LECTURE 7.

[Section 14.1]

Functions of several variables.

1

$f(x_1, x_2, \ldots, x_n) \to \mathbb{R}$

Example: $f(x_1, x_2) = x_1^2 + x_2^2$

$f(x, y) = (\sin x + \cos x) \cdot xy$

Domain and range

Domain - for which variables the function is well defined.

Example: $f(x, y) = \sqrt{x^2 - y}$ ~~is~~

Domain - it is well defined if $x^2 \geq y$

Domain $= \{(x, y) : x^2 \geq y\}$.

Example $f(x, y) = \ln(x + y)$.

Domain $= \{(x, y) : x + y \geq 0\}$

Range of a function

Range: What values does a function take?

Example:

1. $f(x,y) = x^2+y^2$. Range is $[0,\infty)$

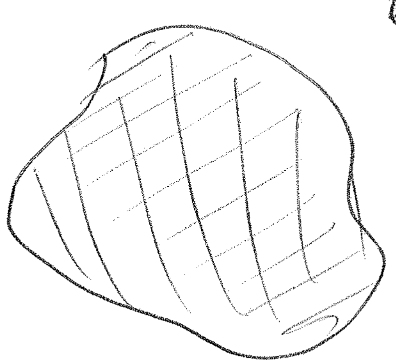
2. $f(x,y) = \ln(x+y)$. Range is $(-\infty,\infty)$

3. $f(x,y,z) = 1 + \cos x + \cos y + \cos z$

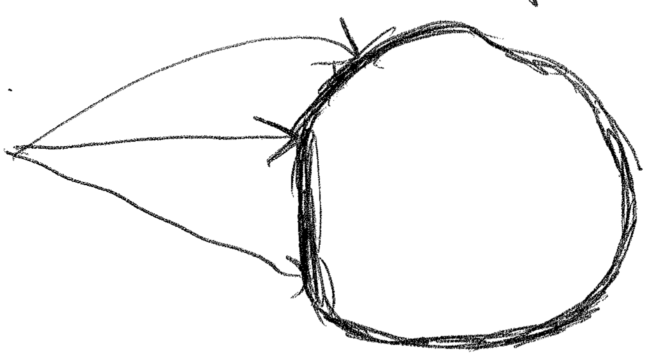
Range is $[-2,4]$ (why? Because $-1 \le \cos x \le 1$)

Interior points, boundary points and bounded domains.

Assume you have a set in 2D (or 3D) ← it does not matter this is your set.

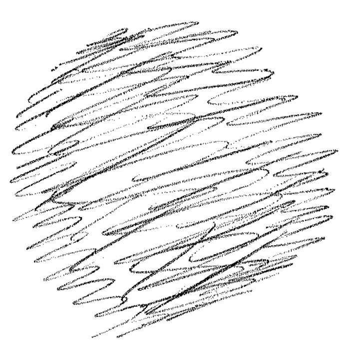


What are boundary points? This is exactly boundary of this domain i.e.



(but not the points which are inside).

2. What are interior points?
This is ~~a~~ set without boundary



does not have boundary.

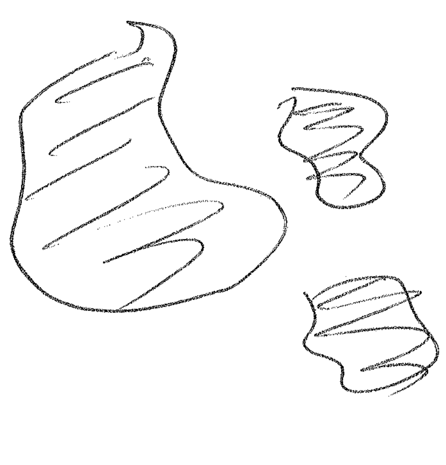
What is bounded domain (set)?

Answer: which is bounded i.e can be included im a ball of finite radius.

