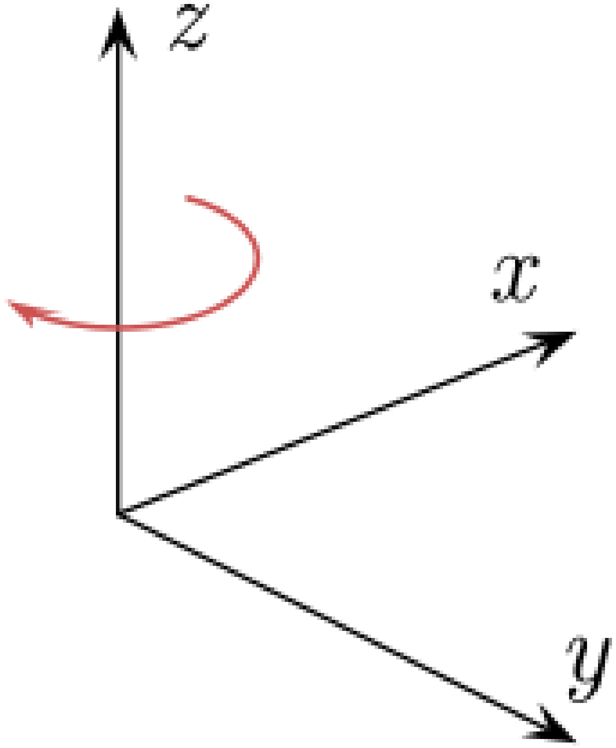
Multivariable calculus

- MTH132 & 133: $f: R \to R$
  one-dimensional functions
  Examples: $y = x^2, y = \sin x, \ldots$

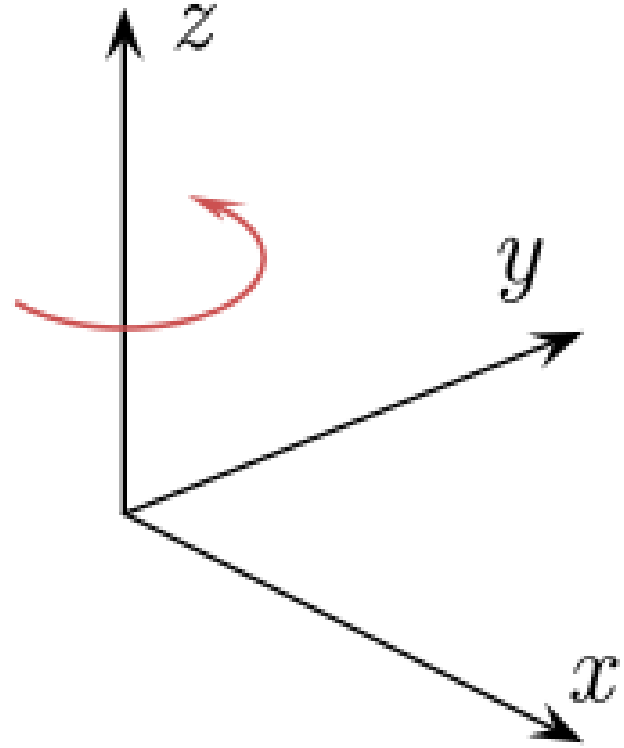
- MTH234: $f: R^2 \to R \;\; or \; f: R^3 \to R$
  or $f: R \to R^2, R \to R^3$
  Examples: $z = x^2 + y^2, (x, y) = (\sin t, \cos t)$

