MTH 234 Sections 010-018,021

Fall 2013

Exam-4

Name: Key (Please print)

Student ID: ____________________

Section #: ____________________

Problem #	Points	Student scores
1	15	
2	25	
3	20	
4	20	
5	20	
Total	**100**	

Important Note:

Please show ALL your work. NO CREDIT for any correct answer/solution alone.

<u>Please simplify all your solutions/numbers.</u>

1. (15 pts) Evaluate the line integral:

$$\int_C (x + y + z)ds,$$

where C is the circle: $r(t) = \langle 2\cos t, 0, 2\sin t\rangle$, $0 \le t \le \pi$.

Solution $\vec{r}\,'(t) = \langle -2\sin t, 0, 2\cos t \rangle$

$$\int_C (x+y+z)\,ds = \int_C (x(t)+y(t)+z(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^{\pi} [2\cos t + 2\sin t]\cdot\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt$$

$$= 4\int_0^{\pi} (\cos t + \sin t)\,dt$$

$$= 4\left[\sin t - \cos t\right]_0^{\pi}$$

$$= \boxed{8}$$

2. (25 pts) (a) Find the area of the surface :

15pts

$$\boldsymbol{r}(u,v) = (u+v)\boldsymbol{i} + (u-v)\boldsymbol{j} + v\boldsymbol{k}, \qquad 0 \le u \le 1, \qquad 0 \le v \le 1.$$

Solu.: $\vec{r}_u = \langle 1, 1, 0 \rangle$, $\vec{r}_v = \langle 1, -1, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \vec{i} - \vec{j} - 2\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$\text{surface area} = \int_0^1 \int_0^1 |\vec{r}_u \times \vec{r}_v| \, du\, dv = \sqrt{6} \int_0^1 \int_0^1 du\, dv$$

$$= \boxed{\sqrt{6}}$$

(b) Integrate $f(x,y,z) = xy - z^2$ over the surface given in (a)

10pts

Soln.: $$\iint_S f(x,y,z)\, d\sigma = \int_0^1 \int_0^1 (xy - z^2)\, |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$= \sqrt{6} \int_0^1 \int_0^1 [(u+v)(u-v) - v^2]\, du\, dv$$

$$= \sqrt{6} \int_0^1 \int_0^1 [u^2 - 2v^2]\, du\, dv$$

$$= \sqrt{6} \int_0^1 \left[\tfrac{1}{3}u^3 - 2v^2 u\right]_0^1 dv$$

$$= \sqrt{6} \int_0^1 \left[\tfrac{1}{3} - 2v^2\right] dv = \sqrt{6}\left[\tfrac{1}{3}v - \tfrac{2}{3}v^3\right]_0^1$$

$$= \boxed{-\frac{\sqrt{6}}{3}}$$

3. (20 pts) Use the Green's Theorem to evaluate:

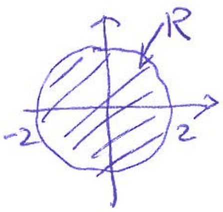
$$\oint_C (x^3 + \sin(y^2))dy - (y^3 + \cos(x^2))dx,$$

where C is the boundary of the disk: $R = \{(x, y) | x^2 + y^2 \leq 4\}$ oriented counterclockwise.

Note: No credit for not using the Green's Theorem.

Solu.1: Normal form of the Green's theorem:

$$\oint_C \underbrace{F_x\,dy - F_y\,dx}_{\vec{F}\cdot\vec{n}\,ds} = \iint_S \underbrace{(\partial_x F_x + \partial_y F_y)}_{\mathrm{div}\,\vec{F}}\,d\sigma$$



Not nec.

$$F_x = x^3 + \sin(y^2), \quad \partial_x F_x = 3x^2$$

$$F_y = y^3 + \cos(x^2), \quad \partial_y F_y = 3y^2$$

$$\iint_S (\partial_x F_x + \partial_y F_y)\,d\sigma = \iint_S 3(x^2 + y^2)\,dx\,dy$$

$$= 3\int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta = 6\pi \cdot \frac{1}{4} r^4 \Big|_0^2 = \boxed{24\pi}.$$

