LECTURE 18

section 16.2

line integral of F along curve C

You parametrize a curve C like

$r(t) = \langle g(t), h(t), k(t) \rangle = g(t)\cdot i + h(t)\cdot j + k(t)\cdot k$

when $a \le t \le b$

and you have a vector field

$$F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle =$$
$$= M(x,y,z)\cdot i + N(x,y,z)\cdot j + P(x,y,z)\cdot k$$

and you compute:

$$\int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt$$

where $F(r(t)) = F(g, h, k) = \langle M(g,h,k), N(g,h,k), P(g,h,k) \rangle$

$= \langle M(g(t), h(t), k(t)), N(g(t), h(t), k(t)), P(g(t), h(t), k(t)) \rangle$.

depends on t

Example:

this is how do we denote it

Evaluate $\int_C F \cdot dr$ where

$F(x,y,z) = \langle x, 0, z \rangle$ and $r(t) = \langle t, t^2, t^3 \rangle$

$0 \le t \le 1$

Solution

$F(r(t)) = \langle t, 0, t^3 \rangle$, $\frac{dr}{dt} = \langle 1, 2t, 3t^2 \rangle$

then

$$\int_0^1 F(r(t)) \cdot \frac{dr}{dt}\, dt = \int_0^1 \langle t, 0, t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt =$$

$$= \int_0^1 (t + 3t^5)\, dt = \left(\frac{t^2}{2} + \frac{3t^6}{6} \right) \Big|_{t=0}^{t=1} = \frac{1}{2} + \frac{3}{6} = \frac{1}{2} + \frac{1}{2} = 1$$

Definition:

This object

$$\int_C F \cdot dr$$ — is called

$\boxed{a \le t \le b}$

~~work~~ work done ~~by a force~~ in moving an object from the point $A = r(a)$ to the point $B = r(b)$

$$\boxed{\int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt}$$ — work

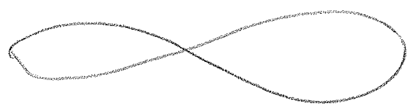
(Flow, circulation) and flux

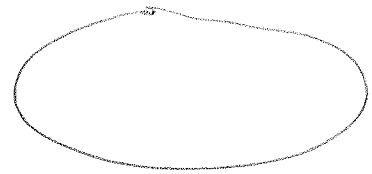
We are in 2D

$F = \langle M, N \rangle$, $r(t) = \langle g(t), h(t) \rangle$

$$\text{Flow} = \int_a^b F(r(t)) \cdot \frac{dr}{dt}$$ the same integral !!!

If the curve is simple, closed





on this pictures both are closed

but this

is simple. (other one is NOT!)

So $r(a)=r(b)$ then flow becomes

<u>circulation</u>

$$\boxed{\text{Circulation} = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt}$$

the same integral difference is that

$r(t)$ - is a closed curve

<u>Example</u>. Let $r(t) = \langle \cos t, \sin t \rangle$, $t = 0 .. 2\pi$

Find circulation over the field

~~$F\langle x^2, y^2 \rangle$~~ $F = \langle x, y \rangle$

Solution:

$$\int_0^{2\pi} F(r(t)) \cdot \frac{dr}{dt}\, dt = \text{circulation}$$

$F(r(t)) = \langle \cos t, \sin t \rangle$

$\frac{dr}{dt} = \langle -\sin t, \cos t \rangle$

$$\oint_0^{2\pi} F(r(t)) \cdot \frac{dr}{dt}\, dt = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt =$$

$$= \int_0^{2\pi} -\cos t \cdot \sin t + \sin t \cdot \cos t\, dt = 0.$$

Example: Let $F = \langle -y, x \rangle$ and

$r(t) = \langle \cos t, \sin t \rangle$ $\quad t = 0, \ldots \pi$

Find the flow.

Solution:

$F(r(t)) = \langle -\sin t, \cos t \rangle$

$$\int_0^{\pi} F(r(t)) \cdot \frac{dr}{dt}\, dt = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt =$$

$$= \int_0^{\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{\pi} dt = \pi$$

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Flux

(all curves must be simple and closed)

Let $F = \langle M, N \rangle$ and $r(t) = \langle g(t), h(t) \rangle$

$a \leq t \leq b$

$$\text{Flux} = \cancel{\int_a^b M(r(t))\, dh}$$

$$= \cancel{\int_a^b M \, dh}$$

$$\boxed{= \int_a^b M(r(t)) \frac{dh}{dt} \, dt - N(r(t)) \frac{dg}{dt} \, dt}$$

Example:

Let $F = \langle x, y \rangle$ and $r(t) = \langle \cos t, \sin t \rangle$

$0 \leq t \leq 2\pi$

Find Flux:

Solution:

$$F(r(t)) = \langle \cos t, \sin t \rangle$$

$$\frac{dr}{dt} = \langle -\sin t, \cos t \rangle$$

$$\text{Flux} = \int_0^{2\pi} \left[\cos t \cdot (\cos t) - \sin t \cdot (-\sin t)\right] dt =$$

$$= \int_0^{2\pi} \cos^2 t + \sin^2 t \, dt = 2\pi$$

different way of writing
circulation and flux

$\boxed{r(t) = \langle x(t), y(t) \rangle}$

$F = \langle M, N \rangle$

C – is a curve

$\cancel{\text{Flux} = \int M\,dx}$

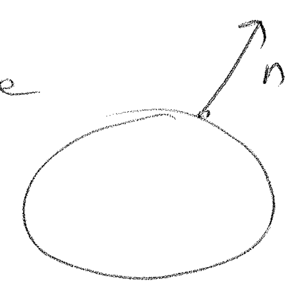
$$\text{Flow} = \int_C M\,dx + N\,dy = \int_C F \cdot dr = \int_C F \cdot T \cdot ds$$

minus

$$\text{Flux} = \int_C M\,dy - N\,dx = \int_C F \cdot \langle dy, -dx \rangle = \int_C F \cdot n\, ds$$

$$\boxed{ds = \cancel{dr}\ \frac{dr}{dt} \cdot dt}$$

n – is a normal (unit) vector to the curve



T – is a tangent unit vector to the curve