Example

this is not bounded

These are bounded



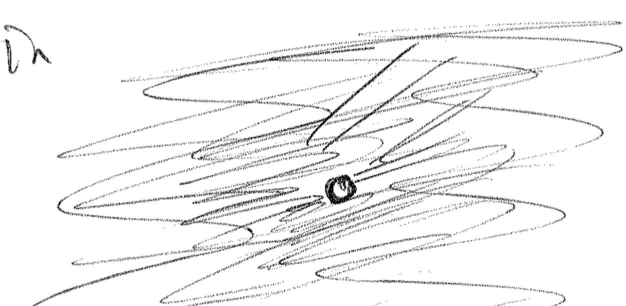
More examples:

1. lines ar unbounded
2. Planes are unbounded

Remark: Interior points can be charecterised in the following way as well:

If you can find a ball which contains this point and is contained inside this domain then the point is interior point.

Example: ~~$\{(x,y)$~~ Consider 2D without ~~a~~ o origin



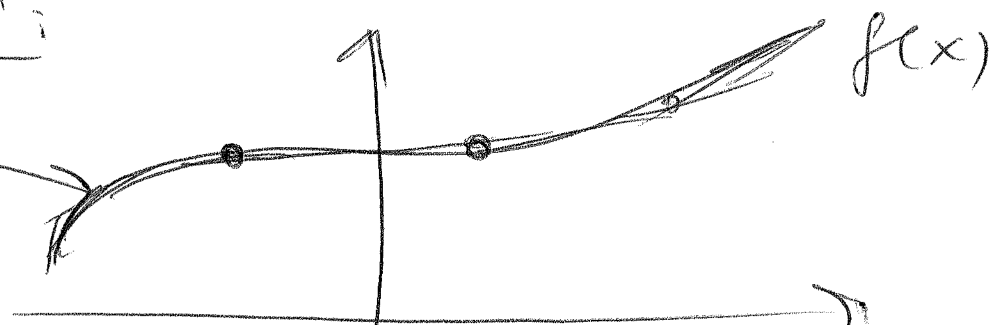
what are interior points?

answer: everything

Graphs, level curves of a function

Assume you have functoo $f(x)$ (of one variable). What is a graph of this function?

