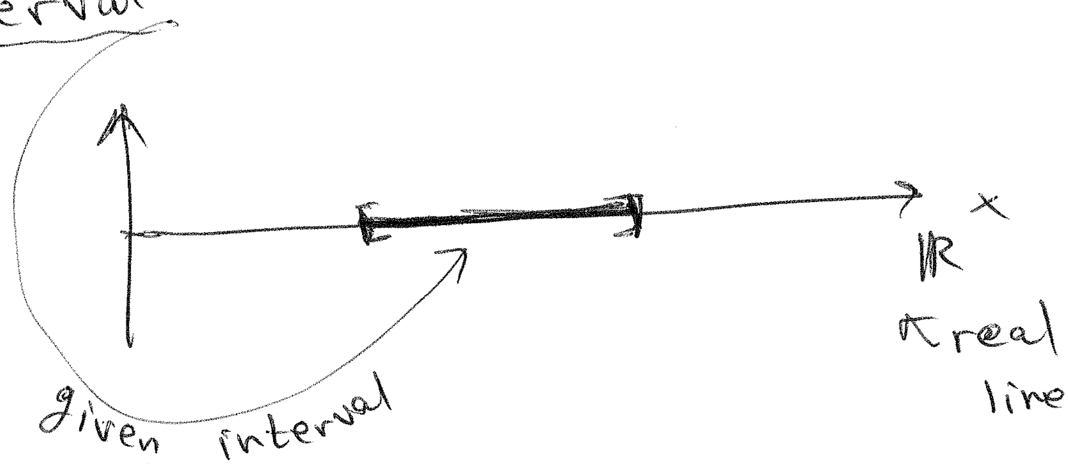
# LECTURE 12 (Important lecture)

Section 15.1 and Section 15.2

## Double integrals

IF YOU HAVE A FUNCTION OF 1 variable, say $f(x)$ then the question find the integral of this function over the given interval

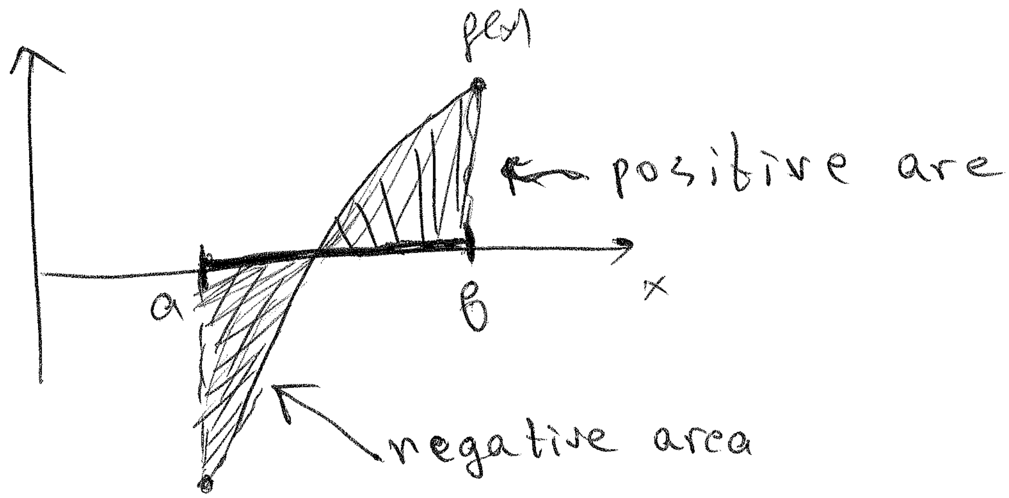



means that you need to find endpoints of this interval say $a$ and $b$ and compute the following integral

