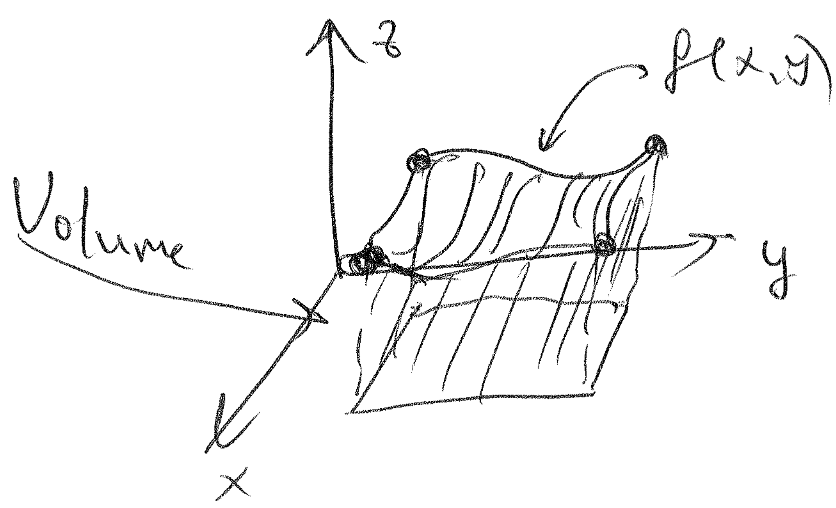
LECTURE 13

1

Sections 15.3 and 15.4

Remark

$\iint_R f(x,y)\,dA$ – is a volume of the subgraph



and $\iint_R 1\,dA$ – is the are of the domain R.

(It is similar to 1 dimensional case)

Average value of f over R

$$= \frac{1}{\text{Area } R} \iint_R f \, dA$$

Example:

Find area of the region eclosed

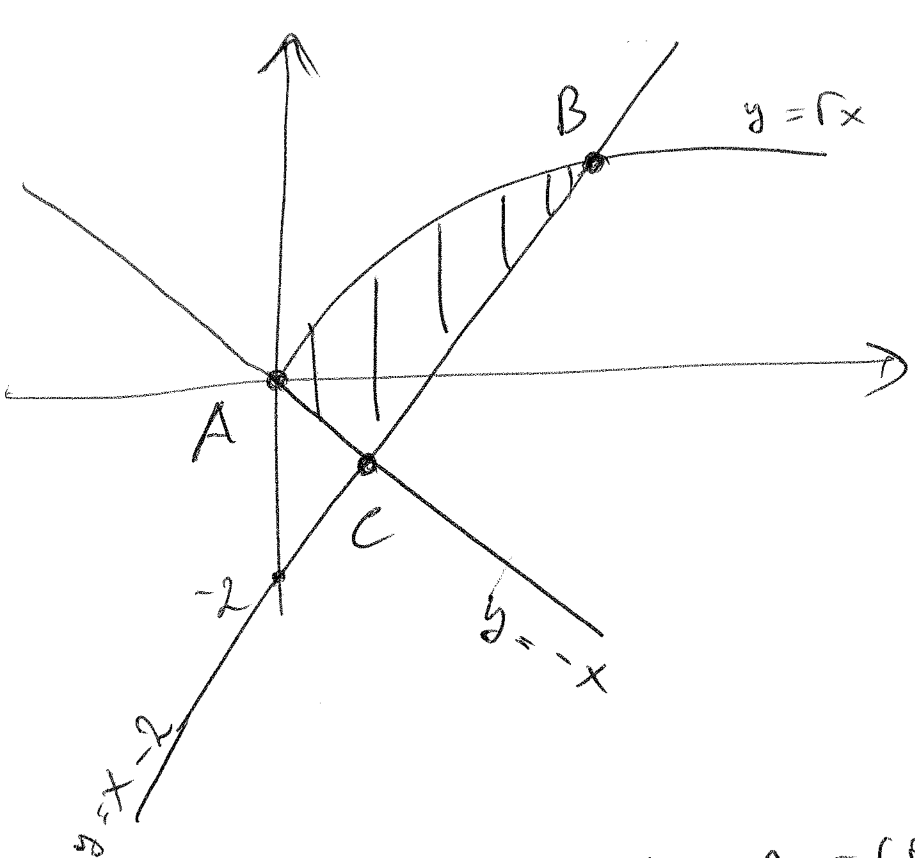
by curves

$y = x - 2$

$y = -x$

$y = \sqrt{x}$

Solution:



It is creal that $A = (0,0)$

Let's find coordinates of C

$$\begin{cases} y = -x \\ y = x - 2 \end{cases} \Rightarrow \begin{cases} y = -x \\ -x = x - 2 \end{cases} \Rightarrow \begin{cases} y = -x \\ x = 1 \end{cases} \Rightarrow \begin{cases} y = -1 \\ x = 1 \end{cases}$$

So $C = (1, -1)$. Let's find coordinates of

B
$$\begin{cases} y = x - 2 \\ y = \sqrt{x} \end{cases} \Rightarrow \begin{cases} \sqrt{x} = x - 2 \\ y = \sqrt{x} \end{cases} \Rightarrow \begin{cases} x = x^2 - 4x + 4 \\ y = \sqrt{x} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x^2 - 5x + 4 = 0 \\ y = \sqrt{x} \end{cases} \Rightarrow \begin{cases} x = \frac{5 \pm \sqrt{25 - 16}}{2} \\ y = \sqrt{x} \end{cases} \Rightarrow \begin{cases} x = \frac{5 \pm 3}{2} \\ y = \sqrt{x} \end{cases}$$

but solution $x = \frac{5-3}{2} = 1$

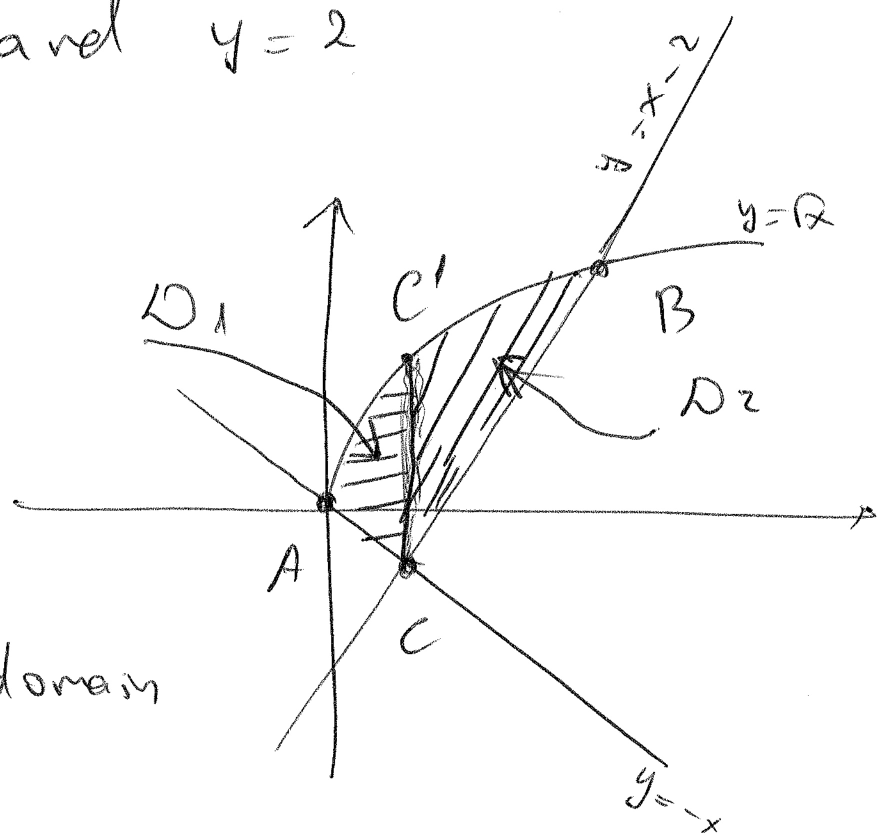
does not count becaus we had

$\sqrt{x} = x - 2$ which means the

$\sqrt{1} = 1 - 2$ wrong!