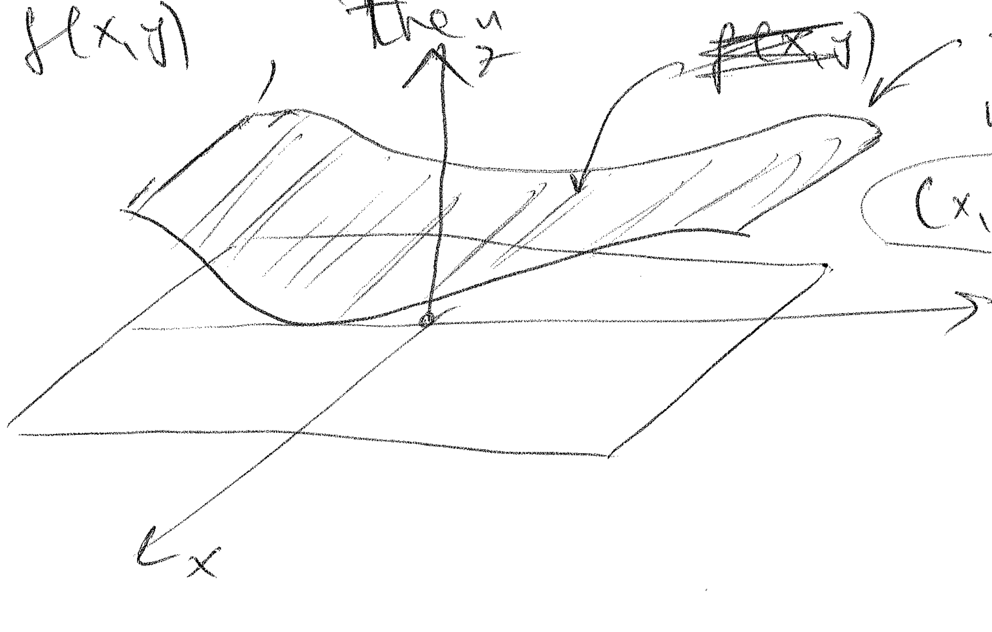
__Answer:__



this is a graph

$(x, f(x)) \rightarrow$ these ~~kird~~ kind of points

If $f(x,y)$, then



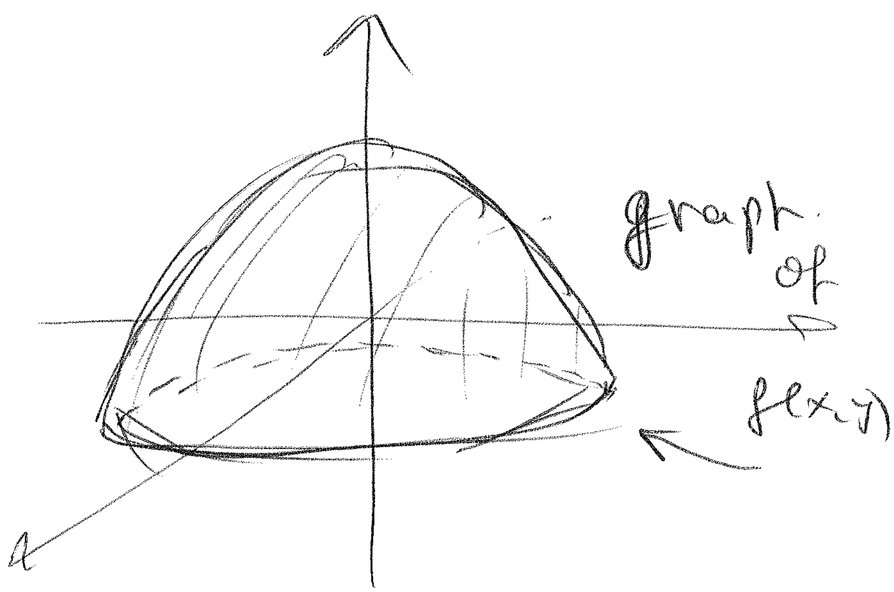
this is a graph

$(x, y, f(x,y))$ points

What is level curve?

It is an intersection of the graph (surface) and the plane $z=c$ when c is constant.

Example:

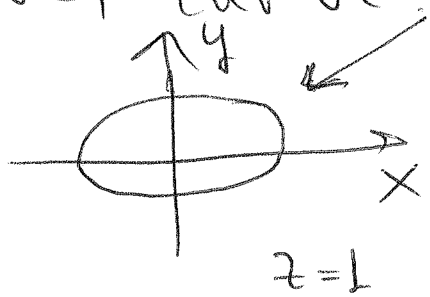


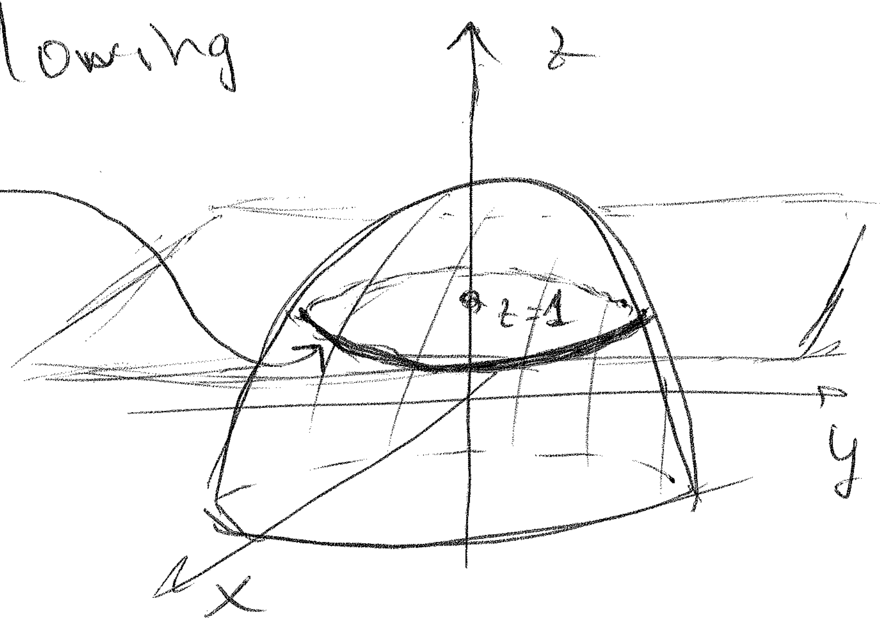
What is level cure when $z=1$?

