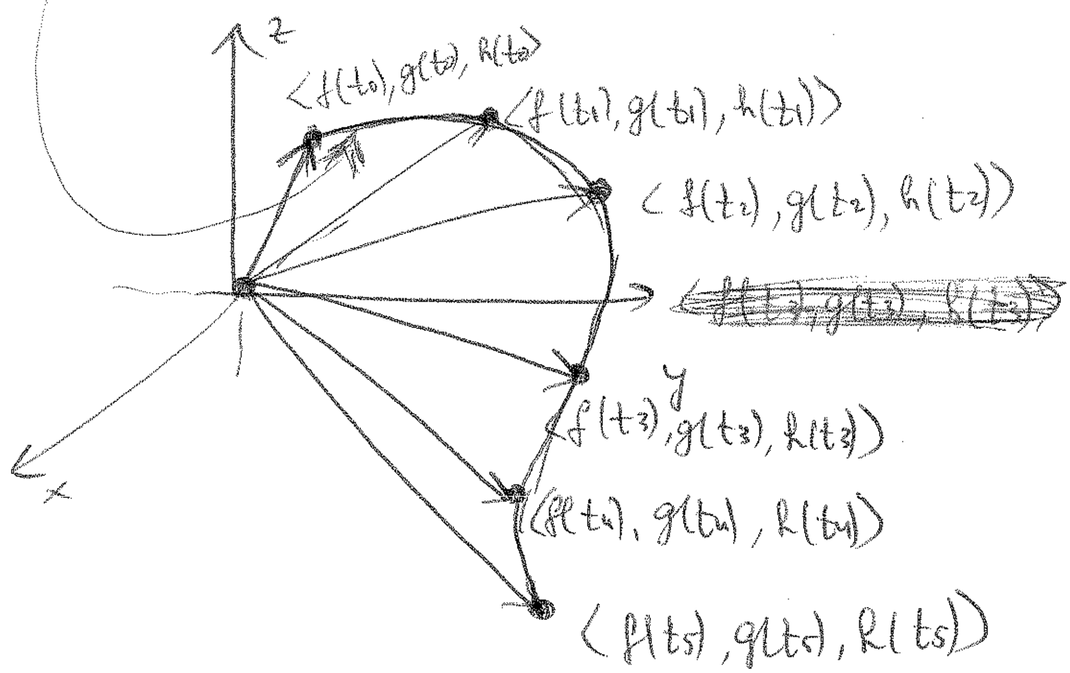
LECTURE 5.

Path in space (3D)



OR WE CAN JUST WRITE

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\cdot i + g(t)\cdot j + h(t)\cdot k.$$

When we say Please parametrize a curve

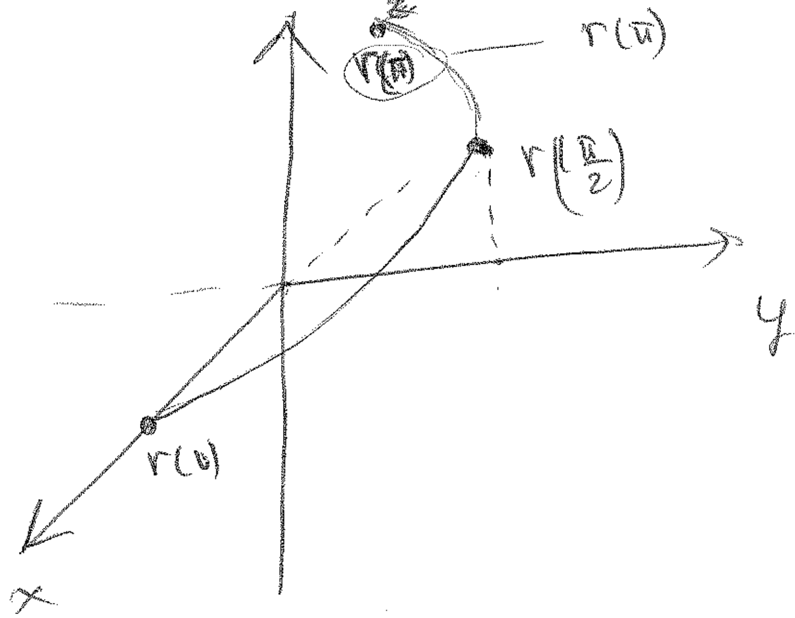
This means that you need to find

$f(t)$, $g(t)$ and $h(t)$ such that

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle.$$

Example:

$$\vec{r}(t) = \cos t \cdot i + \sin t \cdot j + t \cdot k$$



$\vec{r}(0) = \langle \cos 0, \sin 0, 0 \rangle = \langle 1, 0, 0 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle \cos \frac{\pi}{2}, \sin \frac{\pi}{2}, \frac{\pi}{2} \rangle = \langle 0, 1, \frac{\pi}{2} \rangle$