$$\int_a^b f(x)\,dx$$

And the meaning of this expression

$\int_a^b f(x)\,dx$ is the area of the subgraph




Also note that the only natural sets on the real line are <u>intervals</u>

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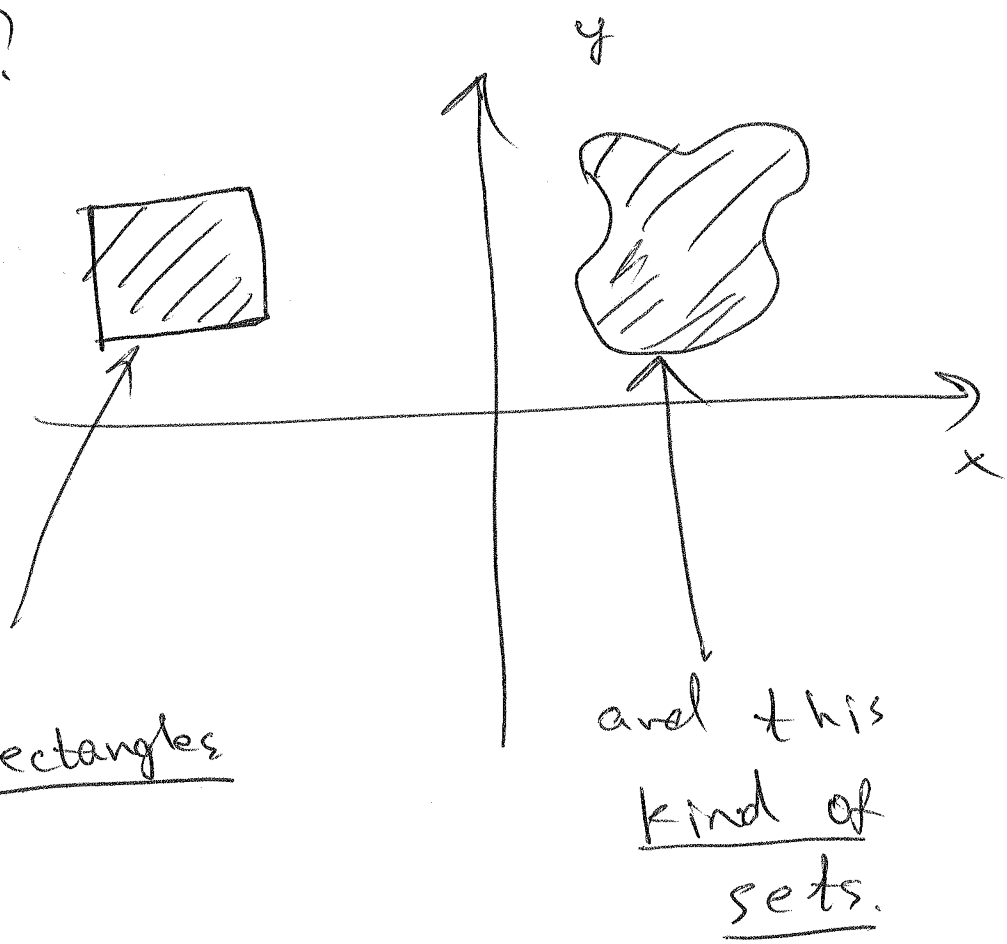
The same question we can ask about the function of two variables $f(x,y)$. But this function is defined on the <u>plane</u> (not on the real line)

So what would be natural sets

Over which we can integrate this functions?

Plane



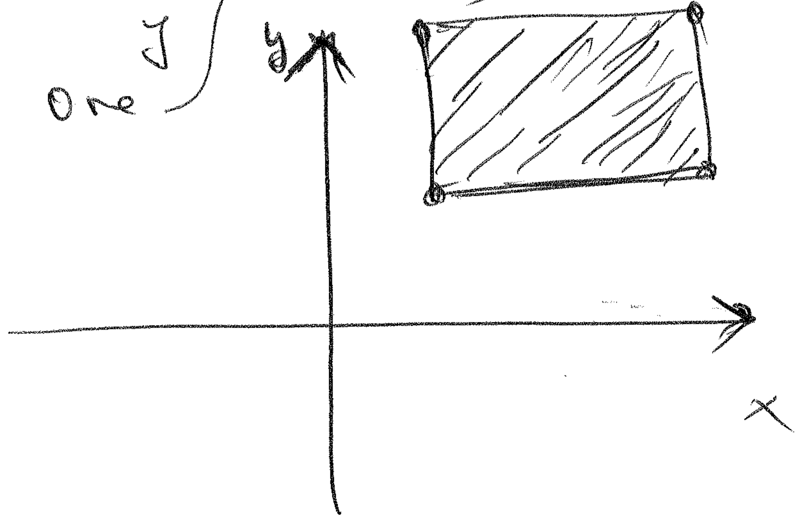

So unlike the real line we have a lot of natural sets on the plane.

