LECTURE 19

Stokes theorem

Let $F(x,y,z) = \langle M, N, P \rangle$

and suppose we have a surface

Stoke's theorem is the following formula

$$\boxed{\iint \nabla \times F \cdot n \, d\sigma = \oint_C F \cdot dr}$$

boundary of the surface

Let me explain what are these symbols.

∇ - is a symbolic expression of

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

F is a vector field

$$F(x,y,z) = \langle M(x,y,z),\ N(x,y,z),\ P(x,y,z) \rangle =$$

$$= i \cdot M(x,y,z) + j \cdot N(x,y,z) + k \cdot P(x,y,z)$$

$$\underline{\underline{\nabla \times F}} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} =$$

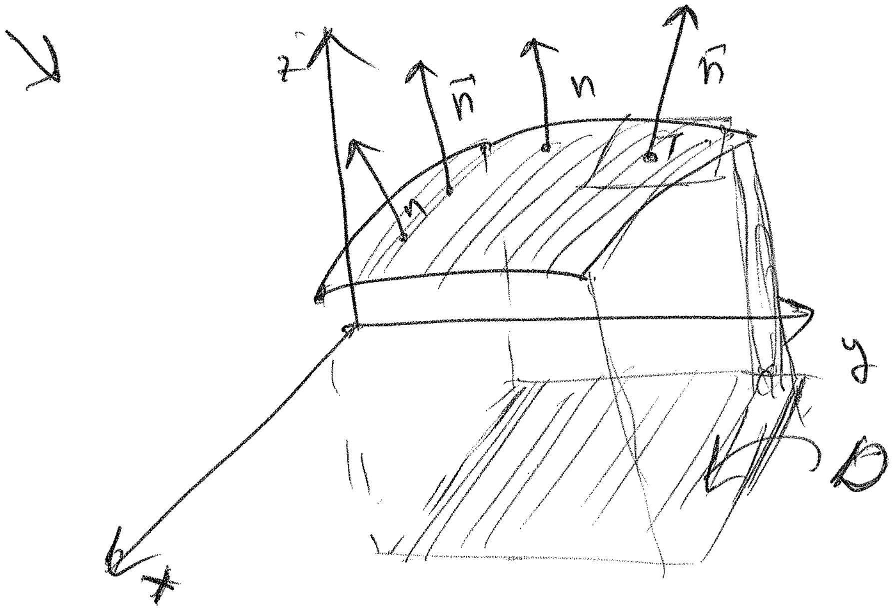
$$= i\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - j\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + k\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \underline{\underline{\left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle}}$$

what is $\vec{n}$? it is a vector of

3 variables $\vec{n}(x,y,z)$ in 3D.

It is a perpendicular vector to the surface



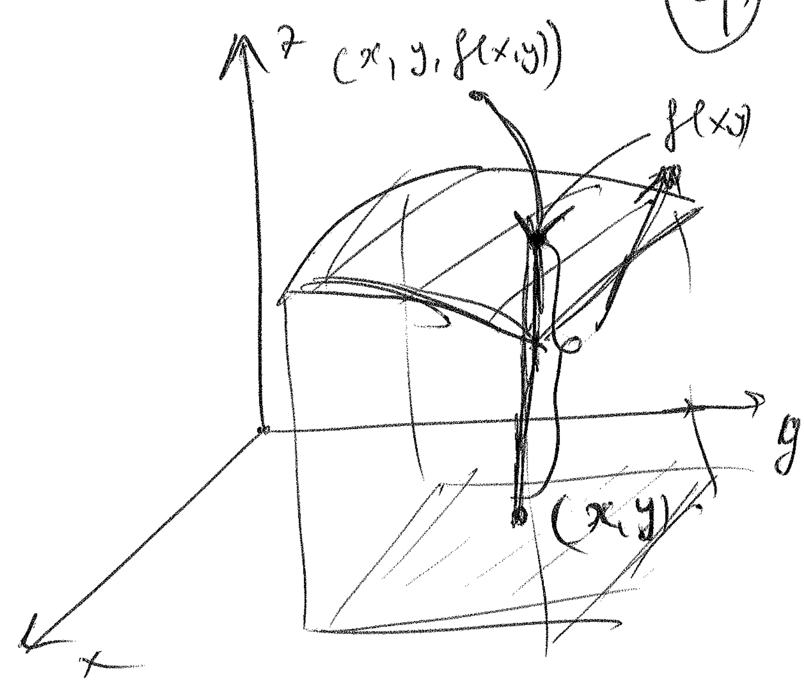
Moreover $\vec{n}$ is a unit vector (Which means that it has a unit ~~Be~~ magnitude)

