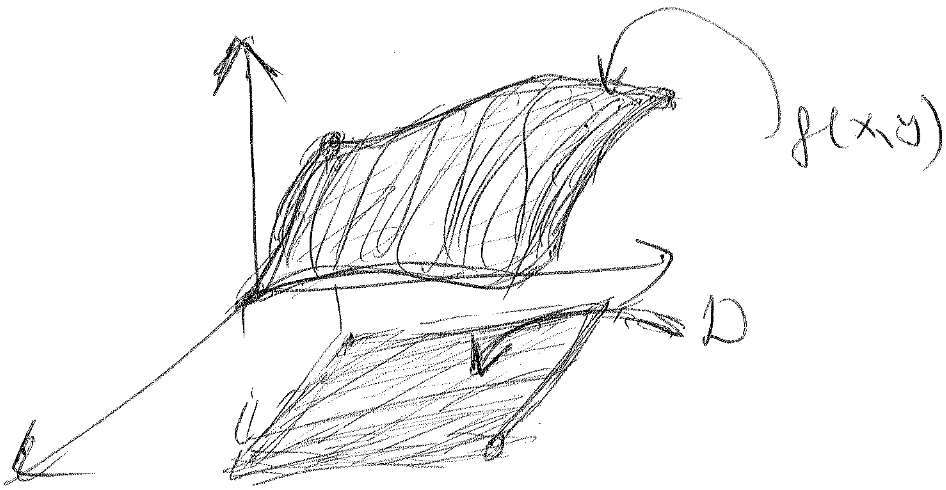
LECTURE 18

Integral of the function over the surface.

Let $F(x,y,z)$ be a function in 3D

(Example: ~~$F(x,y,z) = x^2y^2+z^2$~~) ($F(x,y,z) = xyz$)

~~And~~. So this function is defined in 3D but we would like to restrict this function on the surface. Suppose this surface is $f(x,y)$ (Example $f(x,y) = x^2+y^2$)



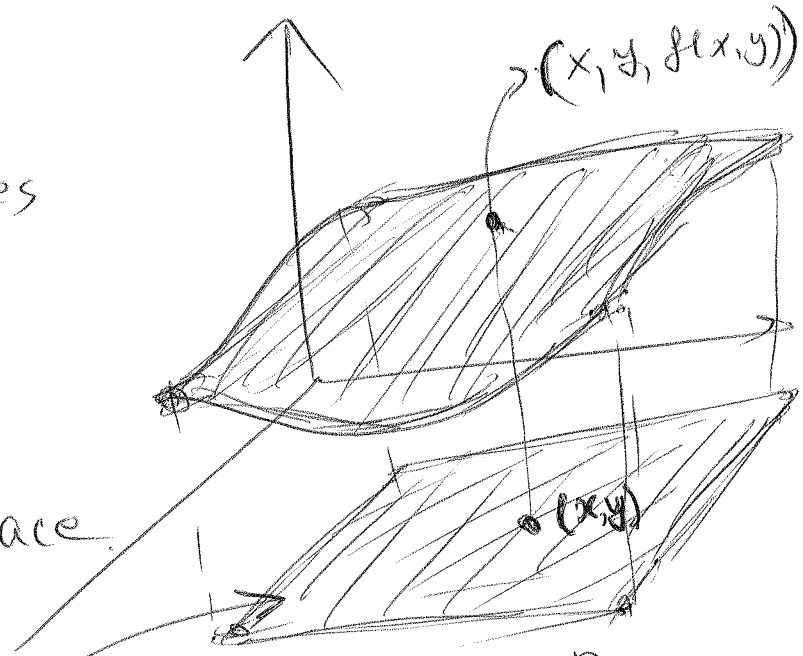
So any point on the surface has a coordinates

$$\boxed{(x, y, z) = (x, y, f(x,y))}$$

where (x, y) are in D

So this object

$F(x, y, f(x,y))$ becomes a function of two variables and it is defined on the surface



this is your domain D

We would like to define integral of the function $F(x,y,z)$ over the surface

$$\boxed{\iint F \, d\sigma}$$

What is this???

