Lecture 3: $\boxed{\text{Section 12.5}}$

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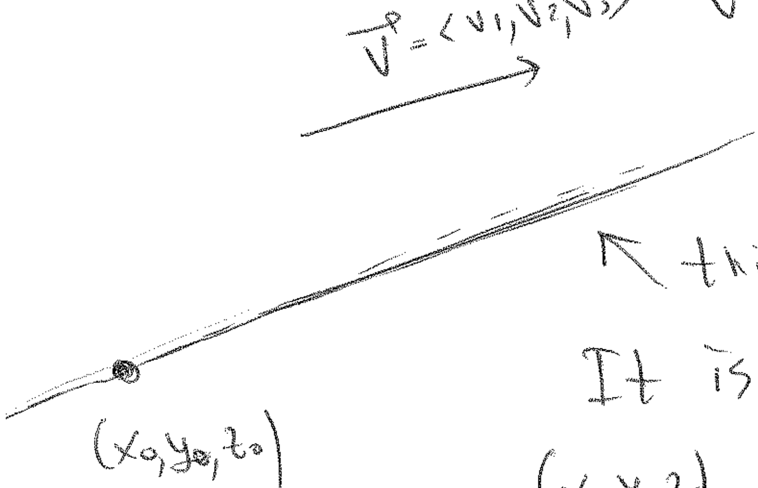
lines in 3D.

In order to determine a line in 3D we need to know direction of this line and a point through which it passes

$\vec{v}$ - direction (any vector)



← this is a line

It is a set of points (x, y, z) such that

$$\begin{cases} x = x_0 + t \cdot v_1 \\ y = y_0 + t \cdot v_2 \\ z = z_0 + t \cdot v_3 \end{cases}$$

← Parametric equation

$-\infty < t < \infty$, t - is a parameter

Vector equation: $\langle x, y, z \rangle =$

$= \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle = \langle x_0, y_0, z_0 \rangle + \langle tv_1, tv_2, tv_3 \rangle =$

$= \langle x_0, y_0, z_0 \rangle + t \cdot \langle v_1, v_2, v_3 \rangle = \vec{r}_0 + t \cdot \vec{v}$

where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

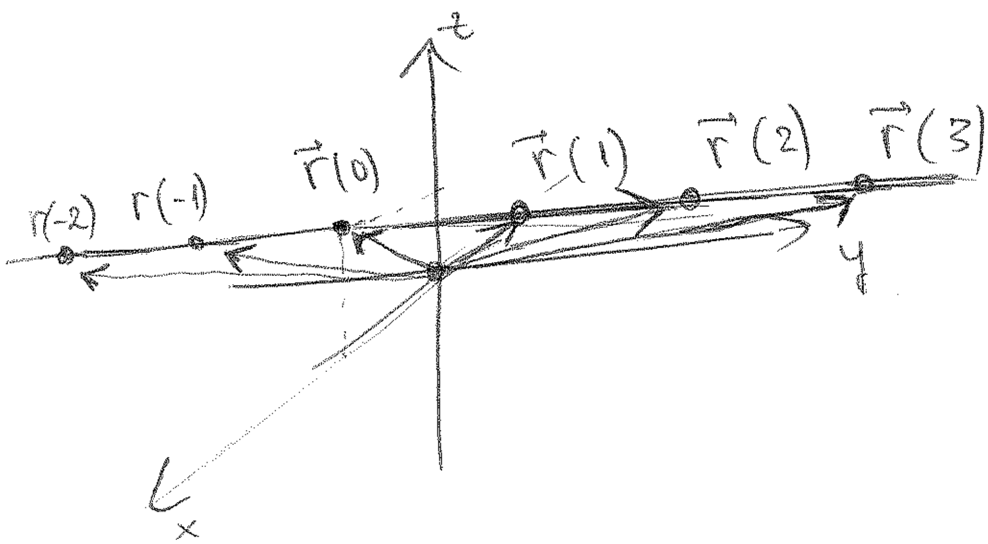
IF instead of $\vec{v}$ we consider $10 \cdot \vec{v}$ then we get the same equation of the line.

