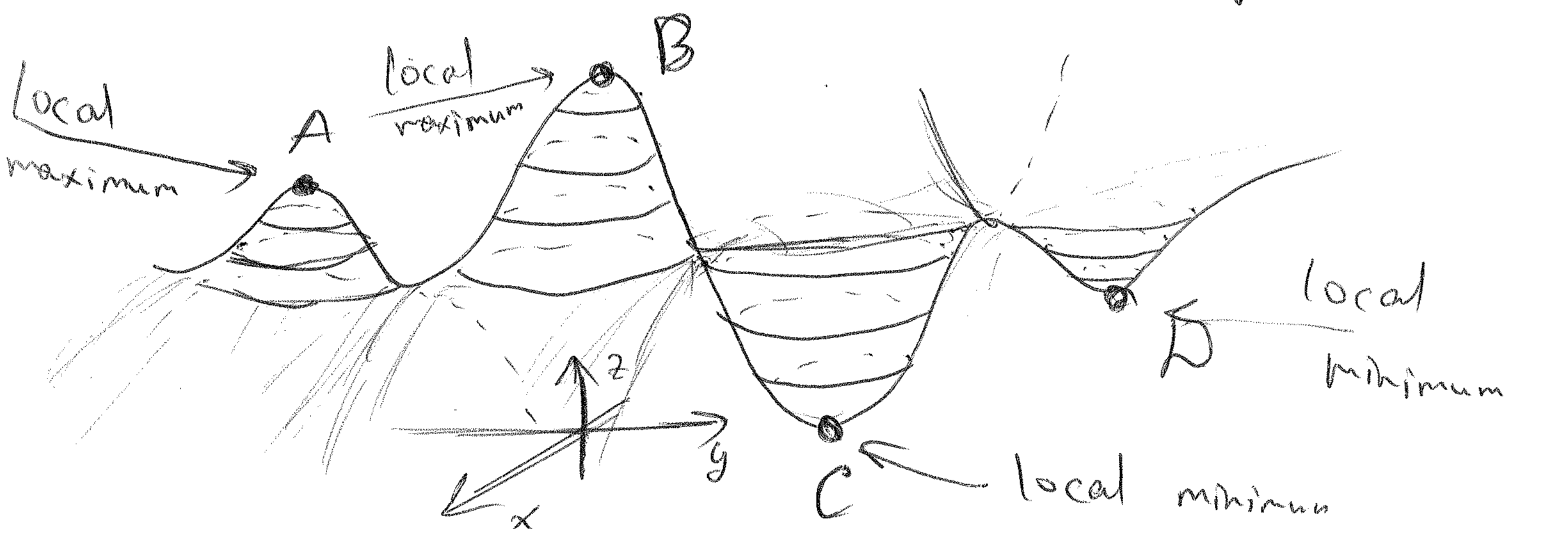
local minimum



~~Eg~~ the

The function might have several points of local maximum and local minimum.

graph of $f(x,y)$



This function ~~of~~ on
the picture has 2 local maximun
(A and B) and 2 local minimum
(C, D) from them

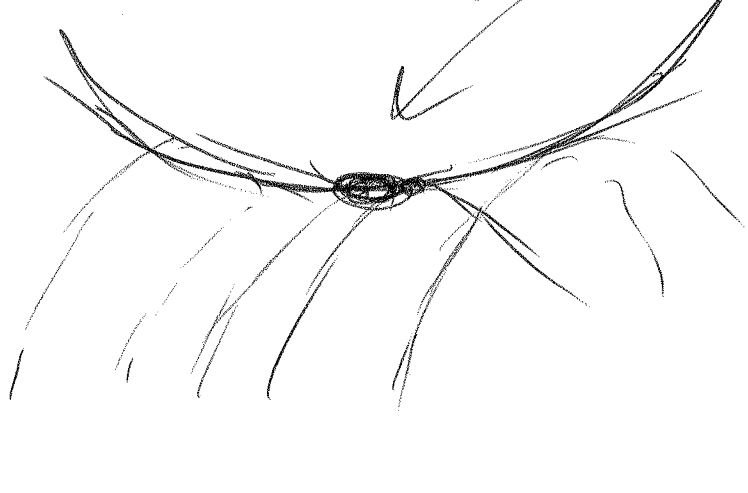
B is also grobal maximum

(because there is no higher point)

And C is global minimum.