So $x = \frac{8}{2} = 4$ and $y = 2$

or $B = (4,2)$



we should split our domain

$C' = (1,1)$

$$\iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA = \int_0^1\int_{-x}^{\sqrt{x}} dy\,dx + \int_1^2\int_{x-2}^{\sqrt{x}} dy\,dx = \int_0^1 (\sqrt{x}+x)\,dx + \int_1^2 (\sqrt{x}-x+2)\,dx = \left(\frac{x^{3/2}}{3/2} + \frac{x^2}{2}\right)\Bigg|_{x=0}^{x=1} +$$

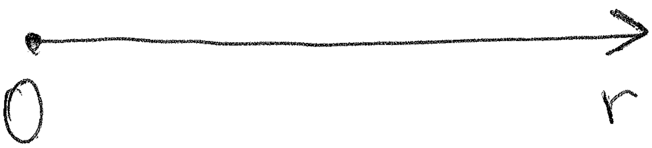
$$+ \left(\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right)\Bigg|_{x=1}^{x=2} = \left(\frac{2}{3} + \frac{1}{2}\right) + \left[\left(\frac{2}{3}\cdot 2^{3/2} - 2 + 4\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)\right] =$$

$= \frac{4+3}{6} + \frac{2}{3}\cdot 2^{3/2} - \frac{1}{6} = \boxed{1 + \frac{4\sqrt{2}}{3}}$

Double integrals in polar coordinates

We have cartesian coordinate system (x,y).

We also can measure things in polar coordinate system. we have two coordinates (r, θ) and the system looks like this



For example curve migth be given like this $r = \cos\theta$ (in polar coordinates)

$r(\theta) =$ smth of θ)

like $y(x) =$ smthing of x

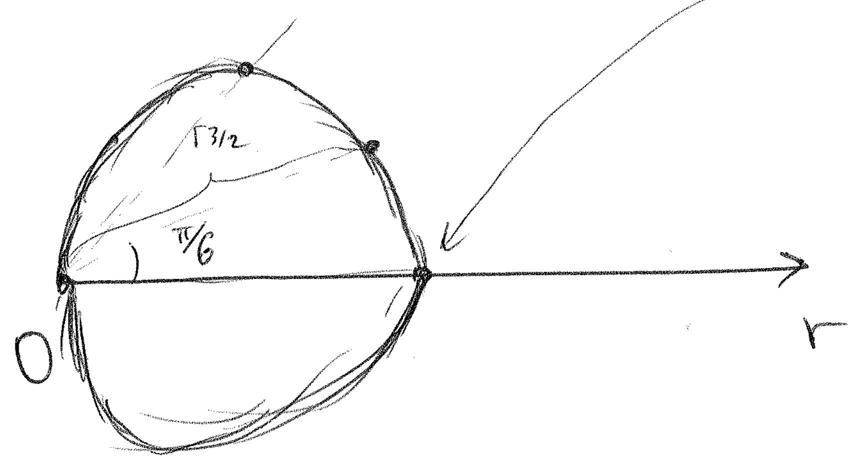
$f(x) =$ something of x.

So how to contsruct that curve?