$*$ $$\boxed{r(t) = \vec{r_0} + t \cdot v}$$

our magic formula

Example: let $\vec{r_0} = \langle 1, 0, 1 \rangle$ and
$\vec{v} = \langle 0, 1, 0 \rangle$

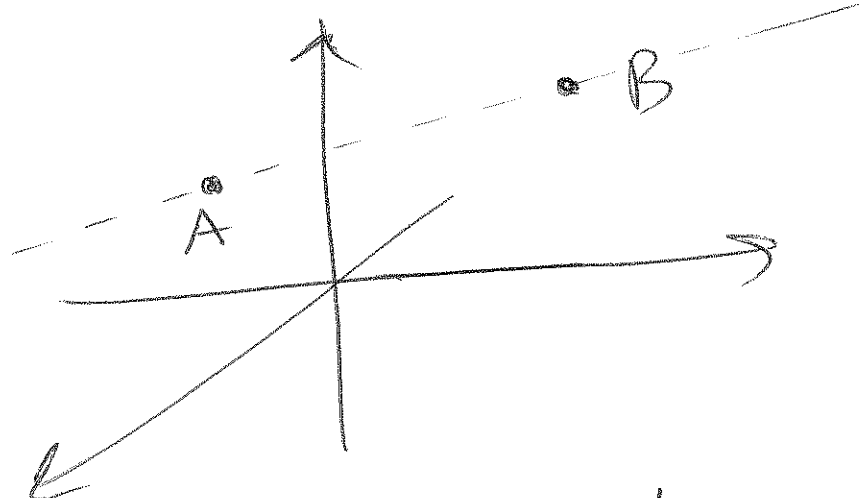
then $\vec{r}(t) = \vec{r_0} + t \cdot \vec{v} = \langle 1, 0, 1 \rangle + t \cdot \langle 0, 1, 0 \rangle =$
$= \langle 1, t, 1 \rangle \quad -\infty < t < \infty$



Example:

Find the equation of a line which passes through the points

$A = (x_0, y_0, z_0)$, $B = (x_1, y_1, z_1)$



Solution: direction is obtained by the vector $\overrightarrow{AB} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

For the ~~starting~~ initial point let's pick

$A = (x_0, y_0, z_0)$

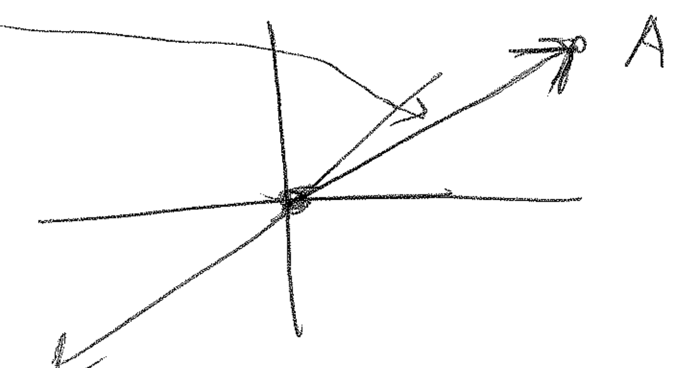
Therefore $\vec{r}(t) = r_0 + t \cdot v =$

$$= \overrightarrow{OA} + t \cdot \overrightarrow{AB} =$$

Important formula

$$= \langle x_0, y_0, z_0 \rangle + t \cdot \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

<u>Remark:</u> If $A = (x_0, y_0, z_0)$ is a point and we want to create a vector with starting point O and endpoint A