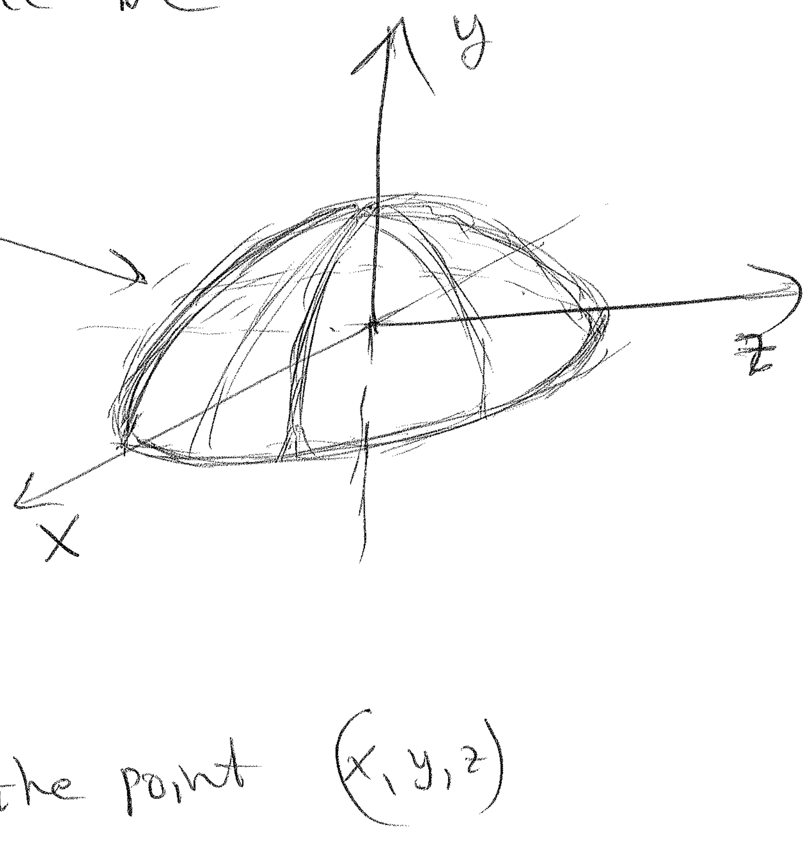
If the surface is parametrized as $f(x,y)$ where (x,y) belong to D

then

$$\boxed{\iint F\,d\sigma = \iint_{D} F(x,y,f(x,y))\cdot\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy}$$