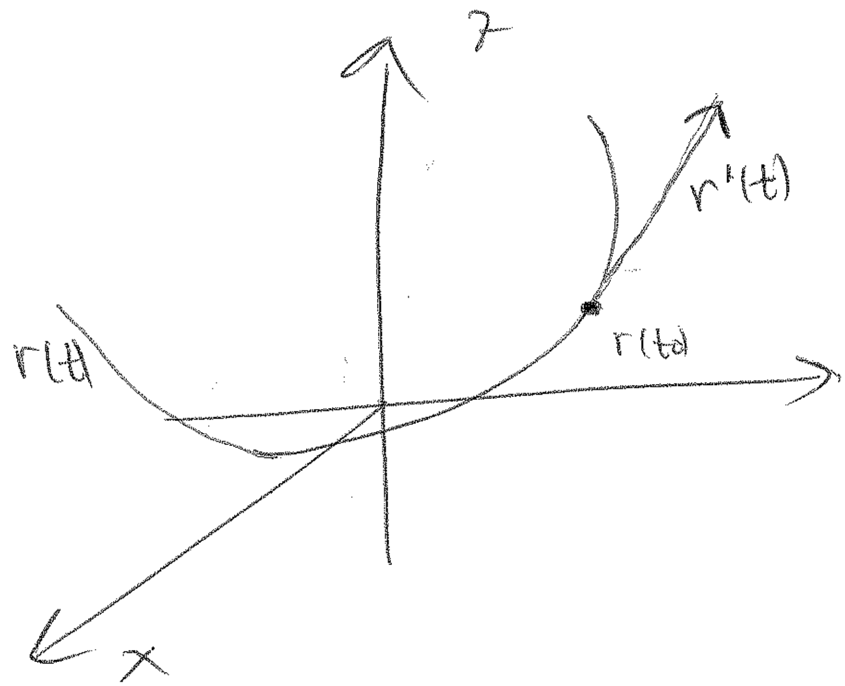
$\vec{r}(\pi) = \langle \cos \pi, \sin \pi, \pi \rangle = \langle -1, 0, \pi \rangle$

Derivatives. $\boxed{r(t) = \langle f(t), g(t), h(t) \rangle}$

Derivative of a curve (or path) $\vec{r}(t)$ we denote it as $r'(t)$ (sometimes $\frac{dr}{dt}$. And it is defined to be equal

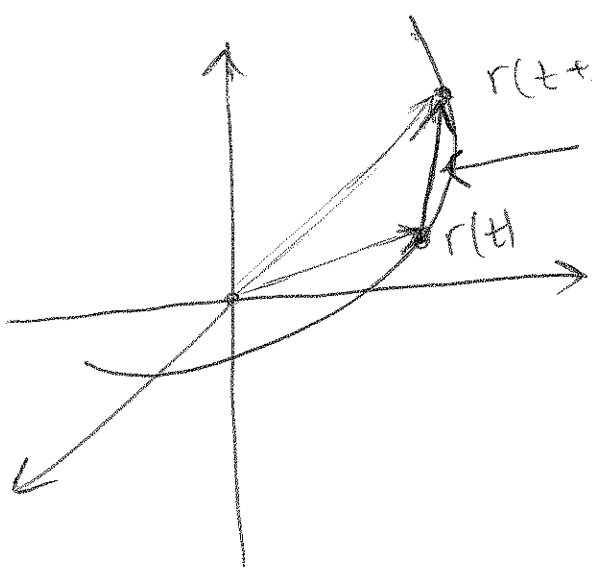
$$r'(t) = f'(t) \cdot i + g'(t) \cdot j + h'(t) \cdot k$$

derivative at a point t_0 is
a <u>tangent vector</u> at that point



<u>Explanation:</u>

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$



and when you divide
by number Δt you
dont change direction.

Definitions:

If $r(t)$ measures position of a particle in 3D. then $r'(t)$ measures velocity and usually we denote it as $v(t) = r'(t)$

The value $|r'(t)|$ measures speed of a particle.

The vector $r''(t)$ measures acceleration of a particle

Example: let $\vec{r}(t) = t\,i + t^2 \cdot j + k$

Q1: Find velocity.

Solution: $r'(t) = i + 2t \cdot j$

Q2: Find acceleration

Solution: $r''(t) = 2 \cdot j$

Q3: Find the speed of a particle (5)

Solution:

$$|r'(t)| = |\,i + 2t \cdot j\,| = \sqrt{1^2 + (2t)^2} = \sqrt{4t^2+1}$$

Q4: Find the tangent unit vector to the curve $\vec{r}(t)$ at time $t = 2$.