then we just write

$$\overrightarrow{OA} = \langle x_0, y_0, z_0 \rangle$$

If $\vec{r}(t) = r_0 + t \cdot v$

then v - is called velocity of the parametrisation of the line

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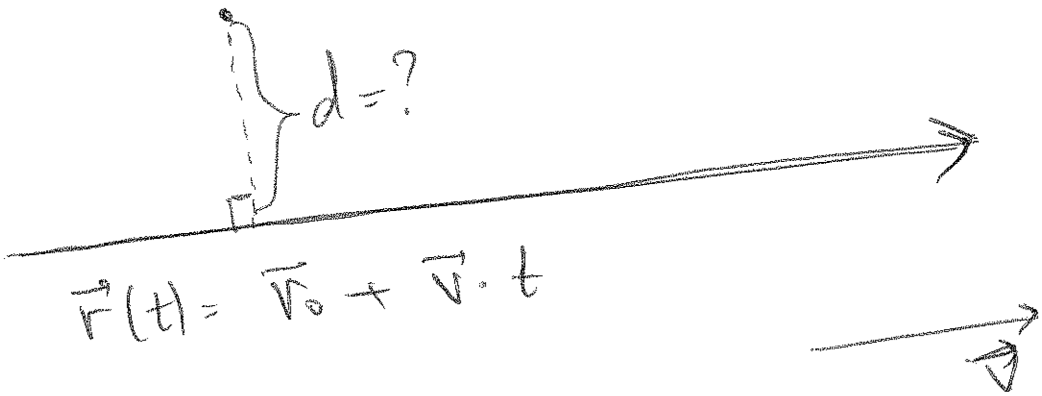
If we consider

$$\vec{r}(t) = r_0 + t \cdot \frac{v}{|v|}$$

then it is the same line (direction is the same) but magnitude of velocity is different

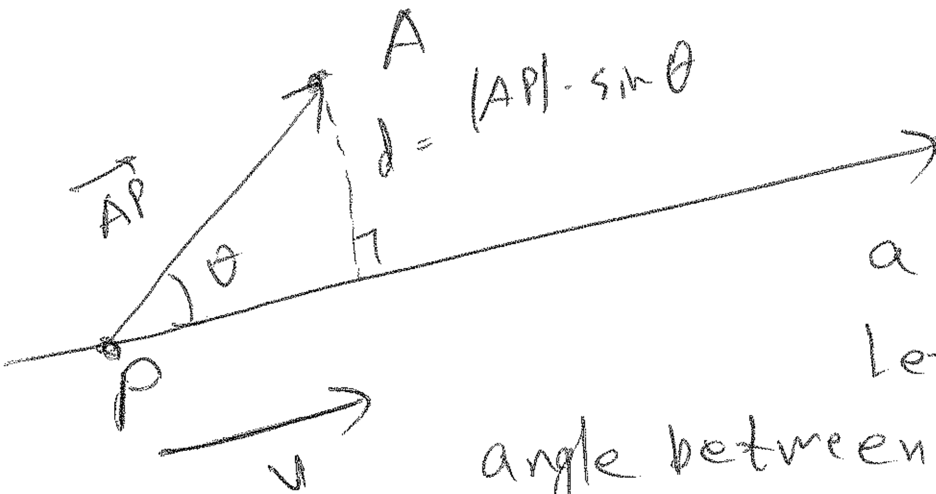
The distance from a point to a line in 3D.

$A = (x_0, y_0, z_0)$



How to derive a formula?

Solution: 1) Pick <u>any</u> point on the line say P.



2) Create a vector $\vec{PA}$ let θ be an angle between $\vec{PA}$ and $\vec{v}$.

then distance $= |\vec{AP}| \cdot \sin\theta = \dfrac{|\vec{AP}| \cdot |v| \cdot \sin\theta}{|v|}$

$$= \frac{|\vec{AP} \times v|}{|v|}.$$

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Thus:

$$\boxed{\text{distance} = \frac{|\vec{PA} \times v|}{|v|}}$$

Example: let $\vec{r}(t) = \langle 1,1,1\rangle + t\cdot\langle 1,2,3\rangle$ and let $A = \langle 0,0,0\rangle$. Find the distance between A and the line $\vec{r}(t)$.