<u>How to find $\vec{n}$?</u>

Surface might be given <u>explicitly</u> like $f(x,y) = x^2 + y^2$

So the points on the surface are the following $(x, y, f(x,y))$ where (x,y) are in <u>D</u>

Surface might be given ~~explicitly~~ **implicitly**



Example:

$x^2 + y^2 + z^2 = 1$

this is a sphere of radius 1.

this sphere is given implicitly and you can ~~find~~ express it explicitly like: $z^2 = 1 - x^2 - y^2$ and

$$z = \pm\sqrt{1 - x^2 - y^2}$$

Ok good. any surface which is given explicitly you can write in a implicit way, Example. $f(x,y) = x^2 + y^2$

then ~~$f(x,$~~ $\boxed{z - x^2 - y^2 = 0}$

Because $\boxed{f(x,y) = z}$

So if the surface is given implicitly like $S(x,y,z)=0$

(for example the sphere $x^2+y^2+z^2=1$ you can write as $\boxed{x^2+y^2+z^2-1=0}$ $S(x,y,z)=0$)

then $\vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{\langle S_x, S_y, S_z \rangle}{|\nabla S|}$

Example: Let $2x+3y+10z=4$

Find $\vec{n}$.

Solution: $2x+3y+10z-4=0$ Therefore

$$\vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{\langle 2, 3, 10 \rangle}{\sqrt{2^2+3^2+10^2}} = \frac{\langle 2, 3, 10 \rangle}{\sqrt{113}}$$

Example.

Let $F = \langle y, x, y^2 \rangle$

Find $\nabla \times F$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & y^2 \end{vmatrix} =$$

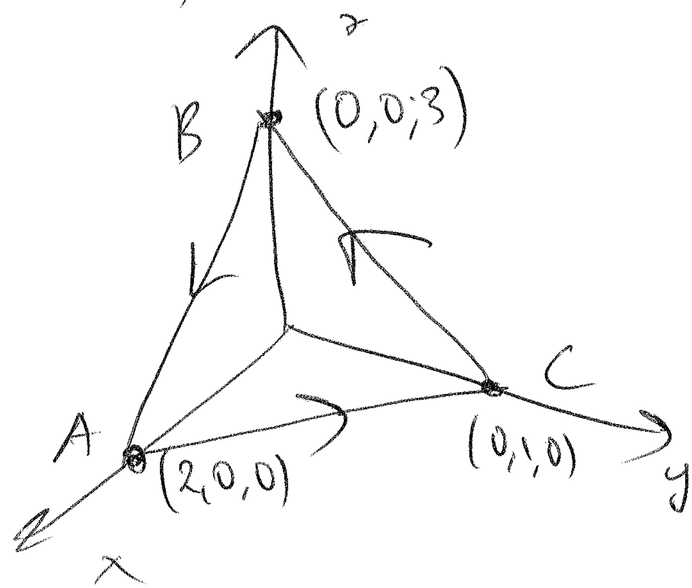
$$= i(2y) - j(0) + k(0) = \underline{\langle 2y, 0, 0 \rangle}$$

Ok. and then you know how to compute this integral

$$\boxed{\iint \nabla \times F \cdot n \, d\sigma}$$

Example: (from the book but with different solution)

Let



Find circulation along the curve joining the points $A \to C \to B$ (~~countercl~~ where ~~$F = \langle y, z^2, y^2\rangle$~~

$F = \langle y, z^2, y^2 \rangle$

Solution:

$$\oint_{\triangle} F\,dr \overset{\text{Stokes theorem}}{=} \iint_{\blacktriangle} \nabla \times F \cdot n \, d\sigma$$

circulation

Solution

Before we start solving

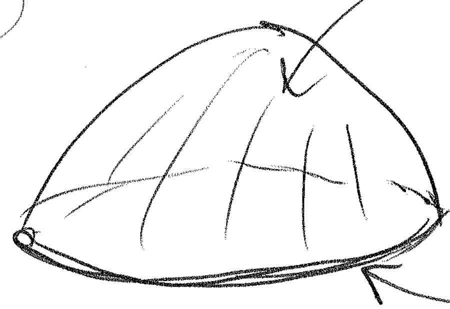
Let's make a little remark:

In the stoke's theorem

$$\iint \nabla \times F \cdot n \, d\sigma = \oint_{C} F \, dr$$

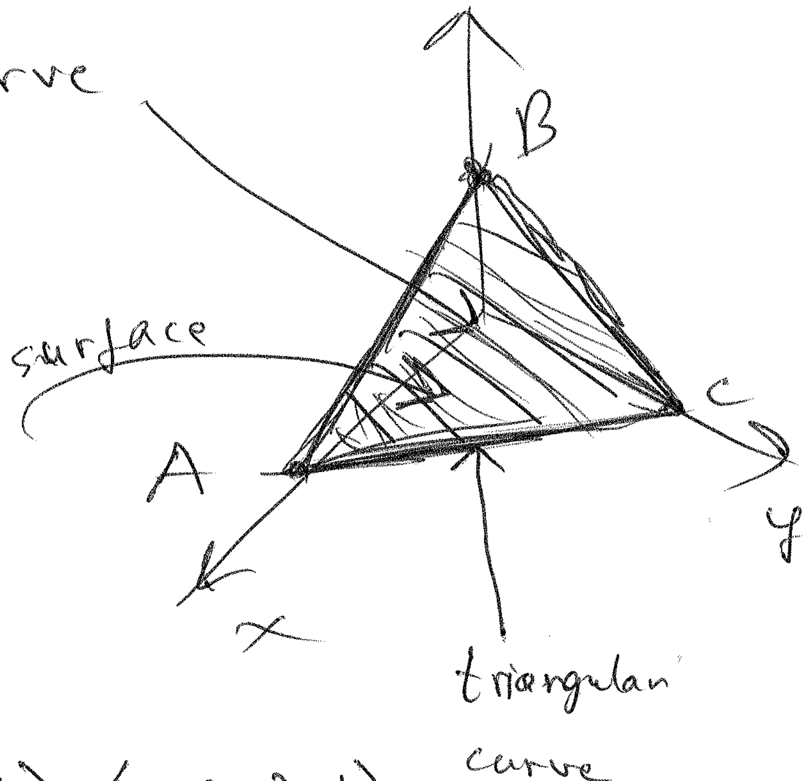
this is a surface and C is a boundary of that surface

example:



Solution (continuation)

$\iint_{\triangle} \nabla\times F\cdot n\, d\sigma$ becomes double integral over the surface attached to the triangular curve



Note that ~~We computed~~ ~~that~~ $\nabla\times F = \cancel{\langle 2y, 0, 1\rangle}\ \langle 2y-2z, 0, -1\rangle$

$$\cancel{n} = \frac{\vec{AB}\times\vec{AC}}{|\cancel{\vec{AB}\times\vec{AC}}|} = \frac{\begin{vmatrix} i & j & k\\ -2 & 0 & 3\\ -2 & 1 & 0\end{vmatrix}}{\cancel{\text{magnitude}}} =$$

$$= i(-3) - j(6) + k(-2) = \langle -3, -6, -2\rangle$$

Now

$$n = \frac{\langle -3, -6, -2 \rangle}{\sqrt{9+4+36}} = \frac{\langle -3, -6, -2 \rangle}{7} =$$

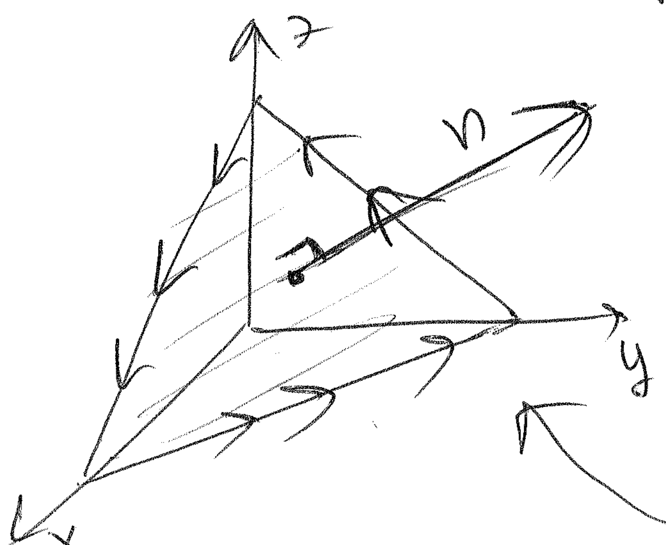
$$= \left\langle -\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7} \right\rangle$$