For simpicity lets consider only rectangles on the plane (and later we will

Cover other sets as well).

Take any rectangle

For example this one




Then how would you define integral of the function $f(x,y)$ over this rectangle (Call it $R$)

$$\int_R f(x,y)\, d$$

of what? $\boxed{dx-?\ dy-?}$

something like this?

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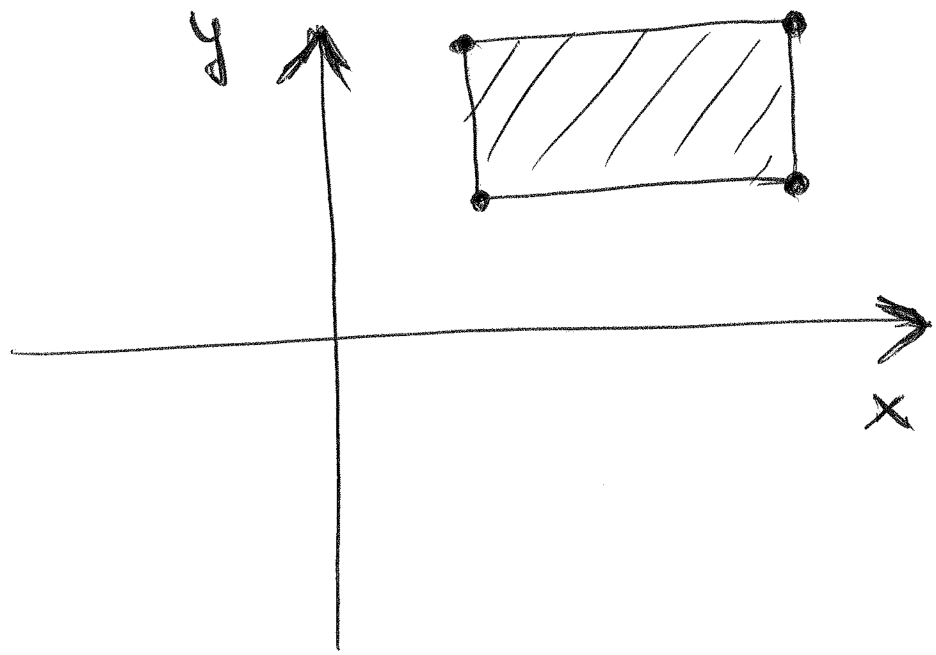
Here is the right way:

Since we have <u>two</u> variables we write double integral.