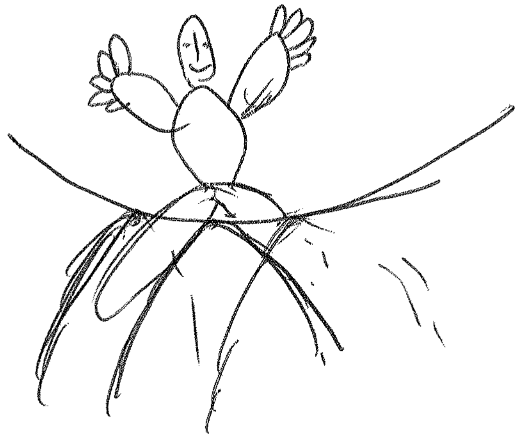
There are also suddle
points

this is a suddle point



suddle point

Imagine this picture as well



The interier point (x_0, y_0) is a suddle point, local minimum or local maximum if and only if

$$\boxed{\nabla f(x_0, y_0) = \langle 0, 0 \rangle}$$

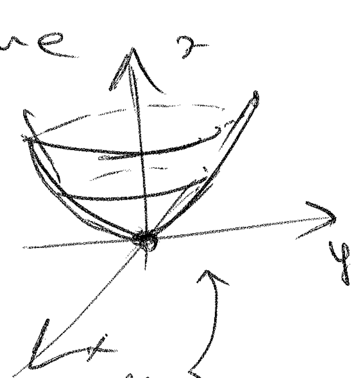
~~Usually~~ Usually suddle point, local minimum or local maximum point is called extreme point.

So (x_0, y_0) (interior) is an extreme point if and only if

$$\boxed{\nabla f(x_0, y_0) = \langle 0, 0 \rangle}$$

Example: Consider $f(x,y) = x^2 + y^2$

Show that $(0,0)$ is an extreme point.



Solution: $\nabla f = \langle 2x, 2y \rangle$.

$\langle 2x, 2y \rangle = \langle 0, 0 \rangle$ if $\boxed{x = y = 0}$

Example:

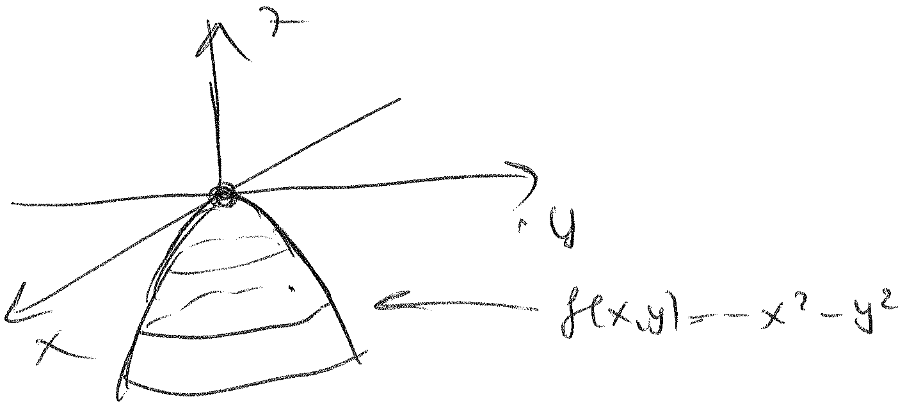
$$f(x,y) = -x^2 - y^2.$$

Show that $(0,0)$ is an extreme point.

Solution:

$$\nabla f = \langle 2x, 2y \rangle$$

$\nabla f = 0$ if and only if $x = y = 0$



Example:

Consider: $f(x,y) = y^2 - x^2$

Show that the point $(0,0)$ is an extreme point.

