LECTURE 17

(1)

Recall: If we have two

Points A and B in 3D

And we have a path

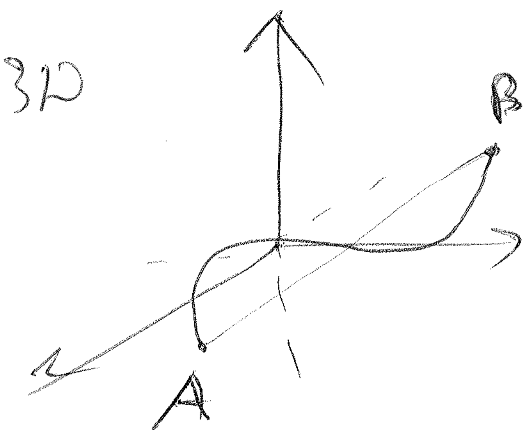
C [or curve] joining

these two points, and

also we have a vector field

$F = \langle M, N, P \rangle$, Then we learned

how to compute $\int_C F \cdot dr$ -



work, ~~circulation~~ flow.

$$\int_C F dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt} \, dt$$

dot product

and $r(t)$ is a parametrization of

a curve C.

But Do we have a formula

like:

$$\int_A^B F \cdot dr = \text{smth}(B) - \text{smth}(A)$$

like in 1 dimensional case?

In general we don't have ☹

But we do have this kind of formula if: there exists a scalar function $f(x,y,z) : 3D \longrightarrow \mathbb{R}$ such that $\boxed{\nabla f = F}$. In this case

$$\boxed{\int_A^B F \cdot dr = f(B) - f(A)}$$

regardless of the choices of the path C joining the points A and B.

This means that integral does not <u>depend on path</u>, so it is <u>path independent</u>.

<u>Example</u>: Let $A=(0,0,0)$, $B=(1,1,1)$ and $F=\langle yz, xz, xy\rangle = yz\cdot i + xz\cdot j + xy\cdot k$

Find $\int_A^B F\cdot dr$

Solution:

Note that if $f(x,y,z)=xyz$ then

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle yz, xz, xy\rangle = F.$$

Hence

$$\int_{(0,0,0)}^{(1,1,1)} F\cdot dr = \int_{(0,0,0)}^{(1,1,1)} \nabla f\cdot dr = f(1,1,1)-f(0,0,0) =$$

$$= 1.$$

We solved the problem!

But how do we find f -?

And how to check if that f exists?

If such f exists then the vector field F is called conservative.

Question: When does such F exist?

Answer: Here is the test:

Let $F = \langle M, N, P \rangle$

If $$\boxed{\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

Then such f exist (so the vector field F is conservative)

How to remember this formula?

We want f such that

$$\langle f_x, f_y, f_z \rangle = \langle M, N, P \rangle$$

Or

$f_x = M$

$f_y = N$

$f_z = P$

but $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$ therefore

for example, this $f_{xy} = f_{yx}$ implies that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and so on you can get others as well

But how to find such f -?

Here is the solution on some particular example:

Example:

Let $F = (2x+y+z)i + (2y+x+z)\cdot j + (2z+y+x)\cdot k$

Find ~~F~~ f such that $\nabla f = F$

Solution: if $\nabla f = F$ it means

① $\langle f_x, f_y, f_z \rangle = \langle 2x+y+z, 2y+x+z, 2z+y+x \rangle$

So it means:

① $f_x = 2x+y+z$

② $f_y = 2y+x+z$

③ $f_z = 2z+y+x$

① implies that

$f_x = 2x+y+z$ integrate over x we get

$f = \int (2x+y+z)\,dx = x^2+yx+zx+C$

when constant C depends on (y,z) So

$f = x^2+yx+zx+C(y,z)$

So we still need to find $C(y,z)$

Lets use ② and ③.

2 . So $f = x^2 + yx + zx + C(y,z)$

but $\to$ $f_y = 2y + x + z$ therefore

$$\left[x^2 + yx + zx + C(y,z)\right]'_y = 2y + x + z$$

$$\cancel{x} + C'_y(y,z) = 2y + x + z \quad \text{or}$$

$$C'_y(y,z) = 2y \cancel{+x} + z = 2y + z$$

Now let's integrate over y.

$$C(y,z) = \cancel{\int (2y + x + z)\,dy} = \int (2y + z)\,dy =$$

$= y^2 + zy + B$, where B is some constant which depends on $\cancel{z}$, So $B = B(x)$

Lets ~~collect~~ collect 1 and 2

1 - implies $f = x^2 + yx + zx + C(y,z)$, 2 implies $C(y,z) = y^2 + zy + B(\cancel{z})$

Here

$$f = x^2 + yx + zx + y^2 + zy + B(\cancel{z})$$

But still we need to find $B(x)$? Let's use 3.