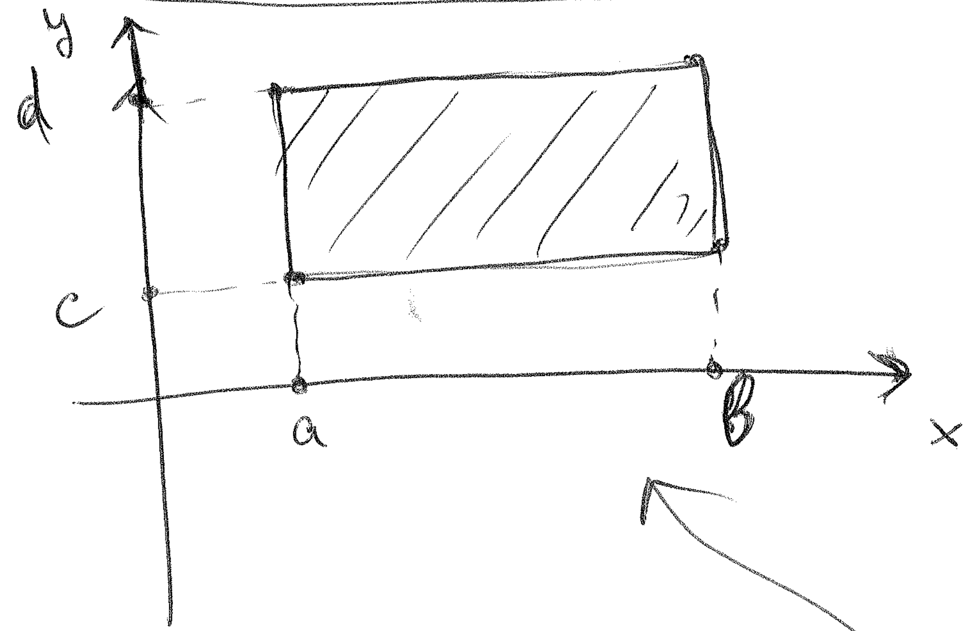
$$\iint_R f(x,y)\, d_{smth}$$ ← this is not crear yet

OK good.




In the caşa of one variable for the given interal we found endpoints of that internal. Now we have a rectangle and we want to find „endpoints" of that rectangle (or we would like to indicate that rectangle somehow)

## How can we indicate the rectangle?




Well, I think this is the best way how can you indicate this rectangle

You see that variable x varies between $a \le x \le b$ and variable y varies between $c \le y \le d$, Then we write

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

smth

Note that:

$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

But one can also write this integral in the following way (change the order)

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

smth

Do you see the difference?

We just changed the order.

Ok. How can one compute this kind of integrals?

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \; - ?$$

First of all, remember that

$$\boxed{\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

So if you integrate over the rectangle and you change the order then nothing changes!

Example:

Find $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy = ?$

Solution :

$$\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$$

1 step: $\displaystyle\int_1^2 (x^2+y^2)\,dx = \left(\frac{x^3}{3}+y^2x\right)\Bigg|_{x=1}^{x=2} =$

$$= \left(\frac{2^3}{3}+y^2\cdot 2\right)-\left(\frac{1^3}{3}+y^2\right) =$$

$$= \frac{7}{3}+y^2$$

2 step: $\displaystyle\int_0^1 \left(\frac{7}{3}+y^2\right)dy = \left(\frac{7}{3}y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=1} =$

$$= \frac{7}{3}+\frac{1^3}{3} = \boxed{\frac{8}{3}}$$

Remark: if you change the order you get the same thing

Indeed

$$\int_1^2 \int_0^1 (x^2+y^2)\,dy\,dx = \int_1^2 \left( x^2 y + \frac{y^3}{3} \right)\Big|_{y=0}^{y=1} dx =$$

$$= \int_1^2 \left( x^2 + \frac{1}{3} \right) dx = \left( \frac{x^3}{3} + \frac{x}{3} \right)\Big|_{x=1}^{x=2} =$$

$$= \left( \frac{8}{3} + \frac{2}{3} \right) - \left( \frac{1}{3} + \frac{1}{3} \right) = \boxed{\frac{8}{3}}$$