Solution: lets choose

~~distance =~~ $P = \langle 1,1,1\rangle$.

then $\vec{PA} = \langle 0,0,0\rangle - \langle 1,1,1\rangle = \langle -1,-1,-1\rangle$

$$\text{distance} = \frac{|\vec{PA} \times v|}{|v|}.$$

$$\vec{PA} \times v = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = i\begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix}$$

$\vec{PA} \times v = i(-3+2) - j(-3+2) + k(-2+1) =$

$= -i + j - k = \langle -1, 1, -1 \rangle$

Therefore $|\vec{PA} \times v| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$|v| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Hence distance $=$ $\dfrac{|\vec{PA} \times v|}{|v|} =$

$= \dfrac{\sqrt{3}}{\sqrt{14}}.$

Equation of a Plane

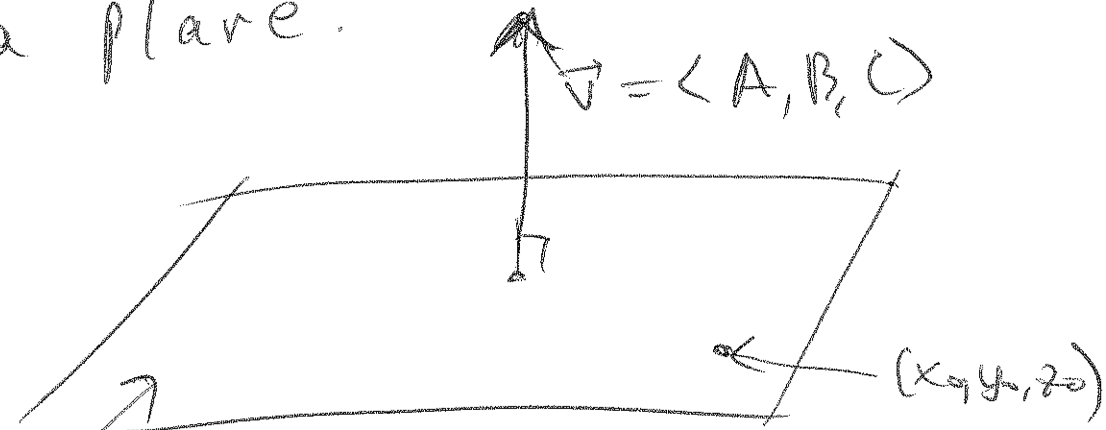
The set of points which satisfy the following equation is ~~and~~ a plane

$Ax + By + Cz = D$

(Another way of writing this equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

This is a plane which passes through the point (x_0, y_0, z_0).

the numbers A, B, C create a <u>vector</u> $\langle A, B, C\rangle$ which is perpendicular to a plane.



$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

Sometimes $\langle A, B, C\rangle$ is called <u>normal vector</u>.

<u>Example:</u> Let $2x - z = 0$.

Find a vector which is perpendicular to ~~the~~ plane.

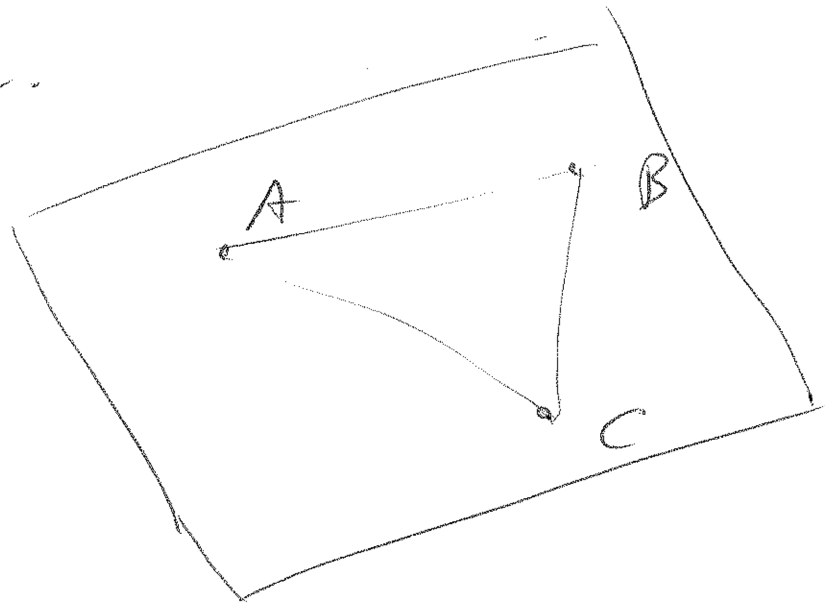
Solution: $\vec{n} = \langle 2, 0, -1\rangle$

Example (Important)