Example: Let $F(x,y,z)=x^2+y^2+z^2$

and Let our surface be this upper half of the sphere of radius $= 3$

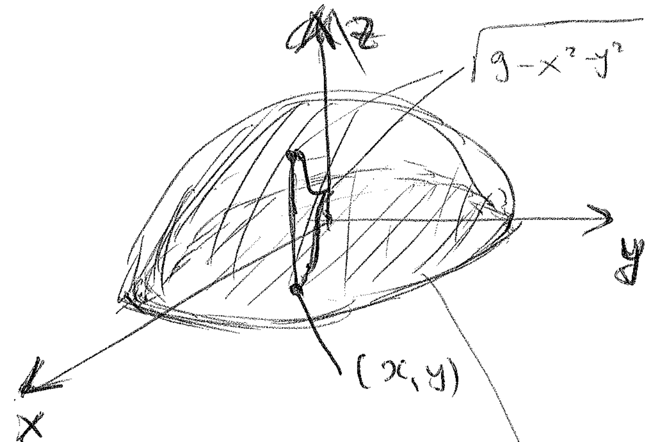


Find $\iint F\,d\sigma$

Solution: If the point (x,y,z) belongs to that sphere then

$x^2+y^2+z^2=9$ or $z=\pm\sqrt{9-x^2-y^2}$

So $f(x,y)=\sqrt{9-x^2-y^2}$



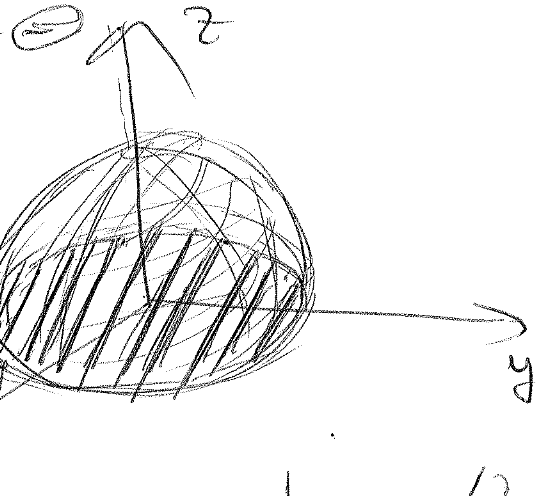
then

$\iint F\,d\sigma =$

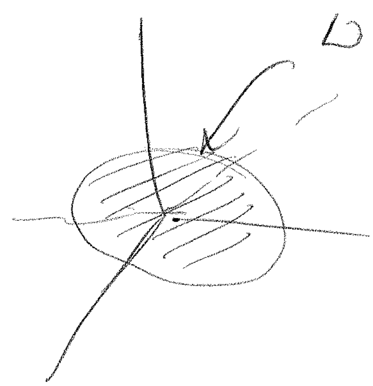
and D is a shadow of that surface onto XY plane

So D : $\boxed{x^2+y^2\le 9}$

$$= \iint_D F(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy =$$



$$= \iint_D \left[x^2+y^2+\left(\sqrt{9-x^2-y^2}\right)^2\right]\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy$$



$$f_x = \frac{-x}{\sqrt{9-x^2-y^2}}$$

$$f_y = \frac{-y}{\sqrt{9-x^2-y^2}}$$

Therefore:

$$\iint\limits_{r=3} \left[x^2+y^2+(9-x^2-y^2)\right]\sqrt{1+\frac{x^2}{9-x^2-y^2}+\frac{y^2}{9-x^2-y^2}}\,dx\,dy =$$

$$= \iint\limits_{r=3} 9\cdot\sqrt{\frac{9}{9-x^2-y^2}}\,dx\,dy = \iint\limits_{r=3} \frac{27}{\sqrt{9-x^2-y^2}}\,dx\,dy$$

Let's go to polar coordinates

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 3$$

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{array}}$$

then

$$\int_0^{2\pi}\int_0^3 \frac{27}{\sqrt{9-r^2}}\,r\,dr\,d\theta$$

Let's do change of variables $9-r^2 = u$

$-2r\,dr = du$

$$\int_0^3 \frac{27\cdot r}{\sqrt{9-r^2}}\,dr = -\int \frac{27}{\sqrt{u}}\cdot\frac{du}{2} = -\frac{27}{2}\int u^{-1/2}\,du =$$

$$= -\frac{27}{2}\cdot 2\cdot u^{1/2}\Big| = -27\cdot\sqrt{9-r^2}\Big|_{r=0}^{r=3} = \boxed{81}$$

Therefore,

$$\int_0^{2\pi}\int_0^{3} \cancel{9}\,\frac{27}{\sqrt{9-r^2}}\, r\,dr\,d\theta =$$