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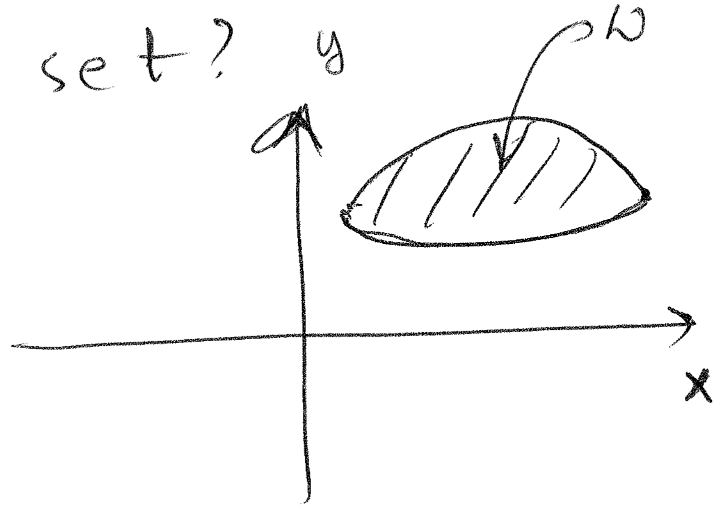
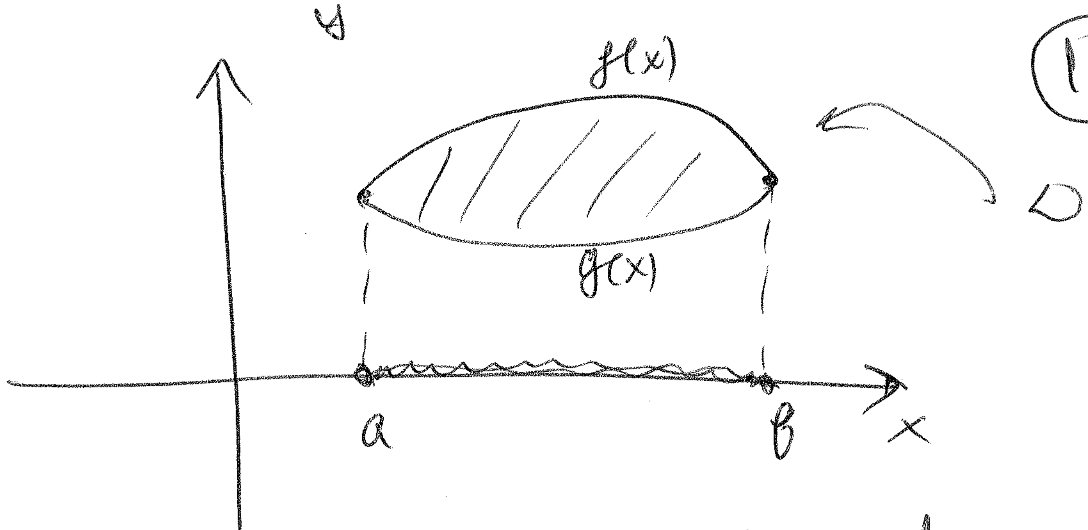
What about the integration over arbitrary set?






You have to indicate this domain mathematicaly.

$a \le x \le b$ and for each fixed $x$

$g(x) \le y \le f(x)$

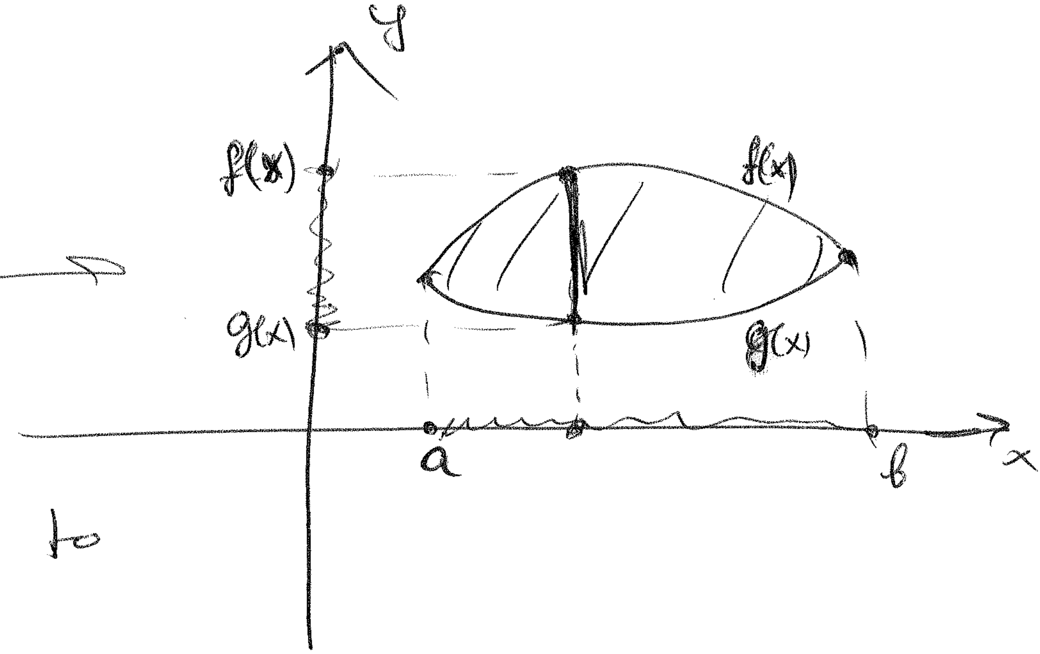
$$\iint_R F(x,y)\,dA = \int_a^b \int_{g(x)}^{f(x)} F(x,y)\,dy\,dx$$

