MTH 234 Sections 010-018,021

Fall 2013

Exam-3

Name: Key. (Please print)

Student ID: ____________________

Section #: ____________

Problem #	Points	Student scores
1	30	
2	15	
3	15	
4	20	
5	20	
6. Extra credit	**5**	
Total	**105**	

Important Note:

Please show ALL your work. NO CREDIT for any correct answer/solution alone.

1. (30 pts)

(a. 8pts) Write down the double integral of a function: $z = f(x, y)$, over a 2-dimensional region:

$$R = \{(x, y) \in R^2 \mid a \le x \le b,\ g_0(x) \le y \le g_1(x)\}.$$

$$\int_a^b \int_{g_0(x)}^{g_1(x)} f(x,y)\, dy\, dx$$

(b. 8pts) Find the double integral in **a.** if

$f(x, y) = x^2 \sin y,\ a = -1, b = 3, g_0(x) = 0$ and $g_1(x) = \frac{\pi}{2}$.

$$\int_{-1}^{3} \int_0^{\frac{\pi}{2}} x^2 \sin y\, dx\, dy$$

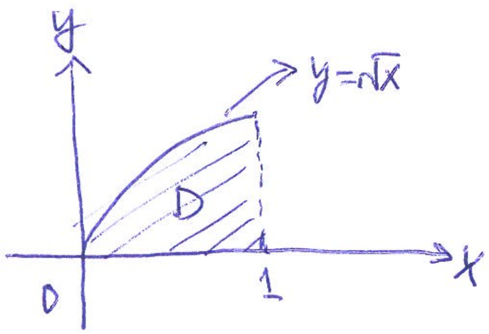
$$= \int_{-1}^{3} x^2 dx \cdot \int_0^{\frac{\pi}{2}} \sin y\, dy \quad (\text{Fubini's thm})$$

$$= \frac{1}{3}x^3\Big|_{-1}^{3} \cdot \Big[-\cos y\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{3}(3^3 - (-1)^3) \cdot \Big[-\overset{\nearrow 0}{\cancel{\cos\tfrac{\pi}{2}}} + \overset{\nearrow 1}{\cancel{\cos 0}}\Big]$$

$$= \boxed{\frac{28}{3}}$$

(**b. 14pts**) Sketch the graph of the 2-dimensional region:
$D: 0 \le x \le 1,\ 0 \le y \le \sqrt{x}$. Find the double integral of
$f(x, y) = 3y\sqrt{x + y^2}$ over D.



$$\int_0^1 \int_0^{\sqrt{x}} 3y\sqrt{x+y^2}\, dy\, dx$$

, $u = y^2 + x \quad du = 2y\, dy$

$y = \begin{cases} \sqrt{x} \\ 0 \end{cases} \Rightarrow u = \begin{cases} 2x \\ x \end{cases}$

$$= \int_0^1 \left[\int_x^{2x} \frac{3}{2}\sqrt{u}\, du \right] dx$$

$$= \int_0^1 \left[\frac{3}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \right]_x^{2x} dx$$

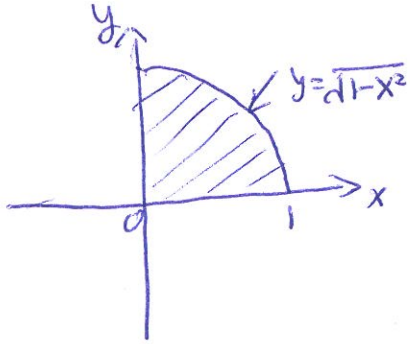
$$= \int_0^1 \left[2^{\frac{3}{2}} \cdot x^{\frac{3}{2}} - x^{\frac{3}{2}} \right] dx$$

$$= (2\sqrt{2} - 1) \cdot \frac{2}{5} x^{\frac{5}{2}} \Big|_0^1$$

$$= \boxed{\frac{4\sqrt{2} - 2}{5}}$$

2. (15 pts) Evaluate the integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (1 + x^2 + y^2)^3 dydx$$



$$= \int_0^{\frac{\pi}{2}} \int_0^1 (1+r^2)^3 r\,dr\,d\theta$$

(change to polar coordinates)

$$= \int_0^{\frac{\pi}{2}} \left[\int_1^2 u^3 \cdot \frac{1}{2} du\right] d\theta$$