You do the following

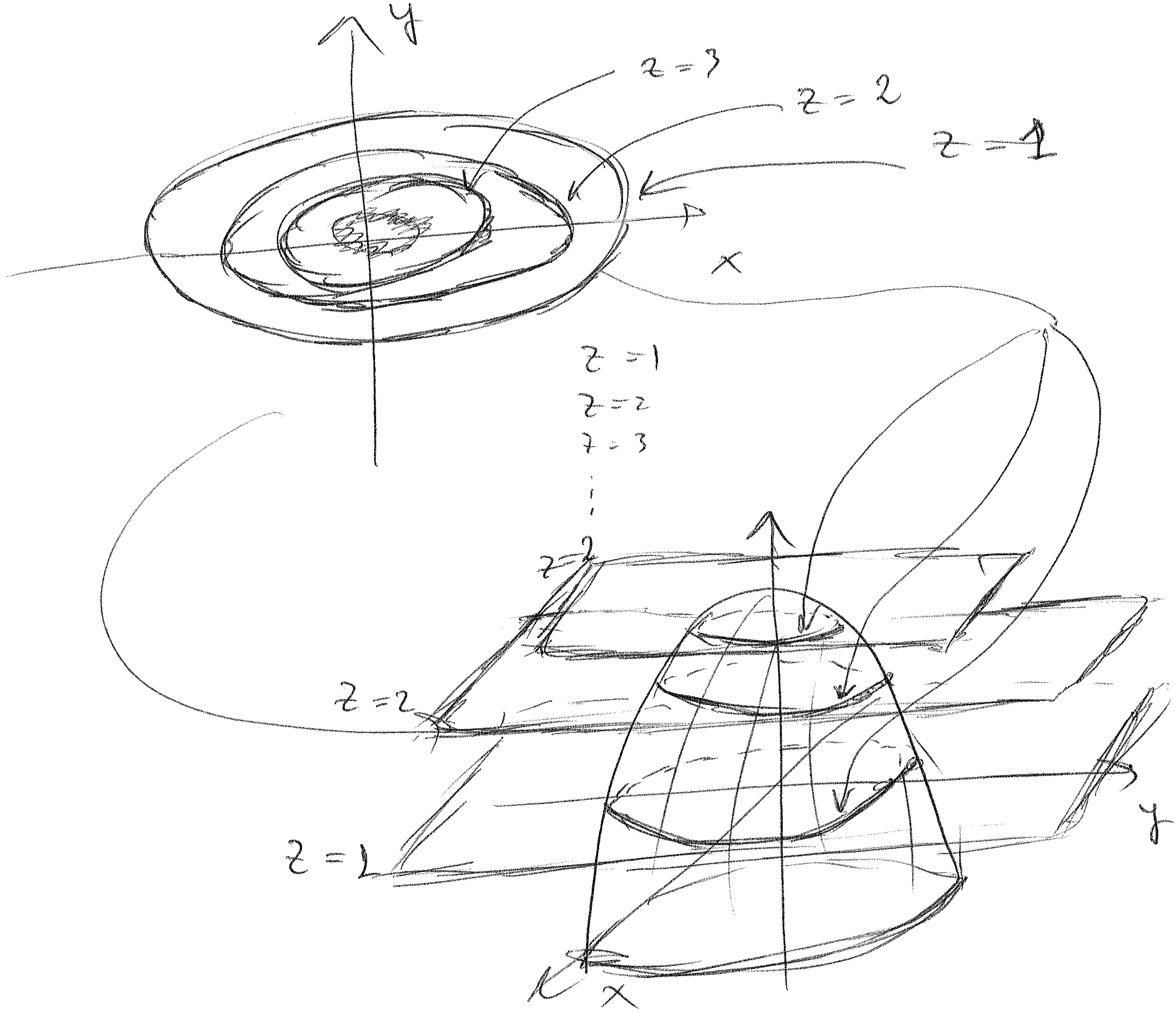
This intersection is level curve

Or





For different values of z
you get different level curves



If you have $f(x,y)$
And the question is sketch the level curve you should do the following

1) you should construct curves $f(x,y) = c$ for different values of c.