Note that if you change the order

$\int_{g(x)}^{f(x)} \int_a^b F(x,y)\,dx\,dy$ you get nonsense

(this is not a number)

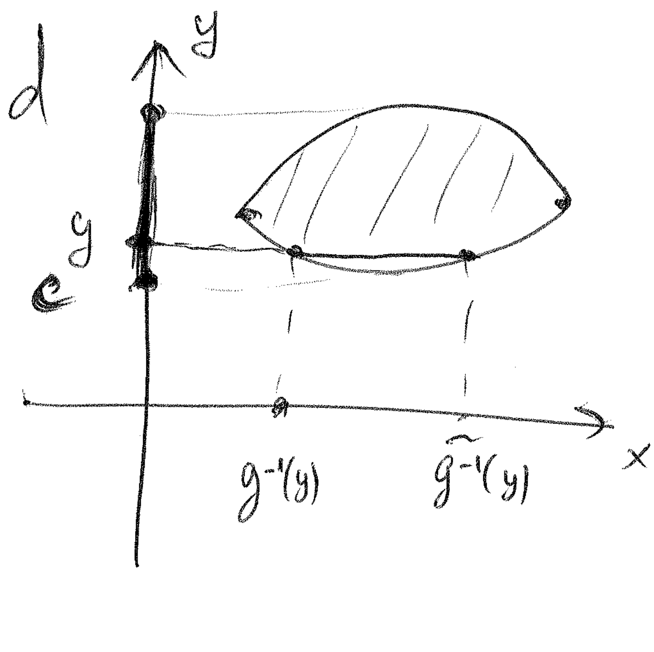
But still can you change the order? Yes, but limits of integral will be different.