3-dimensional coordinate system

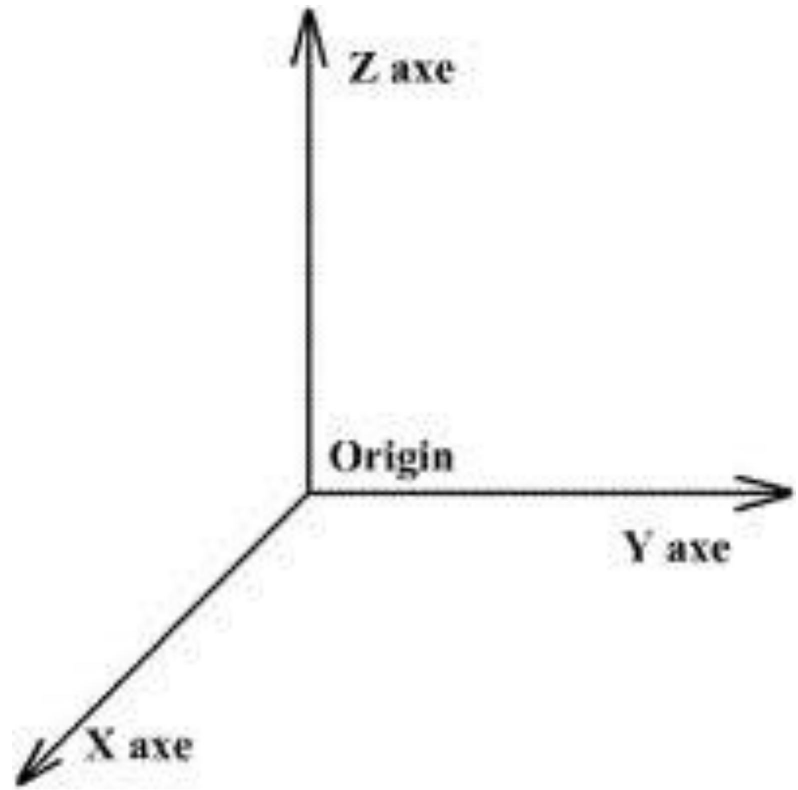
- Left-handed (LH) system



- Right-handed (RH) system

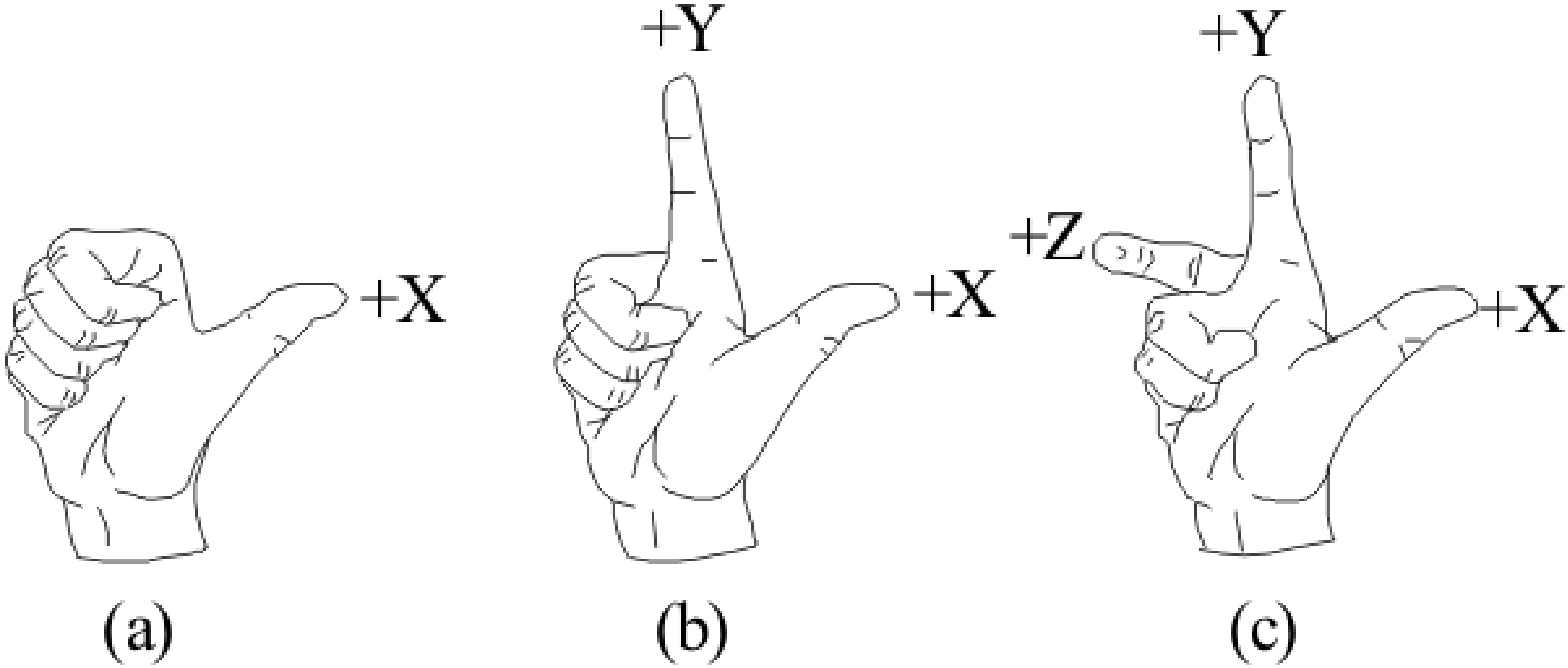


Right-handed coordinate system



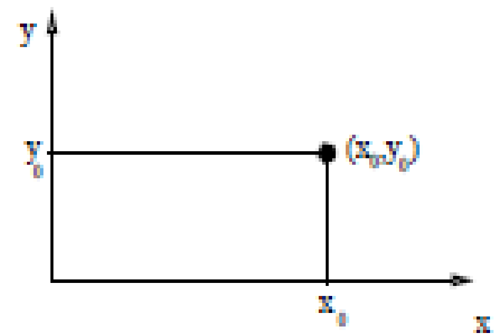
Note: In this course, we will use the right-handed system.

Right-handed coordinate system

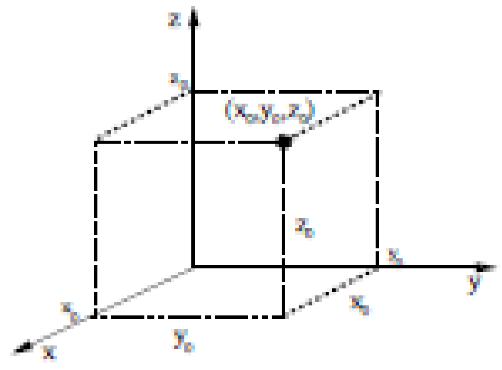


Cartesian coordinates.

Cartesian coordinates on $\mathbb{R}^2$: Every point on a plane is labeled by an ordered pair (x, y) by the rule given in the figure.



Cartesian coordinates in $\mathbb{R}^3$: Every point in space is labeled by an ordered triple (x, y, z) by the rule given in the figure.

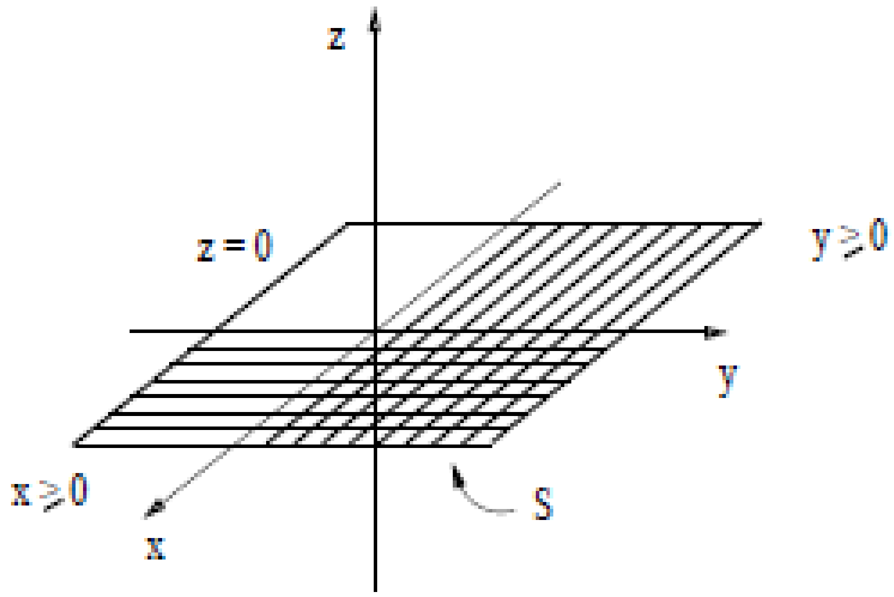


Cartesian coordinates.