$$= \int_0^{2\pi} 81\,d\theta = \boxed{162\cdot\pi}$$

Thus $\iint F\,d\sigma = 162\pi$

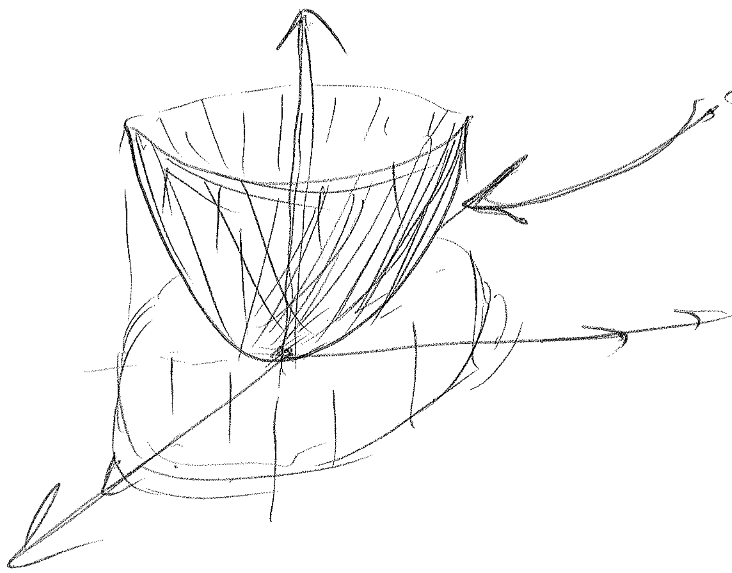
Remark: If $F=1$ then

$\iint d\sigma$ — is the area

of the surface.

Exercise: Let $f(x,y)=x^2+y^2$ is the surface over the domain $x^2+y^2\leq 1$ of the XY plane. Find the area of it.

Picture is the following:



Surface integral in different parametrization.

Sometimes surface might be parametrized in a different way say

$r(u,v) = \langle u\cos v,\ u\cdot\sin v,\ 1-u^2 \rangle$

$0 \le u \le 1$,

$0 \le v \le 2\pi$

How do you construct this kind of surface?

$0 \le u \le 1$
$0 \le v \le 2\pi$

you pick u and v and you find point $r(u,v)$ in 3D and so on

$\boxed{\text{this is your domain } D}$

Then you can compute the same integral $\iint F\,d\sigma$ in a different way:

$$\iint F\,d\sigma = \iint_D F(r(u,v))\,|r_u' \times r_v'|\,du\,dv$$

Now what is $F(r(u,v))$ - ?

$F = F(x,y,z)$

$r(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle$

Therefore

$$F(r(u,v)) = F\big(f(u,v), g(u,v), h(u,v)\big)$$

and what is $|r'_u \times r'_v|$ -?

this is <u>cross</u> product of derivatives

<u>Example:</u>

Let $F(x,y,z) = z$ and

$r(u,v) = \langle u\cos v,\ u\cdot\sin v,\ 1-u^2 \rangle$

$$\boxed{\begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{array}}$$

this is D

Find $\iint F\, d\sigma$

<u>Solution:</u> $\iint F\, d\sigma = \iint_D F(r(u,v))\, |r'_u \times r'_v|\, du\, dv$

$r'_u = \cancel{\langle \cos v, \sin v, 0 \rangle} = \langle \cos v, \sin v, -2u \rangle$

$r'_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$

therefore:

$$r'_u \times r'_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} =$$

$$= i\left(+2u^2\cos v\right) - j\left(-2u^2\sin v\right) + k\left(\underbrace{u\cos^2 v + u\sin^2 v}_{= u}\right) =$$

$$= \langle +2u^2\cos v,\ 2u^2\sin v,\ u\rangle$$

Therefore:

$$|r'_u \times r'_v| = \sqrt{(2u^2\cos v)^2 + (2u^2\sin v)^2 + u^2} =$$

$$= \sqrt{4u^4+u^2} = u\sqrt{1+4u^2}$$

Therefore $\iint F\,d\sigma = \iint_D F(r(u,v))\cdot u\sqrt{1+4u^2}\,du\,dv =$

= Note that $F(r(u,v)) = 1-u^2$ =

= $\iint_D (1-u^2)\,u\sqrt{1+4u^2}\,du\,dv = \int_0^{2\pi}\int_0^1 (u-u^3)\sqrt{1+4u^2}\,du\,dv$

bad integral