Solu.2: Tangential form of the Green's theorem:

$$\oint_C \underbrace{F_x\,dx + F_y\,dy}_{\vec{F}\cdot d\vec{r}} = \iint_S \underbrace{(\partial_x F_y - \partial_y F_x)}_{(\mathrm{curl}\,\vec{F})_z}\,d\sigma$$

$$F_x = -y^3 - \cos(x^2), \quad -\partial_y F_x = 3y^2$$

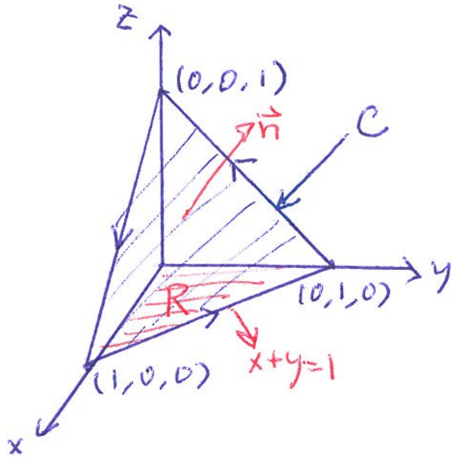
$$F_y = x^3 + \sin(y^2) \quad \partial_x F_y = 3x^2$$

$$\iint_S (\partial_x F_y - \partial_y F_x)\,d\sigma = 3\iint_S (x^2 + y^2)\,dx\,dy$$

The rest is the same as Solu.1.

4. (20 pts) Use the Stokes' Theorem to evaluate the line integral: $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = z\boldsymbol{i} - z\boldsymbol{j} + y^2\boldsymbol{k}$ and C consists of the three line segments that bound the plane $x + y + z = 1$ in the first octant, oriented as shown in the graph.

__Note:__ No credit for not using the Stokes' Theorem.



The Stokes' theorem:

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, d\sigma}$$

① $x+y+z=1 \Rightarrow \underbrace{x+y+z-1}_{f(x,y,z)} = 0$

$\nabla f = \langle 1, 1, 1\rangle$

$\vec{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$

② $\nabla\times\vec{F} = \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z & -z & y^2 \end{vmatrix} = (2y+1)\vec{i} + \vec{j}$.

$\Rightarrow \nabla\times\vec{F}\cdot\vec{n} = \dfrac{1}{\sqrt{3}}(2y+2)$

③ Consider $z = z(x,y)$.

$d\sigma = \dfrac{|\nabla f|}{|\nabla f\cdot \vec{k}|}\,dx\,dy = \dfrac{\sqrt{3}}{1}\,dx\,dy = \sqrt{3}\,dx\,dy$

④ $\displaystyle\iint_S \nabla\times\vec{F}\cdot\vec{n}\, d\sigma = \int_0^1\int_0^{1-y} \frac{1}{\cancel{\sqrt{3}}}(2y+2)\cancel{\sqrt{3}}\,dx\,dy$

$\displaystyle = 2\int_0^1\int_0^{1-y}(y+1)\,dx\,dy$

$\displaystyle = 2\int_0^1 (y+1)(1-y)\,dy = 2\int_0^1 (1-y^2)\,dy$

$\displaystyle = 2\left(y - \frac{1}{3}y^3\right)\Big|_0^1 = 2\left(1-\frac{1}{3}\right) = \boxed{\frac{4}{3}}$

5. (20 pts) Use the Divergence Theorem to find the outward flux for the field:
$\boldsymbol{F} = \langle x\sqrt{x^2+y^2+z^2},\ y\sqrt{x^2+y^2+z^2},\ z\sqrt{x^2+y^2+z^2},\ \rangle$ across the boundary of the region: $D = \{1 \le x^2+y^2+z^2 \le 4\}$.

Note: No credit for not using the Divergence Theorem.
Hint: Use the spherical coordinates.

Solution: The Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_V \operatorname{div}\vec{F}\, dv$$

$$\partial_x F_x = \partial_x\left(x\sqrt{x^2+y^2+z^2}\right) = \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}}$$

$$\partial_y F_y = \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}}$$

$$\partial_z F_z = \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}$$

$$\operatorname{div}\vec{F} = 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}$$

$$\iiint_V \operatorname{div}\vec{F}\, dv = \iiint_V 4\sqrt{x^2+y^2+z^2}\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 4\rho\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left[\int_1^2 4\rho^3\, d\rho\right]\cdot\left[\int_0^{\pi}\sin\phi\, d\phi\right]$$

$$= 2\pi\left[\rho^4\right]_1^2\cdot\left[-\cos\phi\right]_0^{\pi}$$

$$= 4\pi\,[16-1] = \boxed{60\pi}$$