Example

Sketch the set $S = \{x \geqslant 0,\ y \geqslant 0,\ z = 0\} \subset \mathbb{R}^3$.

Solution:



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Distance between 2 points:

Theorem

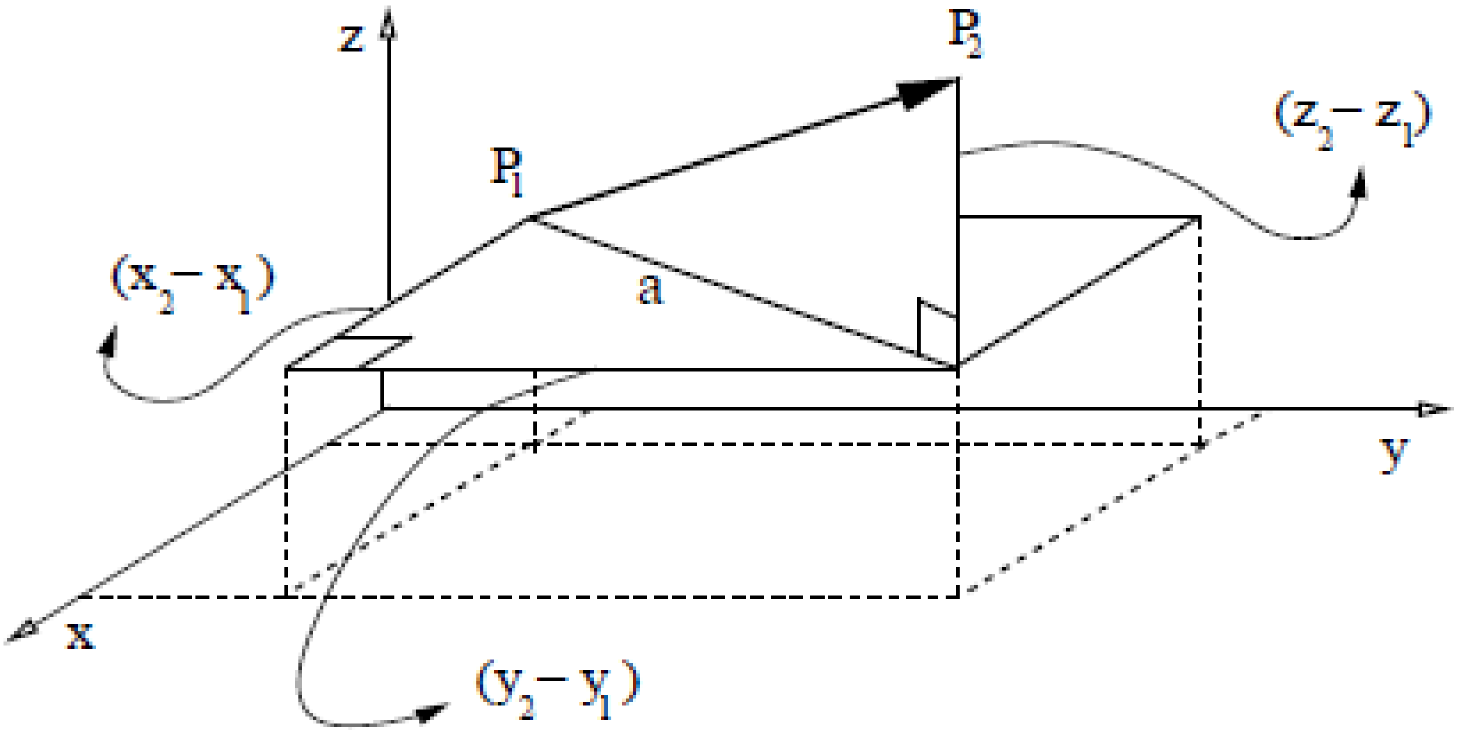
The distance $|P_1P_2|$ *between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The distance between points in space is crucial to define the idea of limit to functions in space.

Proof.

Pythagoras Theorem.

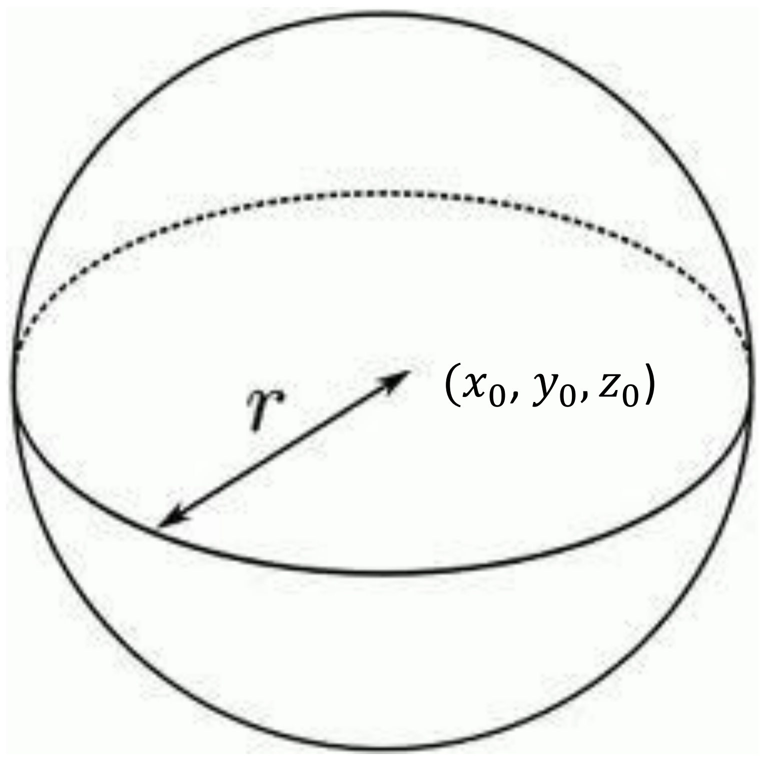


$$|P_1P_2|^2 = a^2 + (z_2 - z_1)^2, \qquad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

