LECTURE 11

Review:

Example

Let $a = 5x^2 + 10y^2$, $x = cd$, $y = \cos(c+d)$

Find $\frac{\partial a}{\partial c}$ at $(c,d) = (\pi, 2)$ (use chain rule)

Solution:

$$\frac{\partial a}{\partial c} = \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial c} + \frac{\partial a}{\partial y}\cdot\frac{\partial y}{\partial c} =$$

$$= 10x \cdot d + 20y \cdot (-\sin(c+d)) =$$

$$= 10(cd)\cdot d + 20(\cos(c+d))(-\sin(c+d)) =$$

$$= 10cd^2 - 20\cos(c+d)\sin(c+d)$$

Hence:

$$\frac{\partial a}{\partial c}(\pi, 2) = 10\cdot\pi\cdot 2^2 - 20\cdot\cos(\pi+2)\sin(\pi+2) =$$

$$= \boxed{40\cdot\pi - 20\cos(\pi+2)\cdot\sin(\pi+2)}$$

Example:

Find limit below or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-10y^2}{(x+y)^2}$$

Solution:

Let $y=kx$ and $x\to 0$

Then $$\lim_{(x,y)\to(0,0)} \frac{x^2-10y^2}{(x+y)^2} = \lim_{(x,kx)\to(0,0)} \frac{x^2-10k^2x^2}{(x+kx)^2} =$$

$$= \lim_{(x,kx)\to(0,0)} \frac{1-10k^2}{(1+k)^2} = \frac{1-10k^2}{(1+k)^2}$$

So the limit does not exist because the result depends on k.

Example:

Let $f(x,y) = x^3 - y^3 + xy$

Find all critical points and label them as local minimum, local maximum, or saddle points

Solution:

$\nabla f = \langle f_x, f_y \rangle = (3x^2+y, -3y^2+x) = 0$

Or $\begin{cases} 3x^2+y=0 \\ -3y^2+x=0 \end{cases} \Rightarrow \begin{array}{l} y = -3x^2 \\ -3(-3x^2)^2 + x = 0 \end{array}$

So $-3 \cdot 9x^4 + x = 0 \qquad x(1-27x^3) = 0$

$x=0$ and $x = \frac{1}{3}$ (2 solutions)

$y=0$ and $y = -\frac{1}{3}$

So critical points are

$(x,y) = (0,0)$ and $(x,y) = \left(\frac{1}{3}, -\frac{1}{3}\right)$

1. Lets investigate first one $(0,0)$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 6x \cdot (-6y) - 1 = -36xy - 1$

at point $(0,0)$ we have

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = -1 < 0$

Therefore it is a suddle point.

2. Lets investigate ~~point~~ critical point $\left(\frac{1}{3}, -\frac{1}{3}\right)$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = -36xy - 1$

So at point $\left(\frac{1}{3}, -\frac{1}{3}\right)$ we have

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = -36 \cdot \frac{1}{3} \cdot \left(-\frac{1}{3}\right) - 1 = 4 - 1 = 3 > 0$

Then $f_{xx} = 6x$

at point $(x,y) = \left(\frac{1}{3}, -\frac{1}{3}\right)$ we have

$$f_{xx} = 6 \cdot \frac{1}{3} = 2 > 0$$

Therefore this is local minimum.