$r = \cos\theta$

r	θ
1	0
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$1/2$	$\frac{\pi}{3}$
0	$\pi/2$

So where is the point $(r,\theta) = (1,0)$



So we can also consider functions $f(r,\theta)$ (in polar coordinate system)

Usuall ~~oreqk~~ of integration in polar coordinare system is $dr\,d\theta$

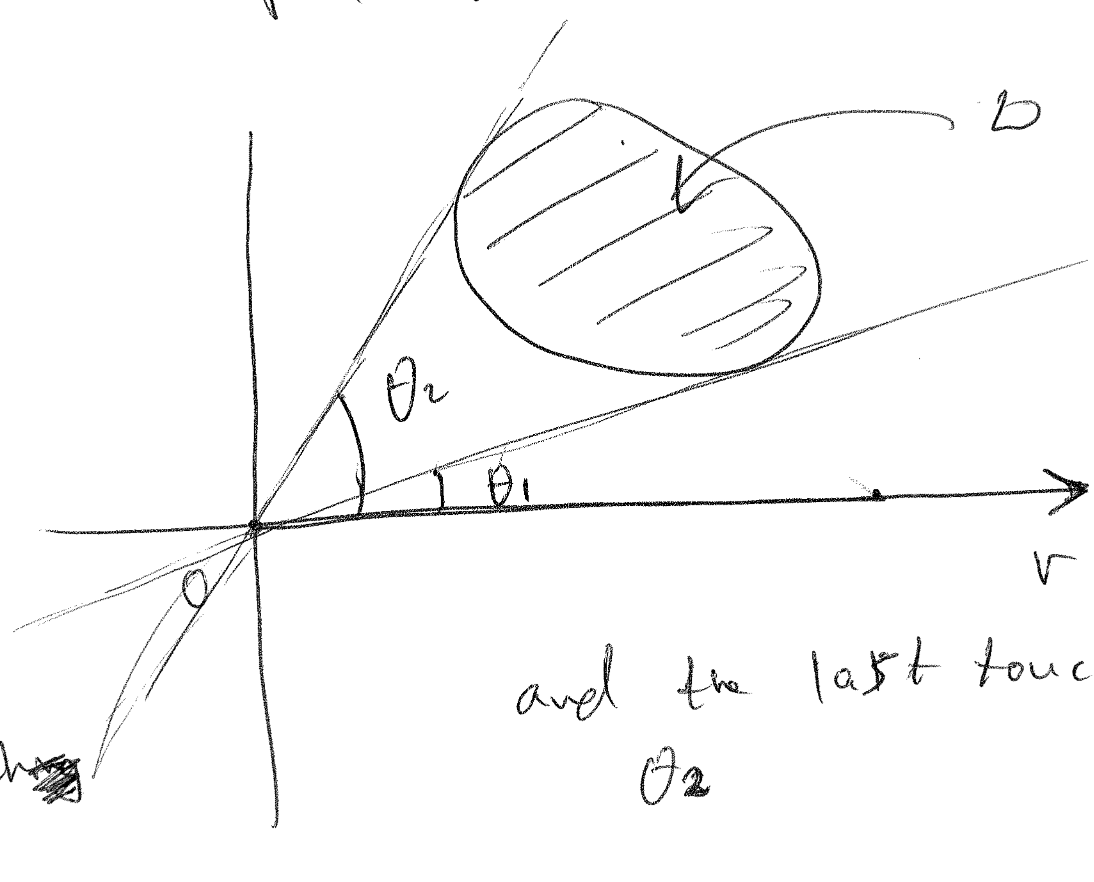
And

$$\iint_D f\,dA = \int_{\circ}^{\circ}\int_{\circ}^{\circ} f(r,\theta)\ \underline{\underline{r\,dr\,d\theta}}$$