□

Sphere equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$



Equation of a sphere

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Complete the square.

$$0 = x^2 + y^2 + 4y + z^2$$

$$0 = x^2 + \left[y^2 + 2\left(\frac{4}{2}\right)y + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + z^2$$

$$0 = x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 - 4.$$

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2.$$

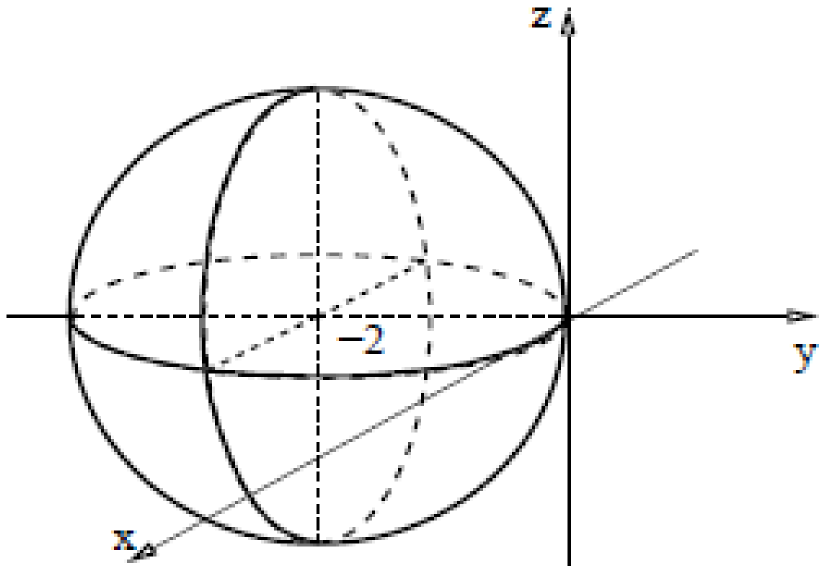
Equation of a sphere

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

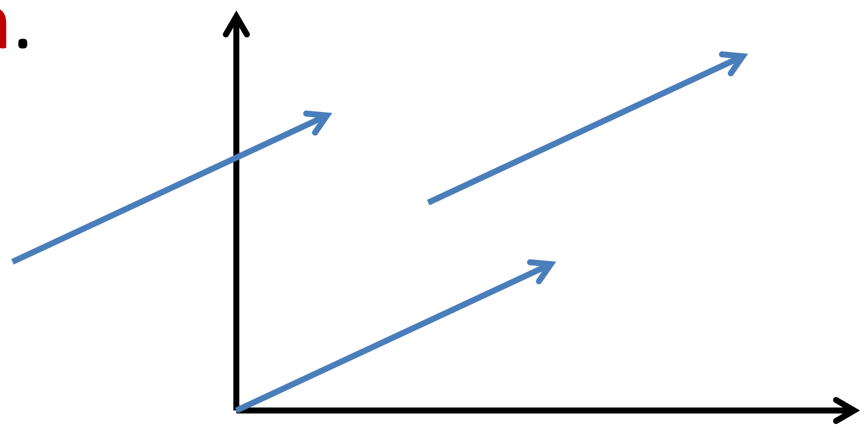
Solution: $x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2$.

Then, we conclude that $P_0 = (0, -2, 0)$ and $R = 2$. Therefore,



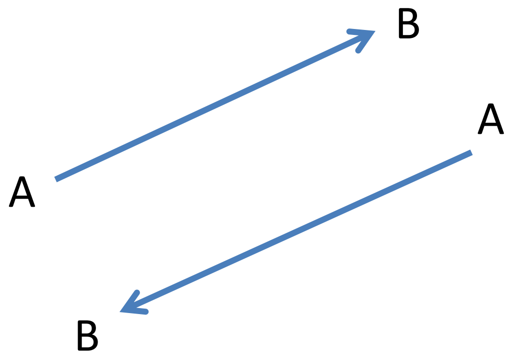
Vectors in R^2 and R^3

<u>**Definition:**</u> A vector is a directed line, $\overrightarrow{AB}$ from point A (initial point) to point B (terminal point) and has its length denoted by $\left|\overrightarrow{AB}\right|$. Two vectors are equal if they have the same length and direction.



Note

- Initial point and terminal point are not unique.
- $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are of the same length but of opposite in directions.



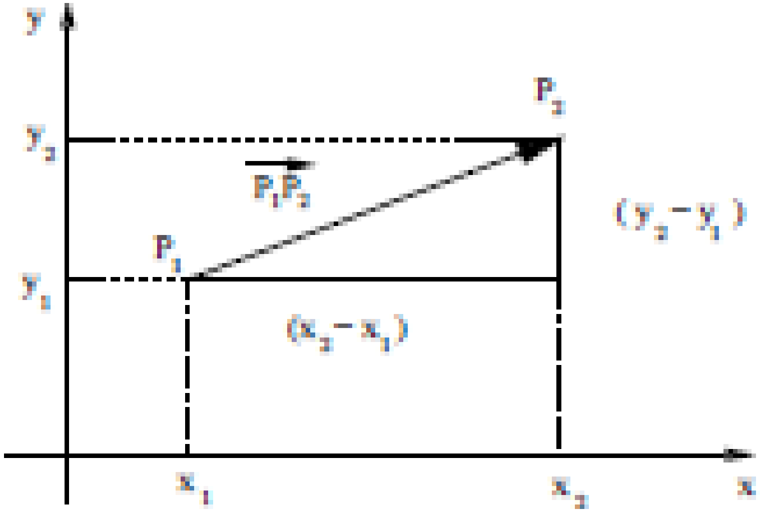
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered pair, called vector components,

$$\langle \overrightarrow{P_1P_2} \rangle = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$

Proof:
Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Remark: A similar result holds for vectors in space.