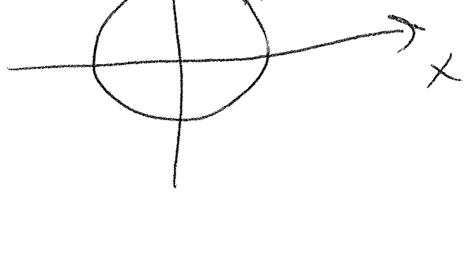
Example:

$$f(x,y) = x^2 + y^2$$

Q: Sketch the level curves

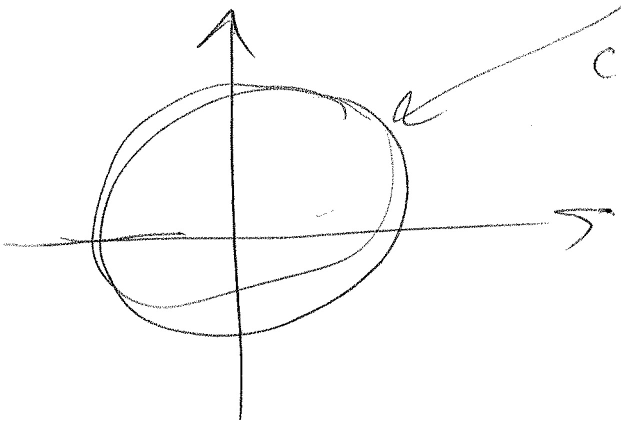
Solution: $f(x,y) = 1$, $x^2 + y^2 = 1$

circle of radius 1.

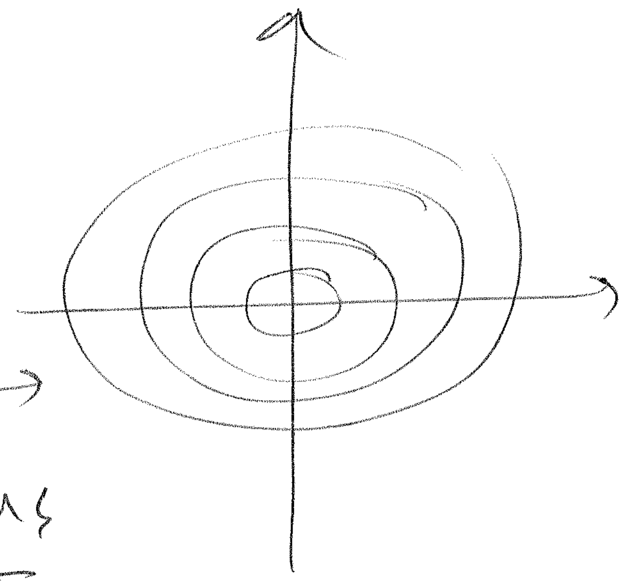


$f(x,y)=2$, $x^2+y^2=2$

circle of radius $\sqrt{2}$



and so on.



cirles of radius $\sqrt{c}$.

If you have a function of 3 variables $f(x,y,z)$. And the question is sketch the level ~~fu~~ surface then you do the following

You should draw a graph $f(x,y,z) = c$ for different values of c.

Example:

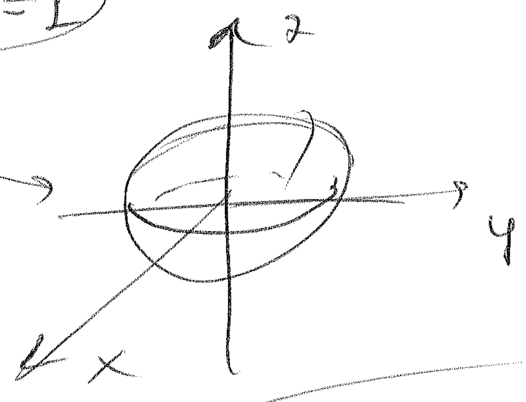
$f(x,y,z) = x^2 + y^2 + z^2$.