limits of θ

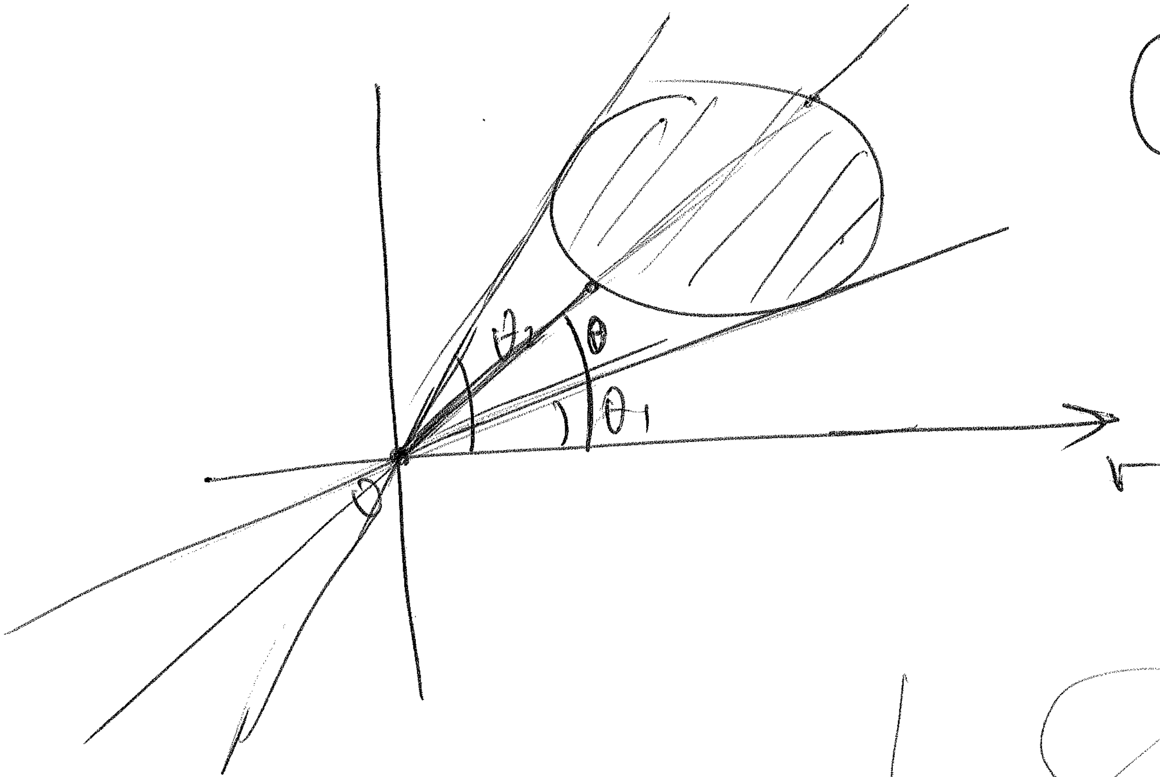
limits of r

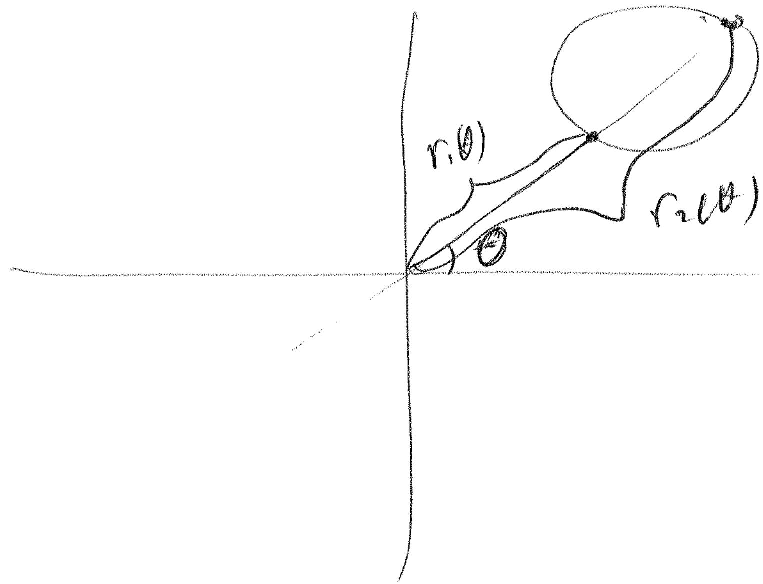
How do we find these limits?