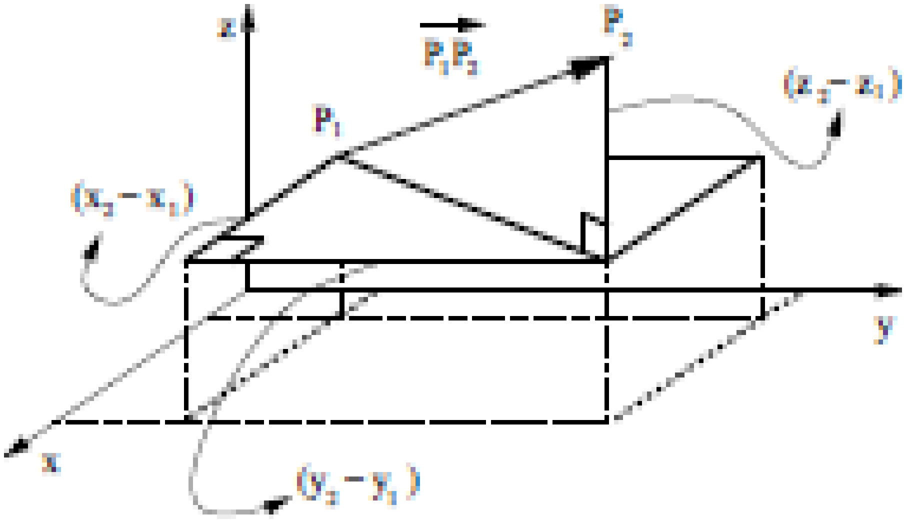
Components of a vector in Cartesian coordinates

Theorem

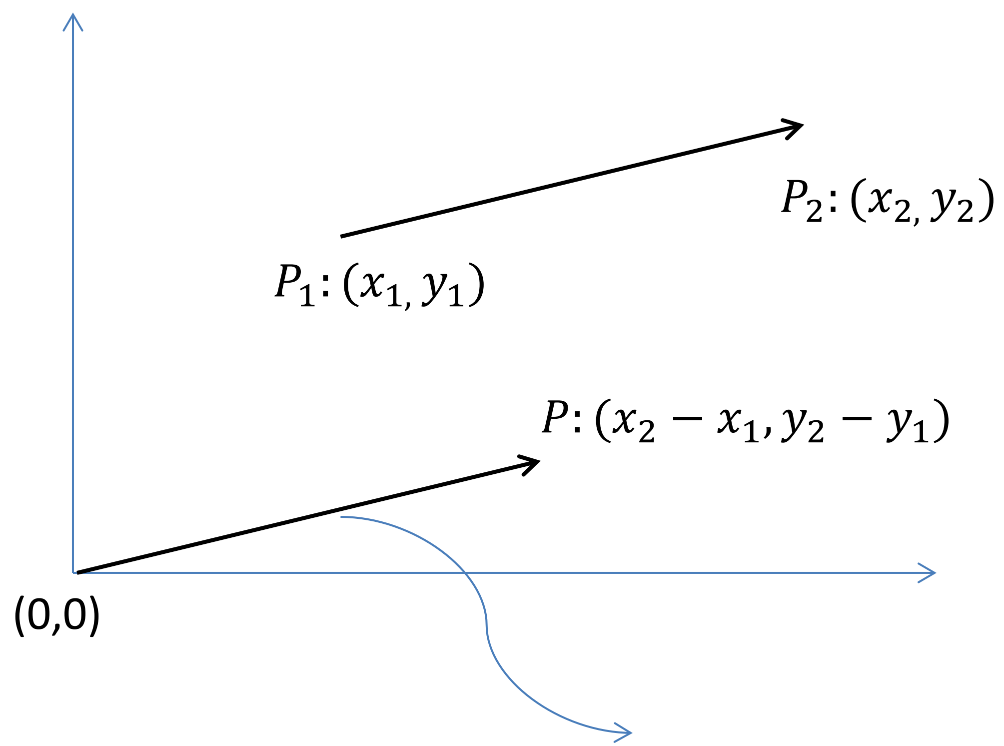
Given the points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the vector $\overrightarrow{P_1P_2}$ fixes a unique ordered triple, called vector components,

$$\langle\overrightarrow{P_1P_2}\rangle = \langle(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\rangle.$$

Proof:
Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Standard position



Standard vector (representative)

$\boldsymbol{v}: \langle v_1, v_2 \rangle$ or $\vec{v}: \langle v_1, v_2 \rangle$ or $\underline{v}: \langle v_1, v_2 \rangle$

Remark: Similar concepts can be defined in 3-D

Vector algebra operations

- Addition:

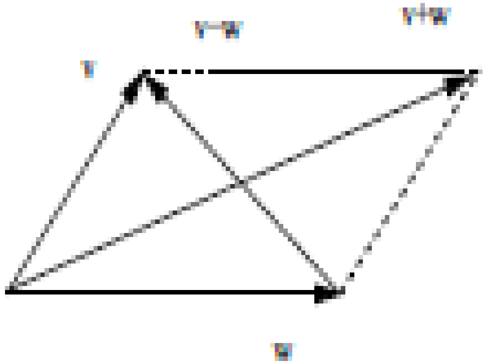
$$\vec{u} =< u_1, u_2, u_3 >, \vec{v} =< v_1, v_2, v_3 >$$
$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

- Scalar multiplications: Let *r* be a real number

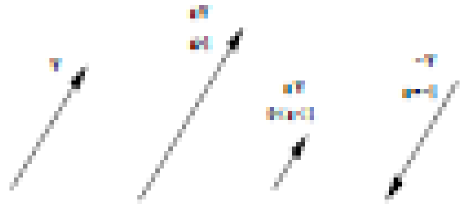
$$r\vec{u} =< ru_1, ru_2, ru_3 >$$

Addition and scalar multiplication.

Remark: The addition and difference of two vectors.



Remark: The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$.



Magnitude of a vector and unit vectors.

Definition

The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.

- If the vector $\overrightarrow{P_1P_2}$ has components

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,$$

then its magnitude, denoted as $|\overrightarrow{P_1P_2}|$, is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- If the vector $\mathbf{v}$ has components $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, then its magnitude, denoted as $|\mathbf{v}|$, is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Magnitude of a vector and unit vectors.