Let $A = \langle 0,0,0 \rangle$, $B = \langle 0,1,2 \rangle$

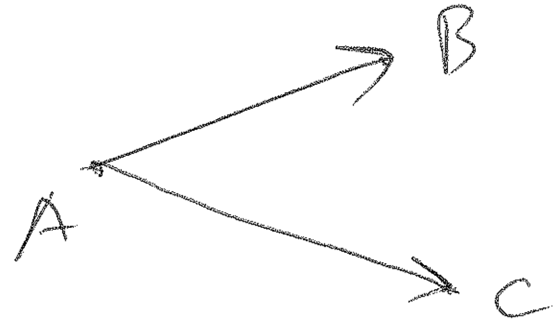
$C = \langle 1,2,2 \rangle$.

Q. Find the equation of a plane which contains the points A, B, C.



Solution:

1 step: we need to find a vector which is perpendicular to a plane



Let's create a vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then we know that $\overrightarrow{AB} \times \overrightarrow{AC}$ is a vector which is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and Hence it is perpendicular to the plane.

$$\overrightarrow{AB} = \langle 0, 1, 2 \rangle$$

$$AC = \langle 1, 2, 2 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -2i + 2j - k = \langle -2, 2, -1 \rangle$$

So instead of normal vector we can choose $\vec{n} = \langle -2, 2, -1 \rangle$

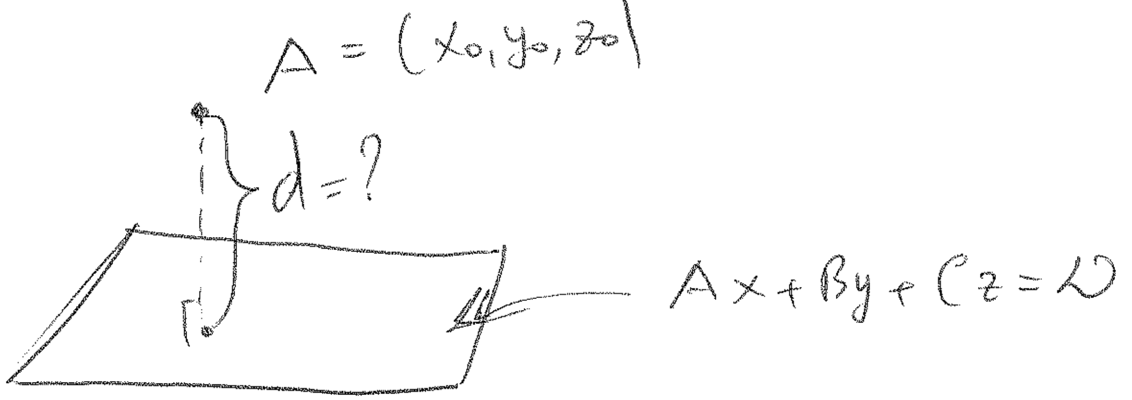
2 step: we need to find a point which belongs to a plane. For example choose $A = \langle 0,0,0 \rangle$ (Any other choises works as well)