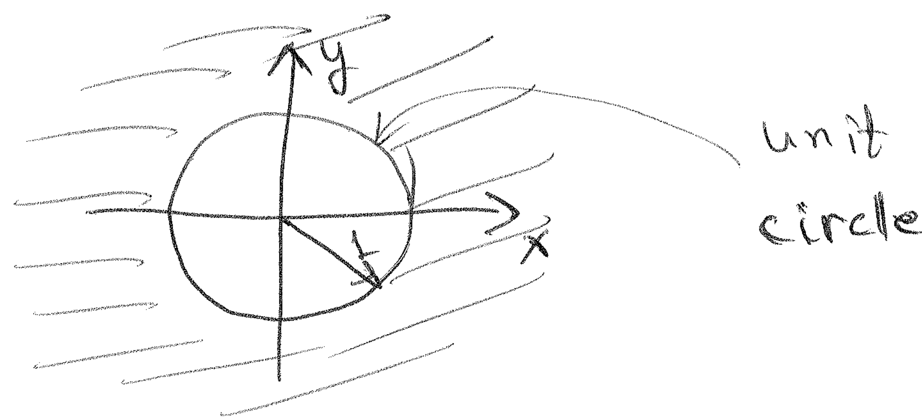
Example

Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-1}}$

Q1: Find domain of $f(x,y)$

Solution: ~~Squar~~ $x^2+y^2-1 \geq 0$ or

$x^2+y^2 \geq 1$

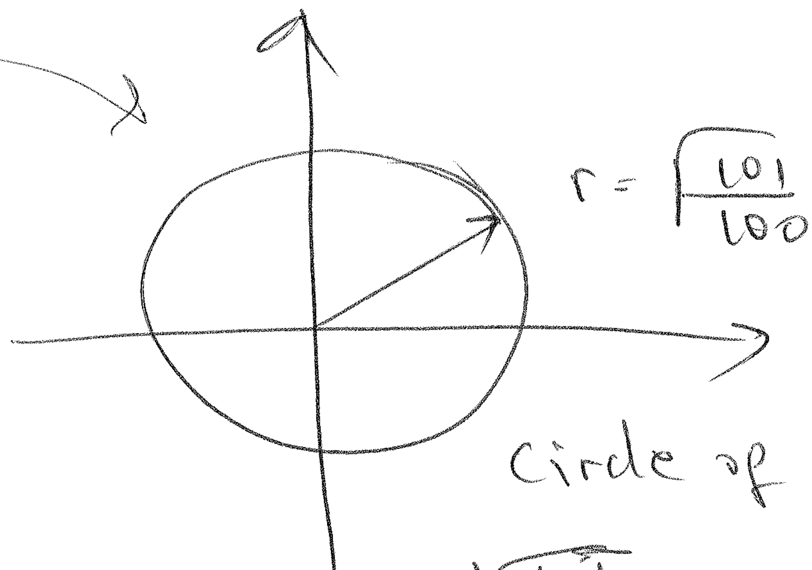


Q2: Sketch the level curve $f(x,y) = 10$

Solution:

$$\frac{1}{\sqrt{x^2+y^2-1}} = 10 \iff \frac{1}{100} = x^2+y^2-1$$

or $x^2+y^2 = 1 + \frac{1}{100} = \frac{101}{100}$



Circle of radius $\sqrt{\frac{101}{100}}$

Q3: Find the range of f

Solution: Range is $(0, \infty)$

Remark:

Remember that

① $f(x) = \sqrt{x}$ ~~rang~~ domain is $[0, \infty)$

and the range is $[0, \infty)$

② $f(x) = \ln x$ domain is $(0, \infty)$

range - $(-\infty, \infty)$

3) $f(x) = \frac{1}{x}$ domain ~~$(-\infty)$~~ $(-\infty, 0) \cup (0, \infty)$

range $(-\infty, 0) \cup (0, \infty)$

3) ~~sin~~ $f(x) = \sin x$ or ($\cos x$) doman $(-\infty, \infty)$

rang $[-1, 1]$

4) $f(x) = e^x$ domain $(-\infty, \infty)$

range $(0, \infty)$

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Example:

Let $f(x,y,z) = x^2 + \ln xy + y^2 + \ln xz$
$+ z^2 + \ln zy$
and $P_0 = (1,2,3)$

Q1: Find

∇f at point P_0

Solution:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x + \frac{2}{x},\ 2y + \frac{2}{y},\ 2z + \frac{2}{z} \rangle$$

$$\nabla f(1,2,3) = \langle 2+2,\ 2\cdot 2 + \frac{2}{2},\ 2\cdot 3 + \frac{2}{3} \rangle =$$

$$= \langle 4, 5, 6 + \frac{2}{3} \rangle = \langle 4, 5, \frac{20}{3} \rangle$$

Q2: Find derivative in the direction

$10i + 11j + 19k$ at point P_0

Solution: (9)

$$\nabla f(1,2,3) \cdot \underbrace{\frac{\langle 10, 11, 19\rangle}{\sqrt{10^2+11^2+19^2}}}_{\text{direction}} =$$

$$= \left\langle 4, 5, \frac{20}{3}\right\rangle \cdot \frac{\langle 10, 11, 19\rangle}{\sqrt{100+121+361}} =$$

$$= \frac{40+55+\frac{380}{3}}{\sqrt{582}} = \frac{1}{\sqrt{582}}\left(95+\frac{380}{3}\right) =$$

$$= \frac{285+380}{3\sqrt{582}} = \boxed{\frac{665}{3\sqrt{582}}}$$