Now instead of $\vec{n} = \left\langle -\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7} \right\rangle$

I would like to pick opposite

vector $\vec{n} = \left\langle \frac{3}{7}, \frac{6}{7}, \frac{2}{7} \right\rangle$ (I changed the signs)

why? Because of the convention



Remark: If the circulation is computed counter clockwise then this must be a direction of normal vector therefore we have

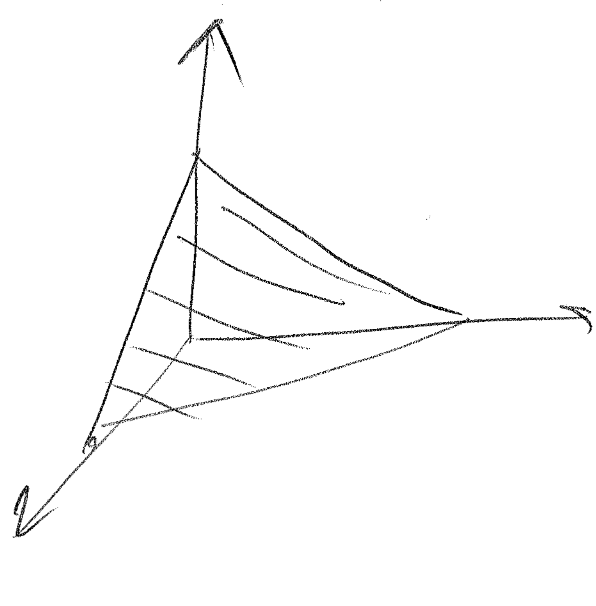
$$\cancel{\nabla \times F \cdot n = \langle 2y, 0, 1 \rangle \cdot \left\langle \frac{3}{7}, \frac{6}{7}, \frac{2}{7} \right\rangle}$$

$$\nabla \times F \cdot n = \langle 2y - 2z, 0, -1 \rangle \cdot \left\langle \frac{3}{7}, \frac{6}{7}, \frac{2}{7} \right\rangle =$$

$$= \frac{1}{7}\left(6y - 6z - 2\right) \quad \text{good.}$$

Now $\displaystyle\iint \frac{1}{7}(6y - 6z - 2)\, d\sigma$ it is

an integral of the function over the surface. So we have



We need to parametrize the surface (Find the equation of the plane)

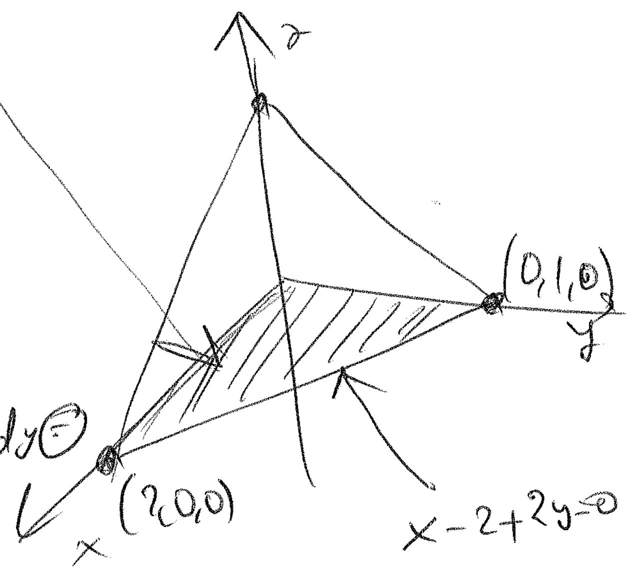
$$3(x-2)+6y+2z=0 \Rightarrow z=\frac{-6y-3x+6}{2}$$

$$f(x,y)=\frac{-6y-3x+6}{2}$$

$$\iint \frac{1}{7}(6y-6z-2)\,d\sigma = \iint \frac{1}{7}\left(6y-6(-6y-3x+6)-2\right)\cdot$$

shadow of

$$\cdot\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy = \iint$$



$$\frac{1}{7}(42y+18x-38)\sqrt{1+\frac{9}{4}+9}\,dx\,dy =$$

$$= \iint \frac{1}{7}(42y+18x-38)\,\frac{7}{2}\,dx\,dy = \iint(21y+9x-19)\,dx\,dy =$$

$$= \int_0^1\int_0^{2-2y}(21y+9x-19)\,dx\,dy = \boxed{\text{compute yourself}}$$