and the last touch θ_2

First touching θ_1





$\theta_1 \le \theta \le \theta_2$

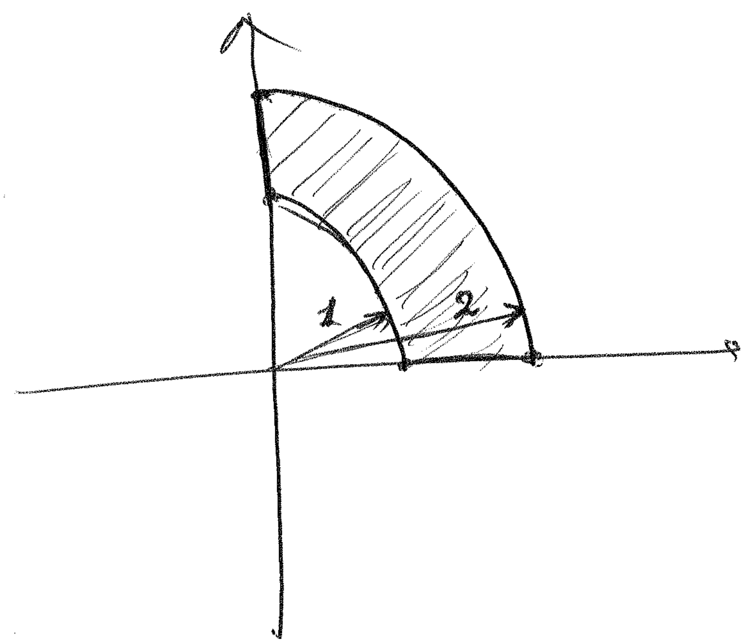
$$\int_{\theta_2}^{\theta_1} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\; r\,dr\,d\theta$$

for each ~~θ θ between~~ fixed θ between θ_1, θ_2 we have

$r_1(\theta) \le r(\theta) \le r_2(\theta)$

Example:

Find the integral of a function in polar coordinate system $f(r,\theta) = r^2 + \theta^2$



$0 \leq \theta \leq \frac{\pi}{2}$

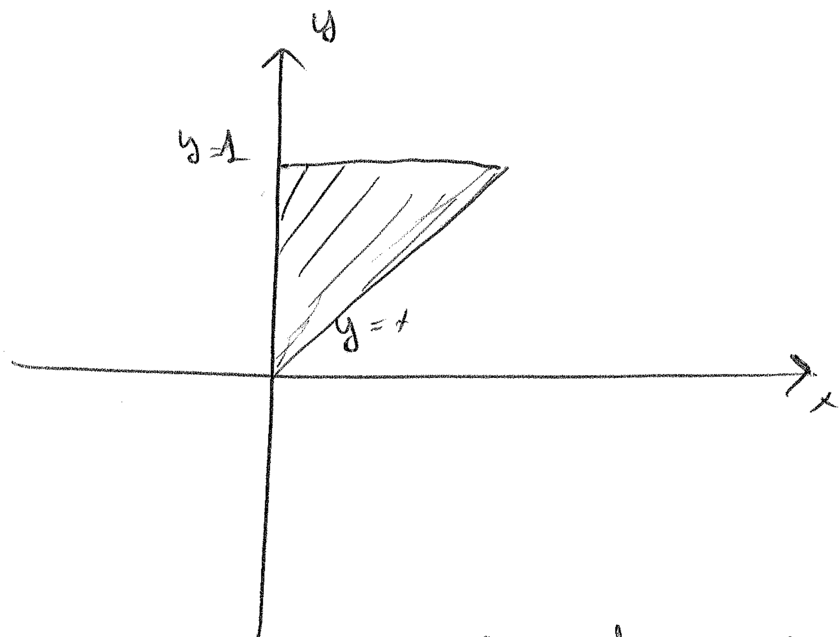
$1 \leq r \leq 2$

Therefore $\displaystyle\int_0^{\pi/2} \int_1^2 (r^2+\theta^2)\, r\, dr\, d\theta =$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4} + \theta^2 \frac{r^2}{2} \right)\Bigg|_{r=1}^{r=2} d\theta = \int_0^{\pi/2} \left(\frac{16}{4} + \theta^2 \cdot 2 - \frac{1}{4} - \frac{\theta^2}{2} \right) d\theta =$$