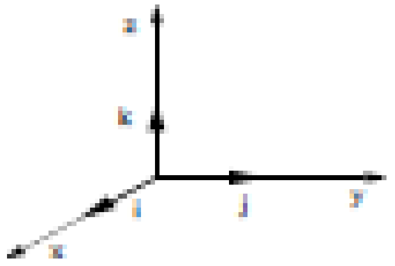
Definition
A vector $\mathbf{v}$ is a *unit vector* iff $\mathbf{v}$ has length one, that is, $|\mathbf{v}| = 1$.

Example
Show that $\mathbf{v} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$ is a unit vector.

Solution:

$$|\mathbf{v}| = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} \quad \Rightarrow \quad |\mathbf{v}| = 1.$$

Example
The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are useful to express any other vector in $\mathbb{R}^3$.



Addition and scalar multiplication.

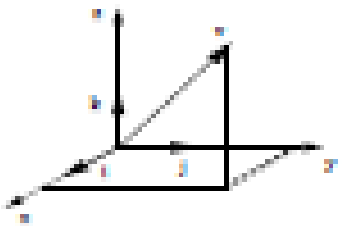
Theorem

Every vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *in* $\mathbb{R}^3$ *can be expressed in a unique way as a linear combination of vectors* $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, *and* $\mathbf{k} = \langle 0, 0, 1 \rangle$ *as follows*

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

Proof: Use the definitions of vector addition and scalar multiplication as follows,

$$\begin{aligned} \mathbf{v} &= \langle v_x, v_y, v_z \rangle \\ &= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\ &= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\ &= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}. \end{aligned}$$



□

Addition and Scalar Multiplication

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

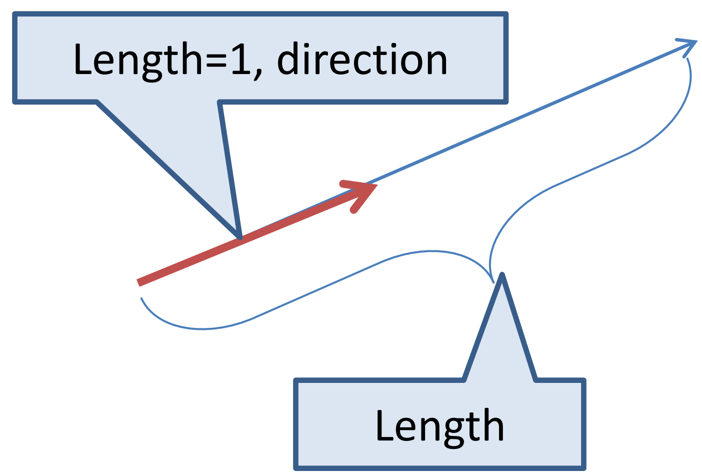
Vector decompositions

- $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$

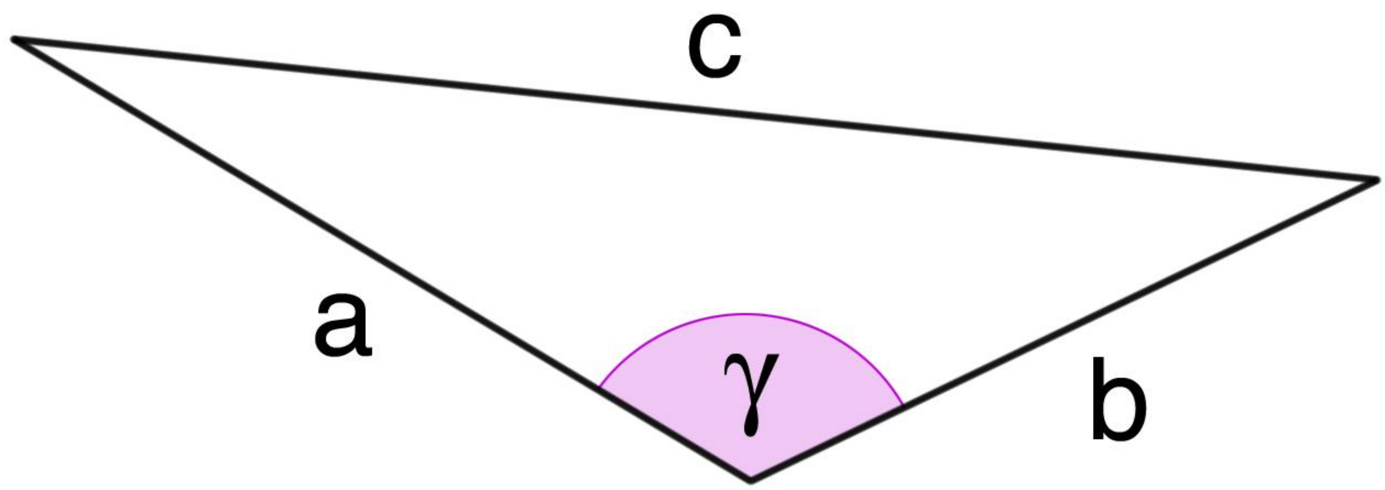
- $\vec{v} = |\vec{v}| \dfrac{\vec{v}}{|\vec{v}|}$