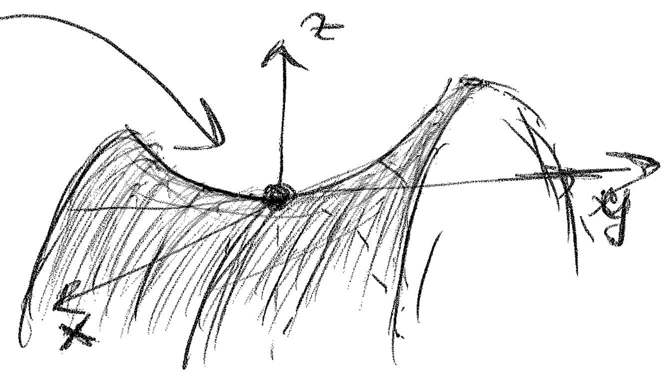
Solution:

$\nabla f = \langle -2x, 2y \rangle$.

So $\nabla f = \langle 0, 0 \rangle$ if and only

if $x = y = 0$

graph ~~picture~~ looks like this



OK we know how to find extreme points (we need to $\boxed{\text{solve } \nabla f = \langle 0, 0\rangle}$)

but how to find out ~~whe~~ whether it is local max local min or suddle point?

Test: Further $\boxed{\text{Assume } (x_0, y_0) \text{ is an extreme point}}$

1) ~~∇f (x_0, y_0)~~

1) If $f_{xx} < 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ then it is a local ~~minimum~~ maximum

2) If $f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ then it is a local minimum

3) If $f_{xx} \cdot f_{yy} - f_{xy}^2 < 0$ then it is a suddle point

4) If $f_{xx} f_{yy} - f_{xy}^2 = 0$ then we don't know (Extra work needed

Example:

Let $f(x,y) = x^2 + y^2 + x + y$

Question: find the extreme points (sometimes we call them critical points) and describe whether it is a suddle, minimum or maximum. Find the value at that point

Solution:

1) $\nabla f = \langle 2x+1, 2y+1 \rangle$

2) $\nabla f = \langle 0,0 \rangle$ iff $\begin{cases} 2x+1=0 \\ 2y+1=0 \end{cases}$ iff $\begin{cases} x = -\frac{1}{2} \\ y = -\frac{1}{2} \end{cases}$

3) $f_{xx} = 2 > 0.$

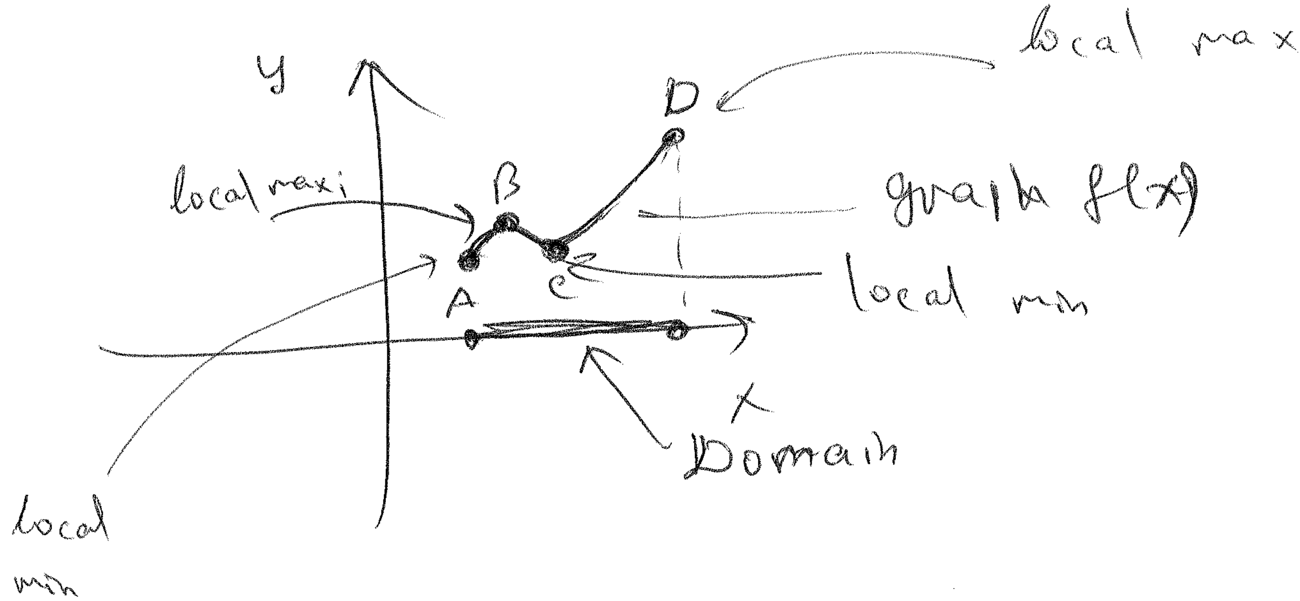
4) $f_{xx} f_{yy} - (f_{xy})^2 = 2 \cdot 2 - (0 \cdot 0) = 4 > 0$