$$= \int_0^{\pi/2} \left(\frac{3\theta^2}{2} + \frac{15}{4} \right) d\theta = \left(\frac{3\theta^3}{3 \cdot 2} + \frac{15}{4}\theta \right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi^3}{16} + \frac{15\pi}{8}$$

Suppose your function is given in Cartesian coordinate system $\boxed{f(x,y)}$ and you have a domain



How to integrate this function in polar coordinate system?

Remember

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

this is the only connection

$$\iint_D f\,dA = \iint_D f(x,y)\,dA =$$

$$\cancel{\iint f(r,\theta)\, r}$$

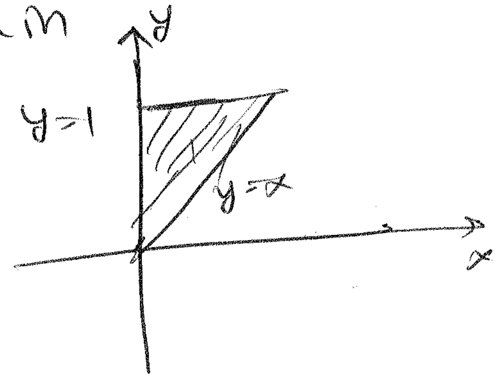
$$= \int \int f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

limits for θ

limits for r

Example: let $f(x,y) = x+y$

Find integral of f in polar coordinate system over the domain



$1 = y = r\sin\theta$

$r = \frac{1}{\sin\theta}$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le r \le \frac{1}{\sin\theta}$

$$\iint f\,dA = \int_{\pi/4}^{\pi/2} \int_0^{1/\sin\theta} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$\int_{\pi/4}^{\pi/2} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_{r=0}^{r=\frac{1}{\sin\theta}} d\theta =$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2}\left(\frac{\cos\theta}{\sin^3\theta}+\frac{1}{\sin^2\theta}\right)d\theta = \frac{1}{3}\int_{\frac{\pi}{4}}^{\pi/2}\frac{\cos\theta}{\sin^3\theta} + \frac{1}{3}\int_{\pi/4}^{\pi/2}\frac{1}{\sin^2\theta}d\theta =$$

change of variables

1 step. $\displaystyle\int \frac{\cos\theta}{\sin^3\theta}\,d\theta = \frac{\sin^{-3}(\theta)}{+3} = -\frac{1}{9\sin^3\theta}$

Hence: $\displaystyle\frac{1}{3}\int_{\frac{\pi}{4}}^{\pi/2}\frac{\cos\theta}{\sin^3\theta}\,d\theta = -\frac{1}{9}\,\frac{1}{\sin^3\theta}\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} =$

$$= -\frac{1}{9}\left(1 - 2^{3/2}\right) = -\frac{1}{9} + \frac{2^{3/2}}{9}$$

2 step.

$$\int \frac{1}{\sin^2\theta}\,d\theta = \int \cancel{(ctg^2\theta+1)\,d\theta} = \cancel{\theta + \int ctg^2\theta\,d\theta}$$

$$= -\cot x$$

Hence $\displaystyle\frac{1}{3}\int_{\frac{\pi}{4}}^{\pi/2}\frac{1}{\sin^2\theta}\,d\theta = \frac{1}{3}\left(-\cot\theta\right)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \frac{1}{3}\cot\frac{\pi}{4} = \frac{1}{3}$

So answer is $-\frac{1}{9} + \frac{2^{3/2}}{9} + \cancel{\frac{1}{3}} = \frac{2}{9} + \frac{2^{3/2}}{9}$