Q: Sketch the level surfaces

Solution:

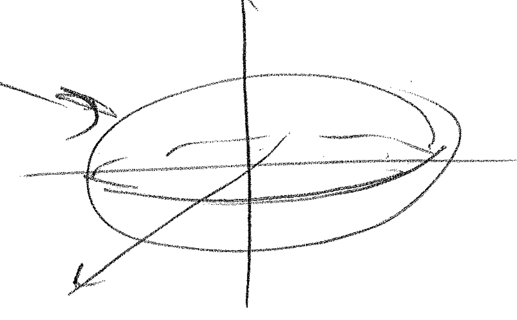
$f(x,y,z) = 1$. $x^2+y^2+z^2=1$

sphere of radius 1



$f(x,y,z) = 2$, $x^2+y^2+z^2=2$

sphere of radius $\sqrt{2}$



and so on!

Limits and continuity of a function in several variables

What does the following expression mean?

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}$$

It says What is the value of $f(x,y)$ as (x,y) approches to the point (x_0,y_0)?

Example:

$$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0.$$

The following $(x,y) \to (0,0)$

means that $x \to 0$ and $y \to 0$

Example:

$$\lim_{(x,y)\to(1,2)} \frac{x+y}{\ln y} = \frac{1+2}{\ln 2} = \frac{3}{\ln 2}$$

Sometimes this value

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ might not exist at all!

Example:

$\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$ does not exist because if we approach to $(0,0)$ from „negative side" like $x \to 0$ but $x < 0$ and $y \to 0$ but $y < 0$, for examle

$$x = \underline{-1}, \underline{-0{,}1}, \underline{-0{,}01}, \underline{-0{,}001}, \ldots \longrightarrow 0$$

and

$$y = \underline{-1}, \underline{-0{,}1}, \underline{-0{,}01}, \underline{-0{,}001} \ldots \longrightarrow 0$$

They approach to zero but they are always negative then $x+y < 0$ as well an therefore

$$\frac{1}{x+y} \approx \frac{1}{\text{something small negative}} = -\infty$$

On the other hand

If we approach from the "positive side" like

$x = \underline{0.1}, \quad \underline{0.001}, \quad \underline{0.0001}, \ldots$

$y = \underline{0.1}, \quad \underline{0.001}, \quad \underline{0.0001}, \ldots$

Then $x, y \to 0$ but $x, y > 0$

Therefore $x + y > 0$ and Hence

$$\frac{1}{x+y} = \frac{1}{\text{something small positive}} = \infty$$

So it takes two different values $-\infty$ and $+\infty$

~~Df~~

~~If these valu~~

Def: If these values always coincide regardless how do we approach to the point then we say that the limit exists.

Otherwize we say that limit dos not exists

Example when limit does not exists

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y}.$$

Solution: Let $y=0$ always and $x \to 0$

then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y} = \lim_{(x,0)\to(0,0)} \frac{x}{x} = \lim_{(x,0)\to(0,0)} 1 = 1$

Now let $x=0$ and $y \to 0$

then

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y} = \cancel{\lim_{(x,0)}}$$

$$= \lim_{(0,y)\to(0,0)} \frac{y^2}{y} = \lim_{(0,y)\to(0,0)} y = 0$$

but $1 \neq 0$ So limit does not exists.

<u>Example</u>

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Lets show that limit does not exists

Indeed lets tak $x = ky$ and (where k is any number) $y \to 0$. Then of cours $x \to 0$ as well

Εσ'

$$\lim_{(x,y)\to 0} \frac{xy}{x^2+y^2} = \lim_{(ky,y)\to 0} \frac{ky^2}{k^2y^2+y^2} =$$

$$= \lim_{(ky,y)} \frac{k}{k^2+1} = \frac{k}{k^2+1}$$

but for different values of k the answer is different. Hence the limit does not exists

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Example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x-y} =$$

$$= \lim_{(x,y)\to(0,0)} x+y = 0.$$

Exists!

If the limit exists and moreover

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Then we say that the function is continuous.

Example:

The function $f(x,y) = \dfrac{x^2-y^2}{x-y}$ is undefined at point $(x,y) = (0,0)$

If you define its value to be 0 like $f(0,0)=0$

then it is continuous function otherwise it is not.