<u>Recall:</u> $f = x^2 + yx + zx + y^2 + zy + B(z)$

1 and 2 $\Rightarrow$

3 $f_z = 2z + y + x$

$$\left[x^2 + yx + zx + y^2 + zy + B(z)\right]'_z = 2z + y + x$$

$$x + y + B'(z) = 2z + y + x$$

$$B'(z) = 2z \quad , \quad B(z) = \int 2z\,dz = z^2 + C$$

which does not depend on anything

So

$$\boxed{f(x,y,z) = x^2 + yx + zx + y^2 + zy + z^2}$$

<u>Remark:</u> Such function f is called

<u>Potential</u> function for vector field F

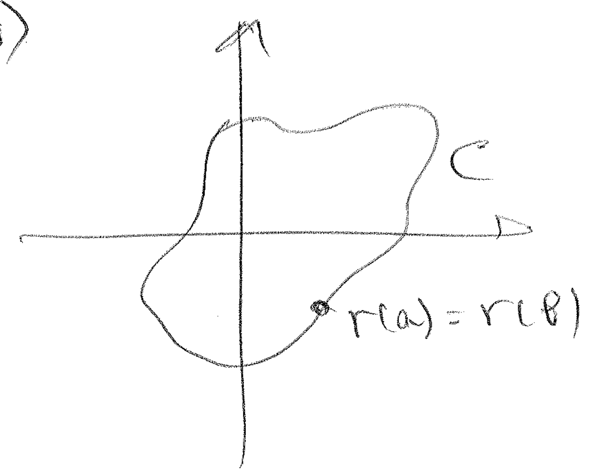
Green's formula

[flux, cirulation]

Sometimes is called outward flux

Let C be closed simple curve | 2D

$F = \langle M, N \rangle$



Then we know that

$$\text{flux} = \int_C F \cdot n\, ds =$$

$$= \int_a^b F(r(t)) \cdot \left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle dt$$

this is the way how do we compute this ~~for~~ flux.

$r(t)$ - is a parametrization of a curve

$a \le t \le b$

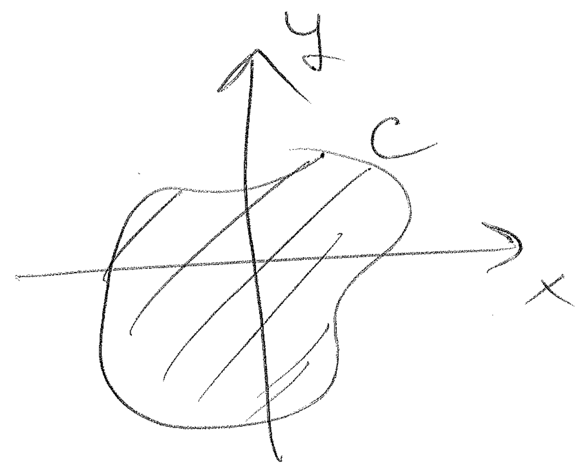
$r(a) = r(b)$

But there is a simple way (Greens formula)

$$\int_C F \cdot n\, ds = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

interior of a curve (everything inside)

So one integral becomes double integral over the domain enclosed ~~By~~ by the cure C.



Circulation:

$$\text{circulation} = \int_C F \cdot T\,ds = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

counterclockwise

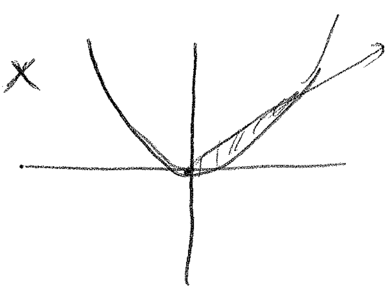
domain enclosed by a curve C.

Examples:

Let $F(x,y) = \langle tg(\cos x) + y,\ ctg(\sin y) - x \rangle$

~~and let~~ Find the circulation over the boundary of $y = x^2$ and $y = 2x$

Domain bounded by these curves



Solution:

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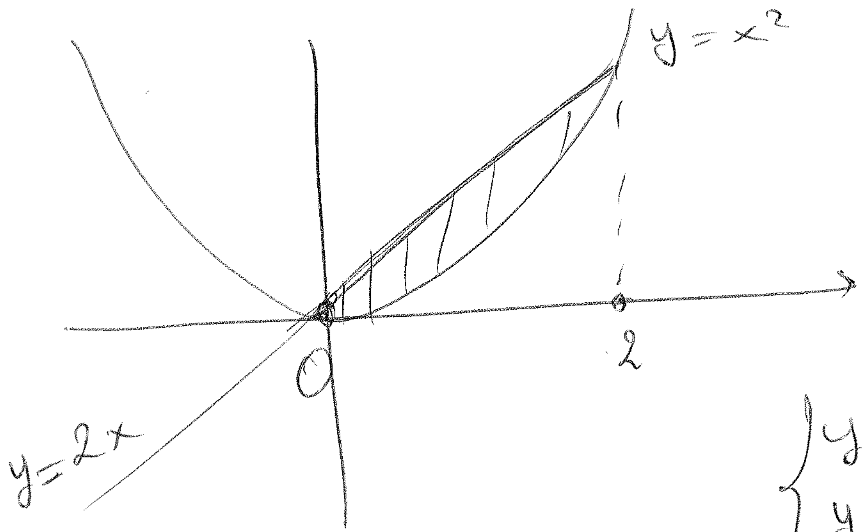

$$\begin{cases} y = x^2 \\ y = 2x \end{cases} \Rightarrow 2x = x^2 \quad x = 2$$

$$\text{circulation} = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy =$$

$$M = \langle \underbrace{tg(\cos x) + y}_{M}, \ \underbrace{ctg(\sin y) - x}_{N} \rangle$$

$$= \iint [-1 - (+1)]\,dx\,dy = -2 \cdot \iint dx\,dy = -2 \cdot \iint dy\,dx =$$

this order is better

$$= -2 \int_0^2 \int_{x^2}^{2x} 1\,dy\,dx = -2 \int_0^2 (2x - x^2)\,dx = -2 \left[x^2 - \frac{x^3}{3} \right]_0^2 =$$

$$= -2 \left[4 - \frac{8}{3} \right] = \boxed{-\frac{8}{3}}$$