$u = 1+r^2$

$du = 2r\,dr$

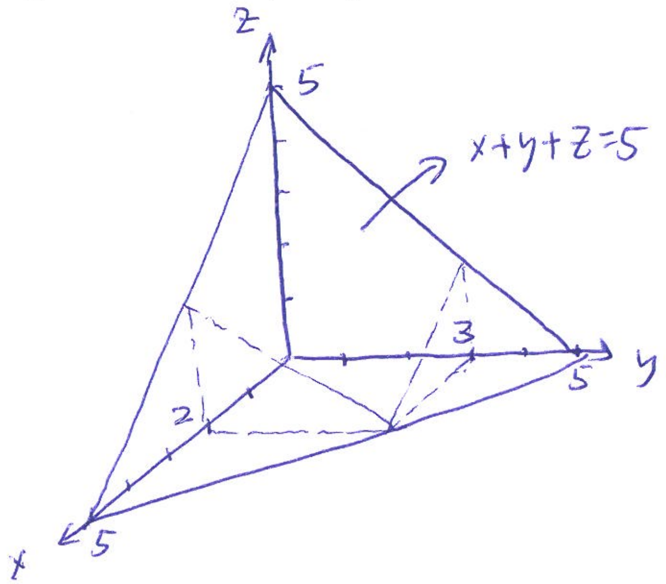
$r = \begin{cases} 1 \\ 0 \end{cases} \Rightarrow u = \begin{cases} 2 \\ 1 \end{cases}$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} u^4 \Big|_1^2$$

$$= \frac{\pi}{16} \left[2^4 - 1^4\right]$$

$$= \boxed{\frac{15\pi}{16}}$$

3. (15 pts) Compute the volume of the 3-dimensional region which is below the surface: $x + y + z = 5$, above the surface: $z = 0$, and over the 2-dimensional region: $0 \le x \le 2, 0 \le y \le 3$.



$$V = \int_0^2 \int_0^3 (5-x-y)\,dy\,dx \quad \text{or} \quad V = \int_0^2 \int_0^3 \int_0^{5-x-y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^3 (5-x-y)\,dy\,dx$$

$$= \int_0^2 \left[5y - xy - \frac{1}{2}y^2\right]_0^3 dx$$

$$= \int_0^2 \left[15 - 3x - \frac{9}{2}\right] dx$$

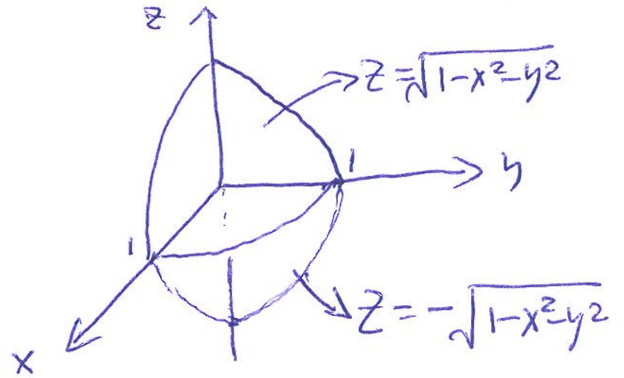
$$= \left(\frac{30-9}{2}x - \frac{3}{2}x^2\right)_0^2$$

$$= 21 - \frac{3}{\cancel{2}} \cdot 2^{\cancel{2}}$$

$$= \boxed{15}$$

4. (20 pts) Consider the triple integral: $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2)\, dz\, dy\, dx$

(a. 10pts) Convert the integral I from the Cartesian coordinate system to the Spherical coordinate system.



$$I = \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^1 \rho^2 \sin^2\phi \cdot \rho^2 \sin^2\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^1 \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

(b. 10pts) Compute the integral I using the Spherical coordinate system.

$$I = \frac{\pi}{2} \left[\int_0^{\pi} \sin^3\phi \, d\phi\right] \left[\int_0^1 \rho^4 d\rho\right] \quad \text{(Fubini's thm)}$$

$$= \frac{\pi}{2} \left\{\int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi\right\} \cdot \left[\frac{1}{5}\rho^5\right]_0^1$$

$$= \frac{\pi}{10} \int_1^{-1} (u^2-1)\, du, \quad u = \cos\phi \quad du = -\sin\phi\, d\phi$$

$$\phi = \begin{cases} \pi \\ 0 \end{cases} \Rightarrow u = \begin{cases} -1 \\ 1 \end{cases}$$