this corresponds to this order

$$\int_a^b \int_{g(x)}^{f(x)} F(x,y) \; \underline{dy\, dx}$$

and this picture



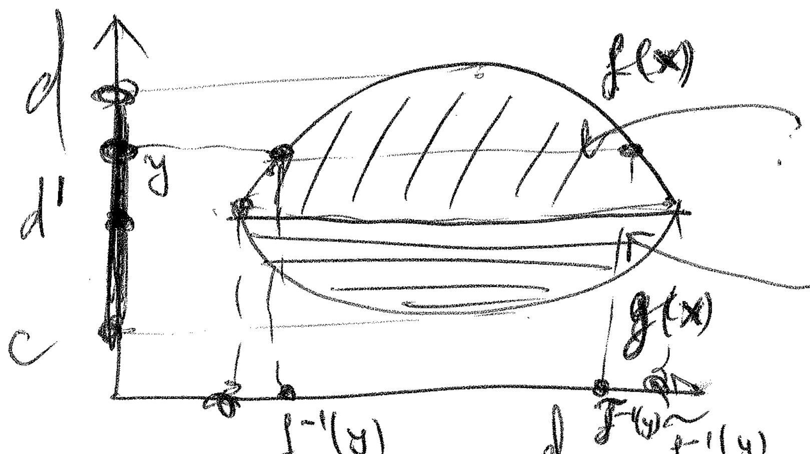

$$\int_c^d \int_{g^{-1}(y)}^{\tilde{g}^{-1}(y)} F(x,y) \; \underline{dx\, dy}$$

~~but it should~~ contain

which is wrong Because it does not contain $f(x)$. So you do the following

$$\iint_D F\,dA = \iint_{D_1} F\,dA + \iint_{D_2} F\,dA$$

additivity

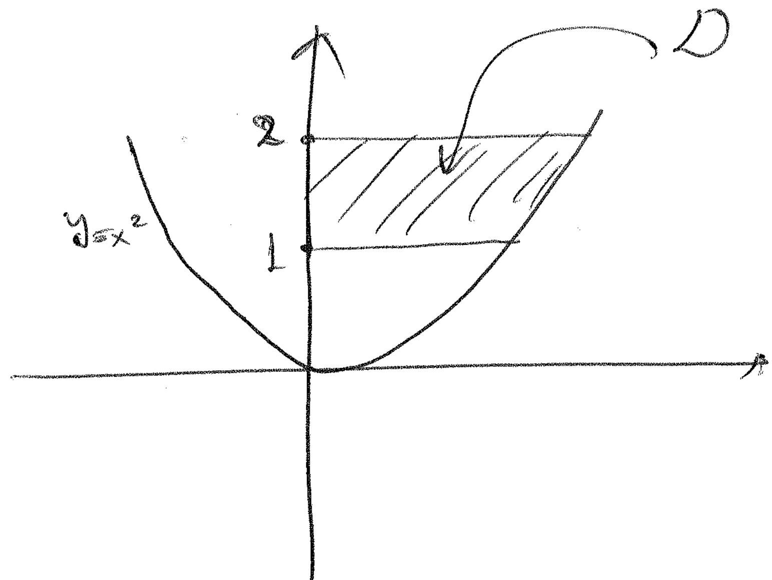



$$\iint_{D_1} F\,dA = \int_{d'}^{d} \int_{f^{-1}(y)}^{\tilde{f}^{-1}(y)} F(x,y)\,dx\,dy$$

$$\iint_{D_2} F\,dA = \int_{c}^{d'} \int_{g^{-1}(y)}^{\tilde{g}^{-1}(y)} F(x,y)\,dx\,dy$$

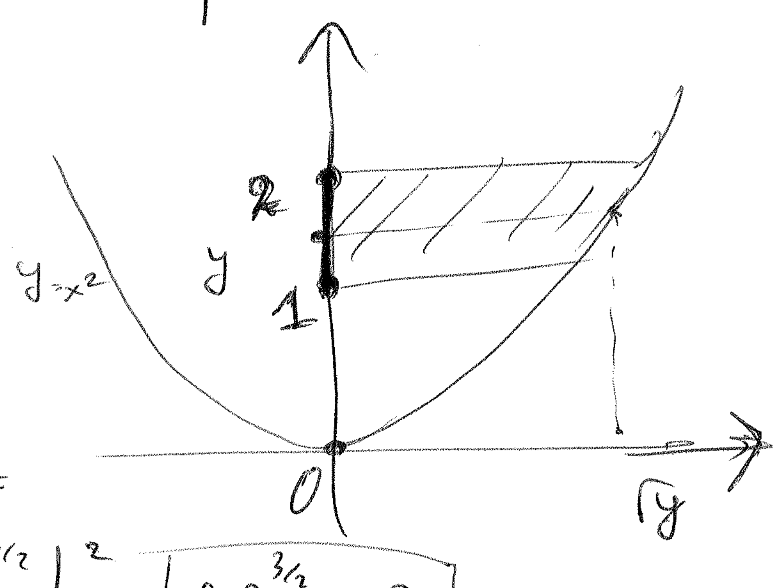
**Remark:** limits of integral define a domain and vice versa domain defines the limit of integrals.

**Example:**




Find $\iint_D dA$ ?