So it is local ~~maximum~~ minimum

The value is $f(-\frac{1}{2}, -\frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = \boxed{-\frac{1}{2}}$

Done

like a function of one variable maximum point (or minimum point) might be on the boundar of the domain of a function.



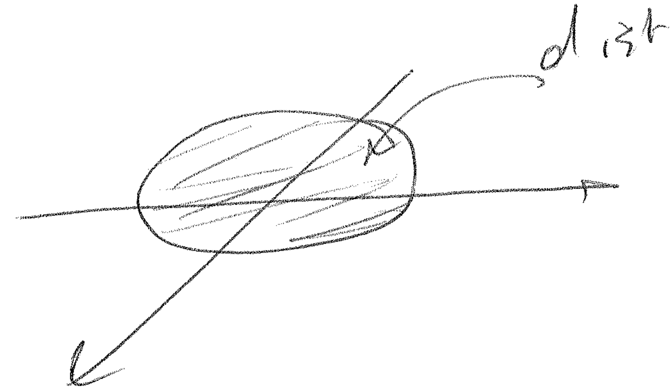
B and C are interior point

D is global max. (but it is not true that $\boxed{f'(D) = 0}$

So we also need to check on the boundary

Example:

Find absolute maximum of a function $f(x,y)=-x^2+y^2$ on the dist $x^2+y^2\le 1$



Solution.

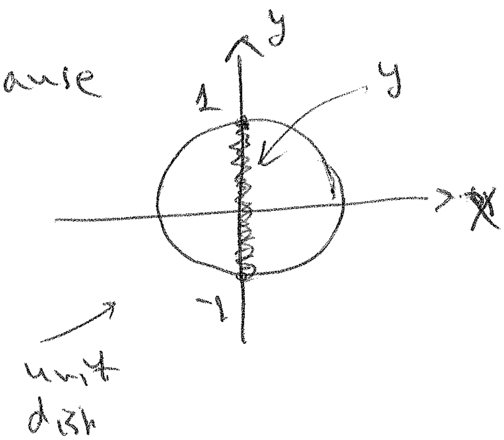
First lets see what is the maximum value on the boundary

boundry $x^2+y^2=1$ or $x^2=1-y^2$

Hence $f(x,y) = -x^2+y^2 = -(1-y^2)+y^2 =$

$= -1+y^2+y^2 = -1+2y^2$. But $-1\le y\le 1$

because



So what is maximum of $-1+2y^2$ when $-1\le y\le 1$

$g(y)=-1+2y^2$

$g'(y)=0$ if $-4y=0$ or $y=0$

but $g''(y) = 4 > 0$ so it is maximum point

Hence $g(0)$ - is maximu onthe boundary $\boxed{g(0) = -1}$

Now are there other extreme points inside the disk $x^2+y^2 \leq 1$?

$\nabla f = \langle -2x, 2y \rangle = 0$ iff $x = y = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = (-2)\cdot(2) - 0 = -4 < 0$

So it is a suddle point

Therefor global maximum is

-1