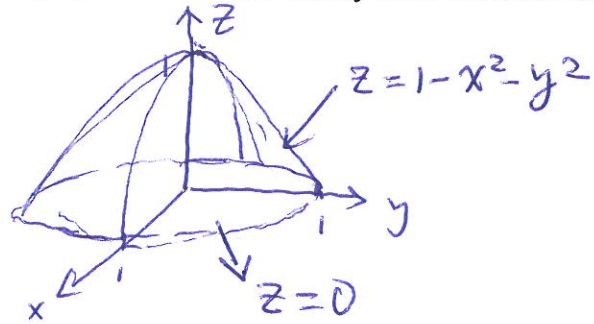
$$= \frac{\pi}{10} \left[\frac{1}{3}u^3 - u\right]_1^{-1}$$

$$= \frac{\pi}{10} \left[\frac{-2}{3} + 2\right]$$

$$= \frac{\pi}{5} \cdot \frac{-1+3}{3} = \boxed{\frac{2\pi}{15}}$$

5. (20 pts) Given a 3-dimensional region: $D = \{(x, y, z) |\ z \geq 0,\ z \leq 1 - x^2 - y^2\}$

(a. 4 pts) Sketch the graph of D. Please clearly label each boundary surface by its mathematical equation.



(b. 4 pts) Let $f(x, y, z) = 1 - z$. Set up $\iiint_D f(x, y, z)dV$ in Cartesian coordinates

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (1-z)\, dz\, dy\, dx$$

(c. 4 pts) Convert the triple integral in **b.** into a triple integral in cylindrical coordinates.

$$I = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (1-z)\, dz \cdot (r\, dr) \cdot d\theta$$

(d. 8 pts) Compute the integral in **c.**.

$$I = 2\pi \int_0^1 \left[z - \tfrac{1}{2}z^2\right]_0^{1-r^2} r\, dr$$

$$= 2\pi \int_0^1 \left[1 - r^2 - \tfrac{1}{2}(1-r^2)^2\right] r\, dr$$

$$= 2\pi \int_0^1 \left[1 \cancel{- r^2} - \tfrac{1}{2} \cancel{+ r^2} - \tfrac{1}{2}r^4\right] r\, dr$$

$$= \pi \int_0^1 \left[r - r^5\right] dr$$

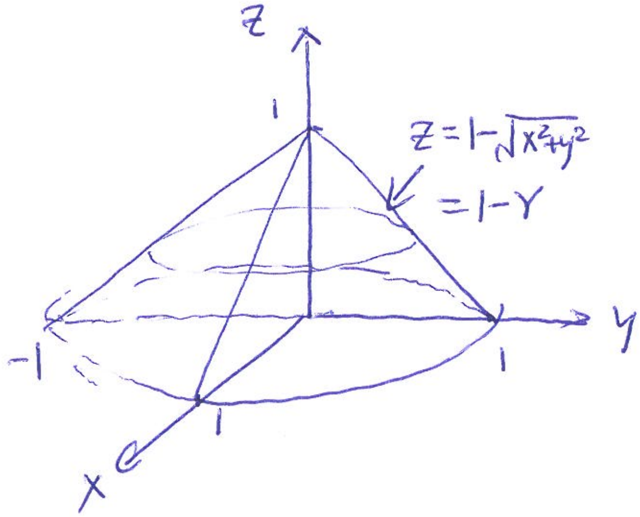
$$= \pi \left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \pi \left[\tfrac{1}{2} - \tfrac{1}{6}\right] = \boxed{\frac{\pi}{3}}$$

6. (Extra credit 5 pts) Find the average value of the function: $f(x, y, z) = z$, over the 3-dimensional region:

$$R = \{(x, y, z) \in R^3 |\, z \le 1 - \sqrt{x^2 + y^2}\ , z \ge 0\}.$$

Please explain your answer.

Remark: No credit for correct answer alone. No partial credit for this problem.



Volume of $R = \frac{\pi}{3} = V$

(volume of a cone: $\frac{1}{3}\pi R^2 h$)

$$I = \iiint_R z\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r} z\, dz\, r\, dr\, d\theta$$

(cylindrical)

$$= 2\pi \int_0^1 \left[\left[\frac{1}{2} z^2\right]_0^{1-r} \right] r\, dr$$

$$= \cancel{2}\pi \int_0^1 \frac{1}{\cancel{2}} (1-r)^2 r\, dr$$

$$= \pi \int_0^1 (r - 2r^2 + r^3)\, dr$$

$$= \pi \left[\frac{1}{2} r^2 - \frac{2}{3} r^3 + \frac{1}{4} r^4\right]_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \pi \cdot \frac{6 - 8 + 3}{12} = \frac{\pi}{12}$$

$$\bar{f} = \frac{I}{V} = \frac{\left(\frac{\pi}{12}\right)}{\left(\frac{\pi}{3}\right)} = \boxed{\frac{1}{4}}$$