**Solution:** $1 \le y \le 2$
$0 \le x \le \sqrt{y}$




$$\iint_D dA = \int_1^2 \int_0^{\sqrt{y}} dx\, dy = \int_1^2 \sqrt{y}\, dy = \frac{2y^{3/2}}{3} \Big|_1^2 = \boxed{\frac{2\cdot 2^{3/2}}{3} - \frac{2}{3}}$$

Lets see what do you get if you start computing in a different order

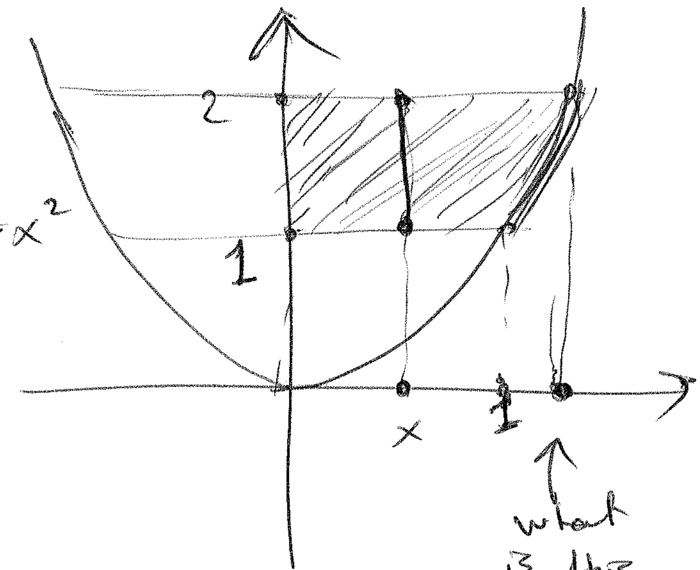
$0 \le x \le \sqrt{2}$

$1 \le y \le 2$

but if $x > 1$

then

$x^2 \le y \le 2$




what is this point?

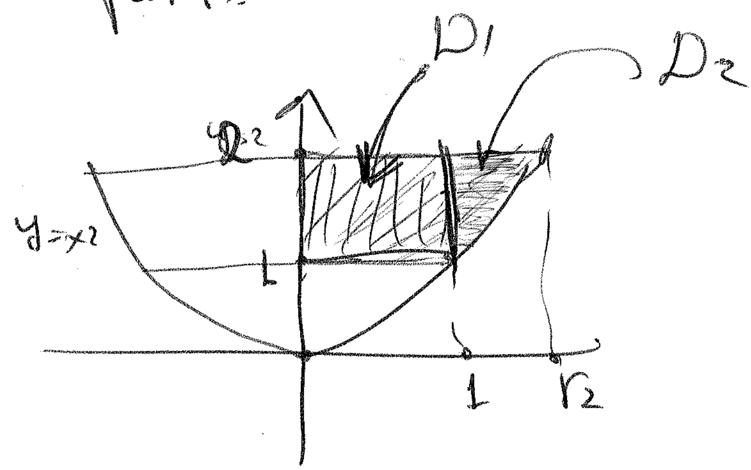
$2 = x^2$

$x = \sqrt{2}$

So what to do?

Split domain in two parts.

$$\iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA$$




in $D_1$ we have $0 \le x \le 1$, $1 \le y \le 2$

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in $D_2$ we have $1 \le x \le \sqrt{2}$, $x^2 \le y \le 2$

Therefore you get:

$$\iint_{D_1} dA + \iint_{D_2} dA = \int_0^1\int_1^2 dy\,dx + \int_1^{\sqrt{2}}\int_{x^2}^{2} dy\,dx =$$

$$= 1 + \int_1^{\sqrt{2}} \left(2 - x^2\right) dx = 1 + 2(\sqrt{2}-1) - \frac{x^3}{3}\Big|_1^{\sqrt{2}} =$$

$$= 2\sqrt{2} - 1 - \left(\frac{2\sqrt{2}}{3} - \frac{1}{3}\right) = \frac{4\sqrt{2}}{3} - \frac{2}{3}$$

the same answer

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Example:

$$\int_1^2 \int_0^{x^3} dy\,dx$$ indicate the domain

Solution:

$1 \le x \le 2$

$0 \le y \le x^3$