Length

Direction



Law of cosines

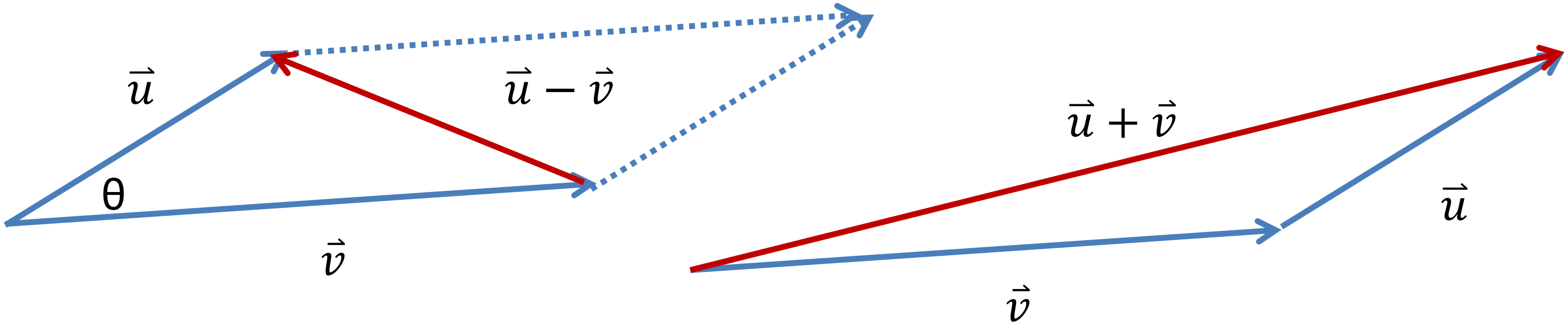


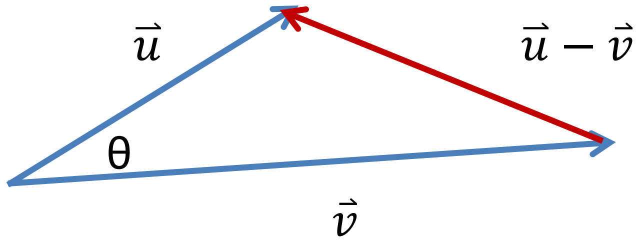
$$c^2 = a^2 + b^2 - 2ab(\cos(\gamma))$$

Theorem

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors (standard position) and θ be the angle between the two vectors. Then,

$$|\vec{u}||\vec{v}| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$





Idea of the proof:

Law of cosines

$$\rightarrow 2|\vec{u}||\vec{v}|\cos\theta = |u|^2 + |\vec{v}|^2 - |\vec{u} - \vec{v}|^2$$

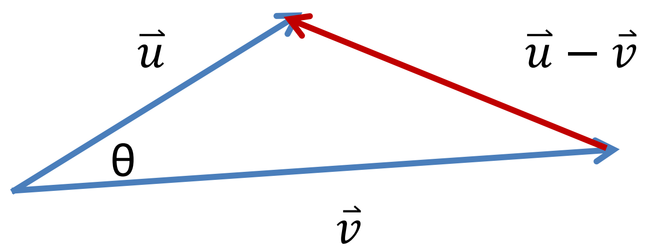
Also recall: $|u|^2 = {u_1}^2 + {u_2}^2 + {u_3}^2$

Definition:

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors (standard position) and θ be the angle between the two vectors. Then,

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$

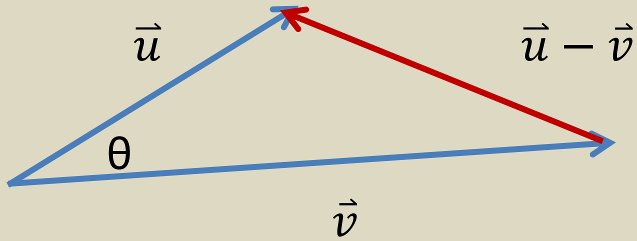
Is call the dot product of these two vectors



Definition

- Geometric definition:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$



- Algebraic definition:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

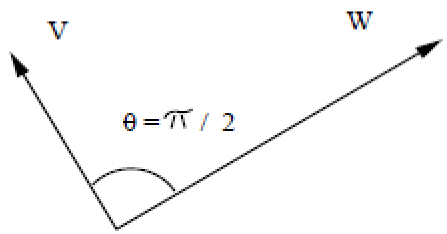
$\triangleleft$

- The angle between two vectors usually is not know in applications.
- It is useful to have a formula for the dot product involving the vector components.

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \cos(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right. \Leftrightarrow \quad \theta = \frac{\pi}{2}.$$

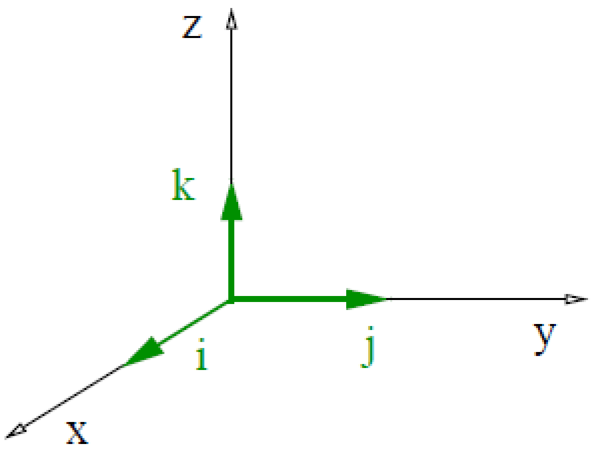
□

The dot product of **i**, **j** and **k** is simple to compute

Example

Compute all dot products involving the vectors **i**, **j**, and **k**.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



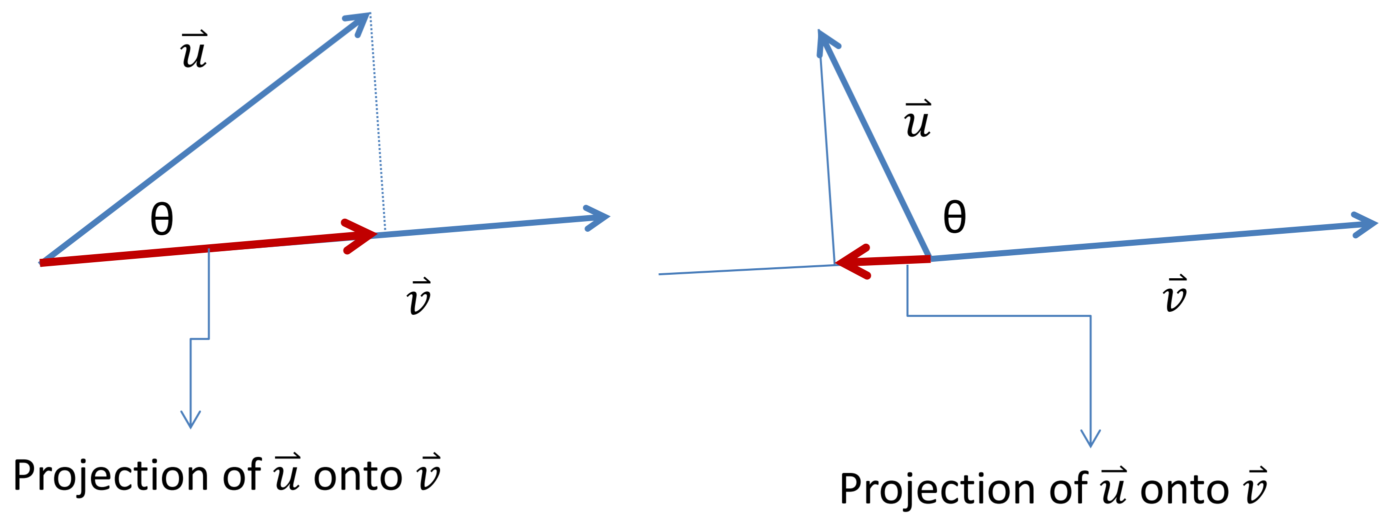
$$\mathbf{i} \cdot \mathbf{i} = 1, \qquad \mathbf{j} \cdot \mathbf{j} = 1, \qquad \mathbf{k} \cdot \mathbf{k} = 1,$$