Therefore

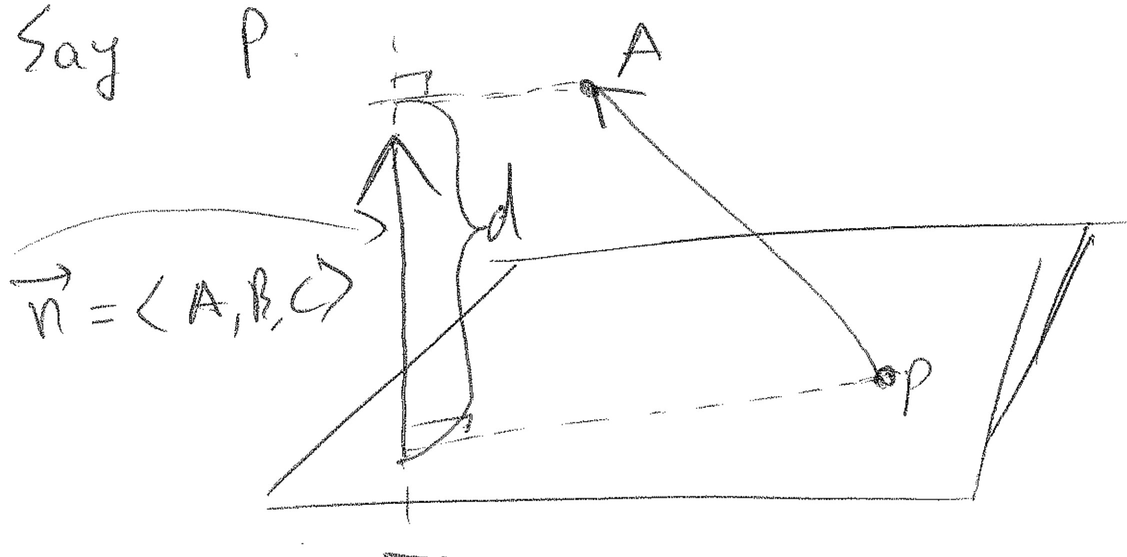
$$-2(x-0) + 2(y-0) - 1(z-0) = 0$$

or $\boxed{-2x + 2y - z = 0}$.

The distance between a point and the plane.



Pick any point in the plane say P.



Create a vector $\vec{PA}$. Let's project $\vec{PA}$ onto vector $\vec{n}$

So consider $\mathrm{Proj}_{\vec{n}} \vec{PA} =$

$= \left(\dfrac{\vec{PA} \cdot \vec{n}}{|\vec{n}|^2} \right) \cdot \vec{n}$. It is clear

that $|\mathrm{Proj}_{\vec{n}} \vec{PA}| =$ distance

Therefore:

$$|\mathrm{Proj}_{\vec{n}} \vec{PA}| = \left| \frac{\vec{n} \cdot \vec{PA}}{|n|^2} \cdot |n| \right| = \frac{|\vec{n} \cdot \vec{PA}|}{|n|^2} \cdot |n| = \frac{|\vec{n} \cdot \vec{PA}|}{|\vec{n}|}$$

Hence

$$\text{distance} = \frac{|\vec{n} \cdot \vec{PA}|}{|\vec{n}|}$$

Example:

let $x+y+z=0$ be a plane

and let $A = \langle 1,1,1 \rangle$

Find the distance between A and the plane:

Solution:

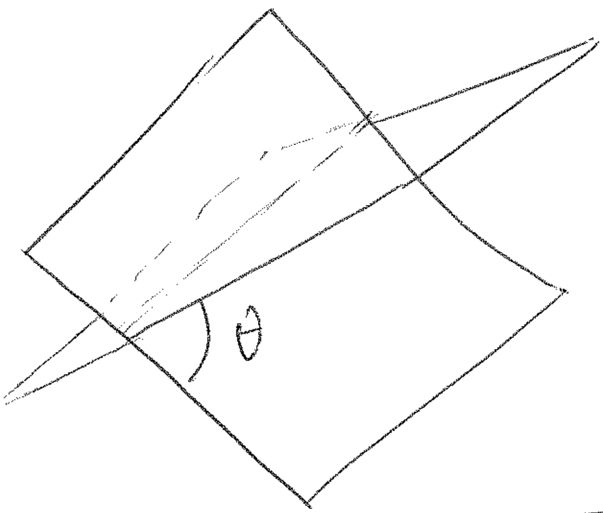
~~dis~~ Pick any point which belongs to a plane, say $(0,0,0) = P$

then $\vec{PA} = \langle 1,1,1 \rangle$. $\vec{n} = \langle 1,1,1 \rangle$

$$\text{distance} = \frac{|\vec{PA} \cdot \vec{n}|}{|\vec{n}|} = \frac{1+1+1}{\sqrt{1^2+1^2+1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Angles between planes

let $x+y+z=0$ are two planes
$x+y+2z=0$



Find the angle between these planes

~~$\vec{n}_1 = \langle 1,1,1\rangle$~~

Solution: $\vec{n}_1 = \langle 1,1,1\rangle$ is a perpendicular vector to the first plane. $n_2 = \langle 1,1,2\rangle$ is a perpendicular vector to the second one.

then angle between these vectors ~~is~~ equals ~~to~~ to the angle between planes