Solution:

1 step: Tangent vector at arbitrary time t is $r'(t)$:

$$r'(t) = i + 2t \cdot j$$

2 step: ~~Unit vector at~~ a

Tangent unit vector at arbitrary time will be

$$\frac{r'(t)}{|r'(t)|} = \frac{i + 2t \cdot j}{\sqrt{1+4t^2}} = i \cdot \left(\frac{1}{\sqrt{1+4t^2}}\right) + \frac{2t}{\sqrt{1+4t^2}} \cdot j$$

3 ~~S~~tep: at time $t=2$ we have

$$\frac{r'(2)}{|r'(2)|} = i \cdot \frac{1}{\sqrt{1+4\cdot 2^2}} + j \cdot \frac{2\cdot 2}{\sqrt{1+4\cdot 2^2}} =$$

$$= i \cdot \frac{1}{\sqrt{17}} + j \cdot \frac{4}{\sqrt{17}}$$

Differentiation rules

1. $\frac{d}{dt} c = 0$

2. $\frac{d}{dt} [c \cdot \vec{r}(t)] = c \cdot r'(t)$

3. $\frac{d}{dt} [\lambda(t) \cdot \vec{r}(t)] = \lambda'(t) \vec{r}(t) + \lambda(t) \cdot r'(t)$

4. $\frac{d}{dt} [\vec{r}(t) - \vec{u}(t)] = r'(t) - u'(t)$

5. $\frac{d}{dt} [\vec{r}(t) \cdot \vec{u}(t)] = r'(t) \cdot \vec{u}(t) + \vec{r}(t) \cdot u'(t)$

6. $\frac{d}{dt}[\vec{u}(t) \times \vec{r}(t)] =$

$= u'(t) \times \vec{r}(t) + \vec{u}(t) \times r'(t)$

7. $\frac{d}{dt}[u(f(t))] = f'(t) \cdot u'(f(t))$

Example:

If $\vec{r} \cdot r'(t) = 0$

Then show ~~that~~ $|r(t)| = \text{const}$

Proof: It is enough to show that $|r(t)|^2 = \text{const}$. It will be true if $\frac{d}{dt}\left(|r(t)|^2\right) = 0$.

$= \frac{d}{dt}\left(|r(t)|^2\right) = \frac{d}{dt}\left(r(t) \cdot r(t)\right) = r'(t) \cdot r(t) +$

$+ r(t) \cdot r'(t) = 2\, r'(t) \cdot r(t) = 0.$ Done.

Examples:

Let $r(t) = \cos t \cdot i + t \cdot j + e^{t} \cdot k$

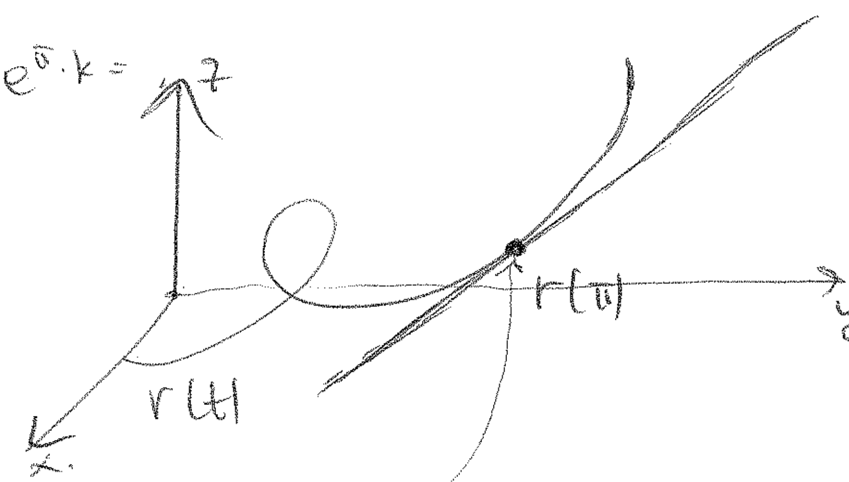
Write the equation of tangent line at point $t = \pi$.

Solution:

We need to find direction of the line and any point which it contains.

1) Instead of a point we can take

$r(\pi) = \cos\pi \cdot i + \pi \cdot j + e^{\pi} \cdot k = \langle -1, \pi, e^{\pi} \rangle$



2) Instead of direction we ~~need to~~ can take a tangent vector

For example $r'(\pi)$

$$r'(\pi) = -\sin t \cdot i + j + e^t \cdot k \Big|_{t=\pi} =$$

$$= -\sin\pi \cdot i + j + e^{\pi} \cdot k =$$

$$= j + e^{\pi} \cdot k = \langle 0, 1, e^{\pi} \rangle$$

Therefore equation of a line will be

$$r(\pi) + t \cdot r'(\pi) =$$

$$= \langle -1, \pi, e^{\pi} \rangle + t \cdot \langle 0, 1, e^{\pi} \rangle =$$

$$= \langle -1, \pi, e^{\pi} \rangle + \langle 0, t, te^{\pi} \rangle =$$

$$= \langle -1, \pi + t, e^{\pi} + te^{\pi} \rangle.$$

Done