$$\mathbf{i} \cdot \mathbf{j} = 0, \qquad \mathbf{j} \cdot \mathbf{i} = 0, \qquad \mathbf{k} \cdot \mathbf{i} = 0,$$

$$\mathbf{i} \cdot \mathbf{k} = 0, \qquad \mathbf{j} \cdot \mathbf{k} = 0, \qquad \mathbf{k} \cdot \mathbf{j} = 0.$$

◁

Projection of one vector onto the other vector



$$|\vec{u}|\cos\theta = \begin{cases} length\ of\ the\ red\ arrow, \theta < \pi/2 \\ -length\ of\ the\ red\ arrow, \theta > \pi/2 \end{cases}$$

----- Scalar component of ***u*** in the direction of ***v***. (Definition)

Definition

$$proj_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta)\frac{\vec{v}}{|\vec{v}|}$$

Signed length
Scalar component

Unit vector in the direction of $\vec{v}$

Definition (continue)

$$proj_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta)\frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|}$$

$$= \boxed{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}}$$

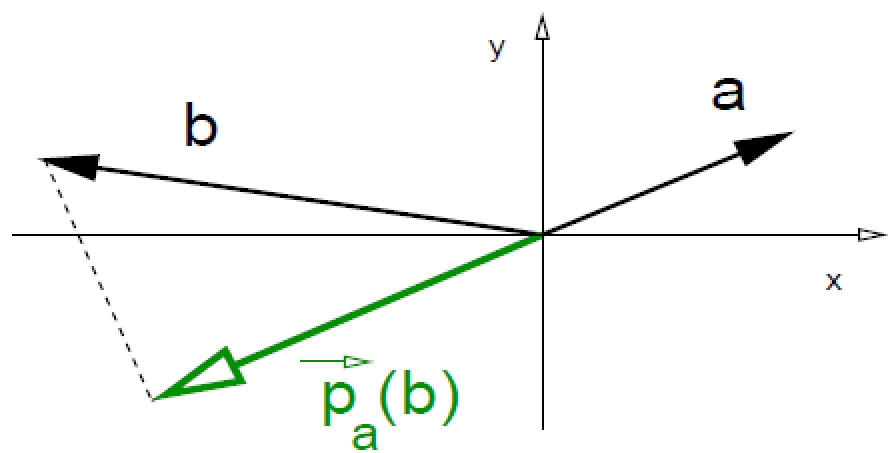
Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(-\frac{2}{\sqrt{5}}\right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = -\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.



Properties of dot product

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

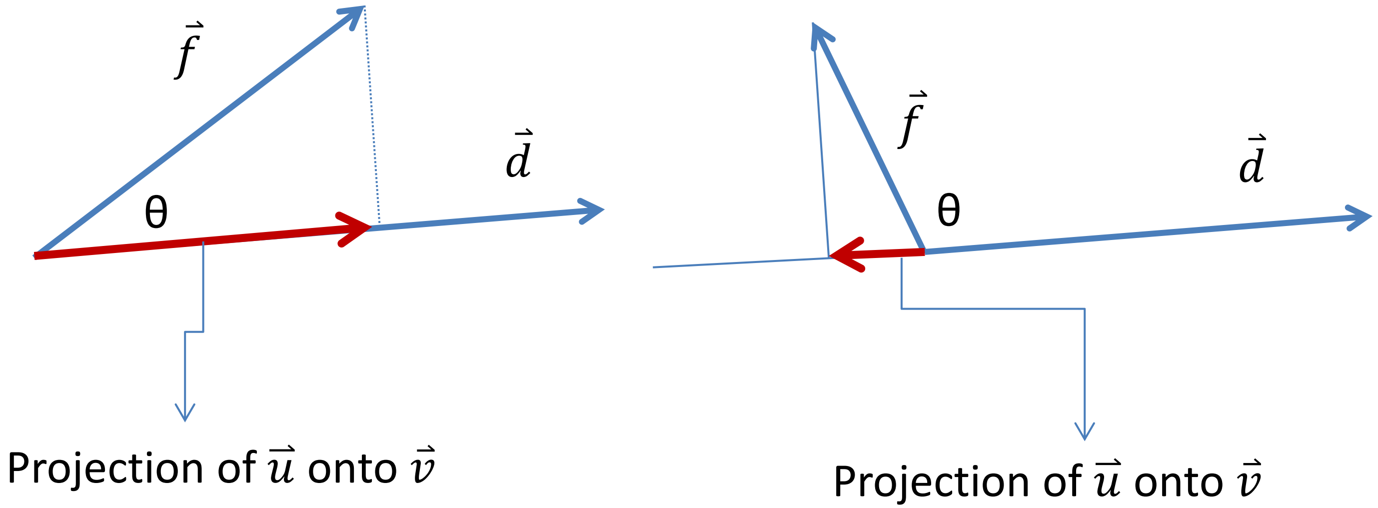
2. $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

5. $\vec{0} \cdot \vec{v} = 0$

Work



$$Work = Scalar\ componet * \left|\vec{d}\right| = \left|\vec{f}\right| \cos\theta * \left|\vec{d}\right|$$

$$= \boxed{\vec{f} \cdot \vec{d}}$$