~~$\theta = \arccos$~~

$$n_1 \cdot n_2 = |n_1| \cdot |n_2| \cdot \cos\theta$$

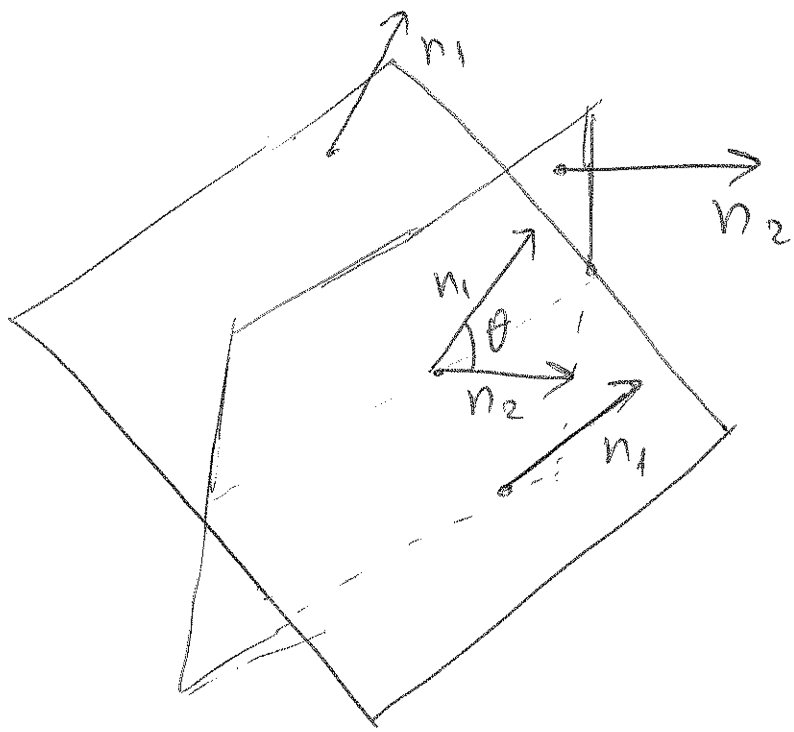
$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|}$$

Therefore

$$\theta = \arccos\left(\frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|}\right) =$$

$$= \arccos\left(\frac{1+1+2}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+2^2}}\right) =$$

$$= \arccos\left(\frac{4}{\sqrt{3}\cdot\sqrt{6}}\right) = \arccos\left(\frac{4}{\sqrt{18}}\right)$$

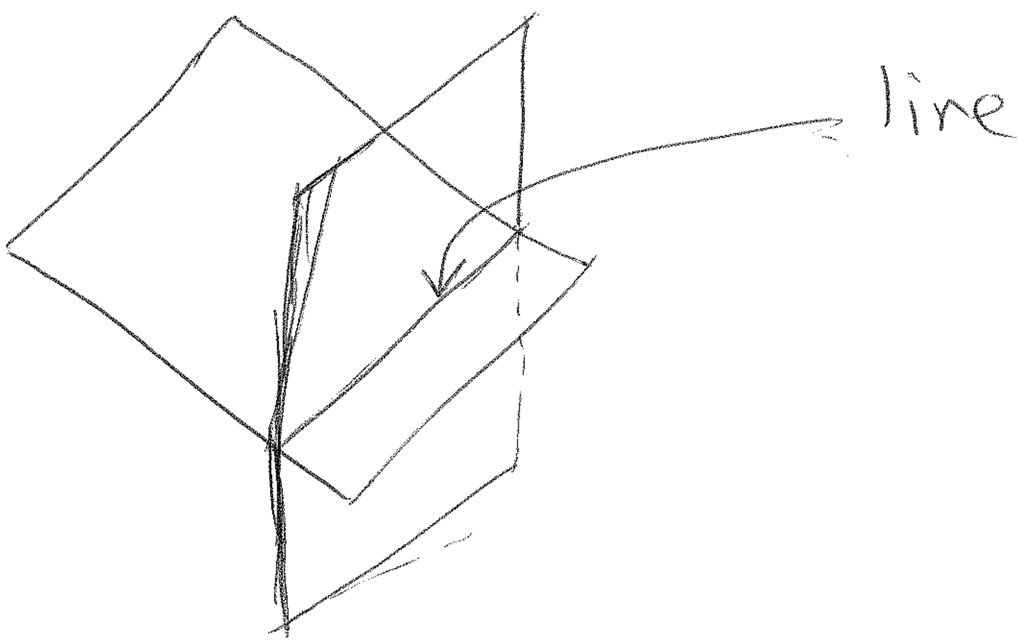


~~Q: Let x+y~~

Example:

Let $x+y=0$ be two planes
$2x+z=0$

Find the equation of the line of intersection



Solution: 1st way:

Pick any point from the intersection

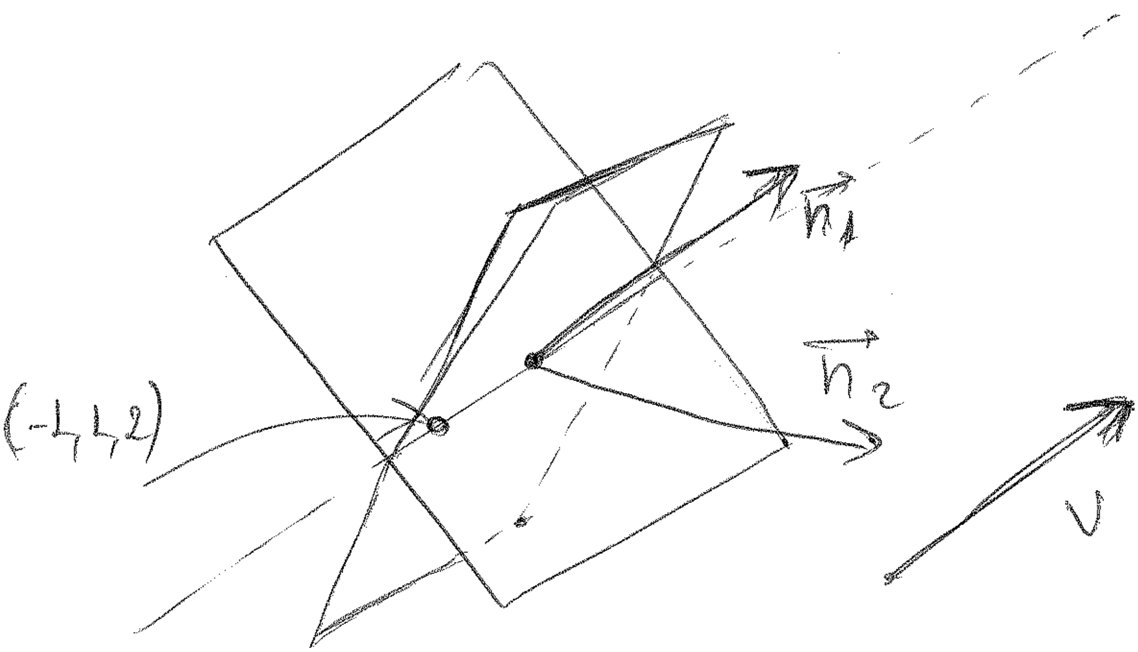
$\begin{cases} x+y=0 \\ 2x+z=0 \end{cases}$ Hence $\begin{cases} x=-y \\ -2y+z=0 \end{cases}$ Hence $\begin{cases} x=-y \\ y=\frac{z}{2} \end{cases}$

So $(-1,1,2)$ works

$n_1 = \langle 1,1,0 \rangle$ is perpendicular vector to the first ~~line~~ plane

$n_2 = \langle 2,0,1 \rangle$ is perpendicular to to the second ~~line~~ plane.

Then the direction of a line is given by a vector $\vec{v}$ which is perpendicular both to $\vec{n}_1$ and $\vec{n}_2$



Therefore we can choose

$\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = i - 2j - 2k = \langle 1, -2, -2 \rangle$$

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therefore $\vec{r}(t) = \langle -1, 1, 2 \rangle + t \cdot \langle 1, -2, -2 \rangle$

$\boxed{\text{2nd}}$ way pick any points (2 points) which belong to the intersection say A, B. Then construct a line which passes through that points.