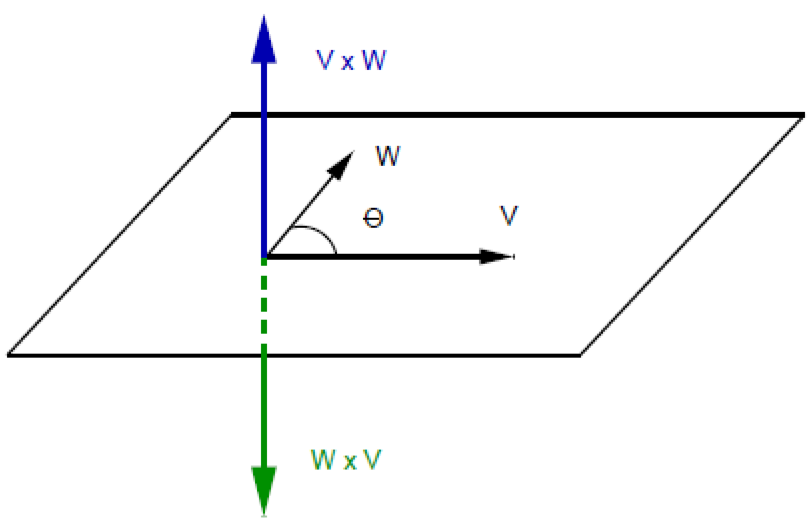
Geometric definition of cross product

Definition

The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

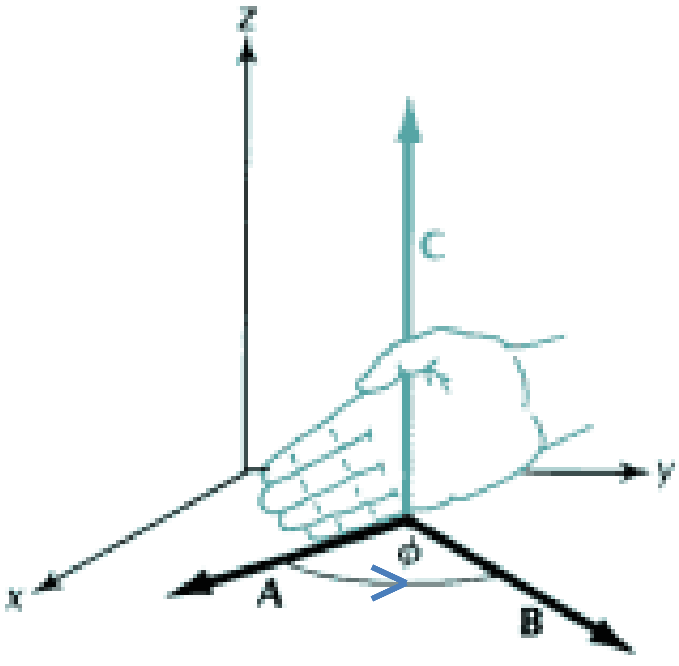
$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin(\theta).$$



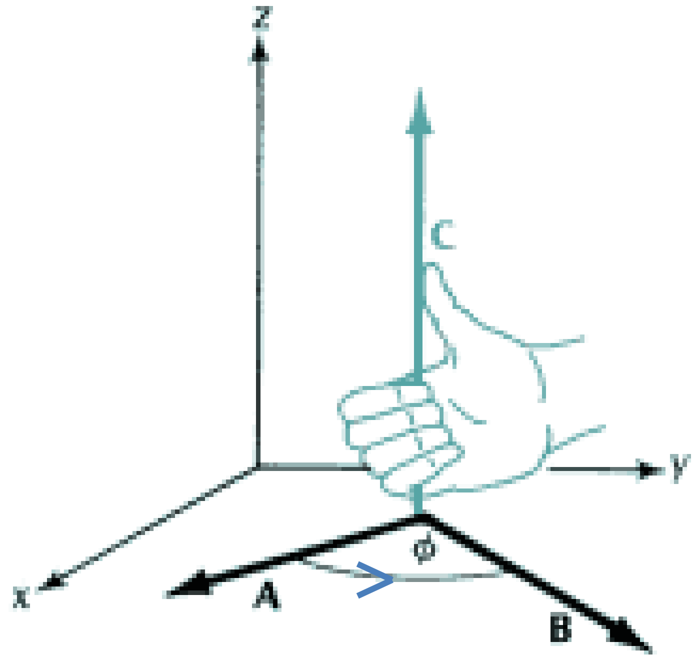
Remark: Cross product of two vectors is another vector; which is perpendicular to the original vectors.

Remark: norm ⟷ length

C = **A**x**B**, in right-hand system



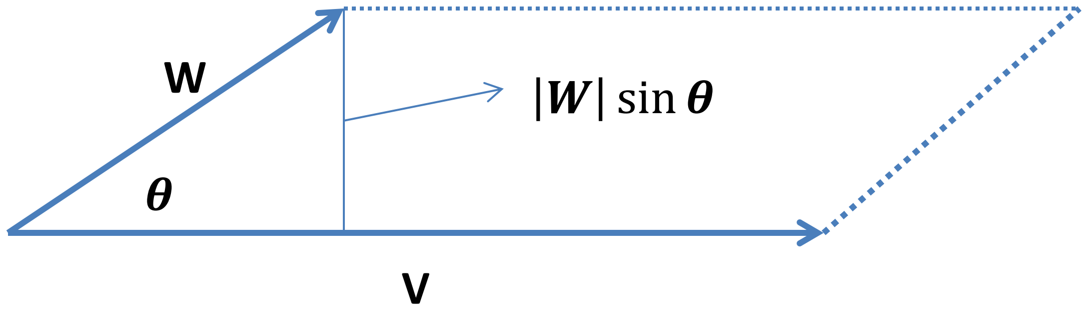
(a)



(b)

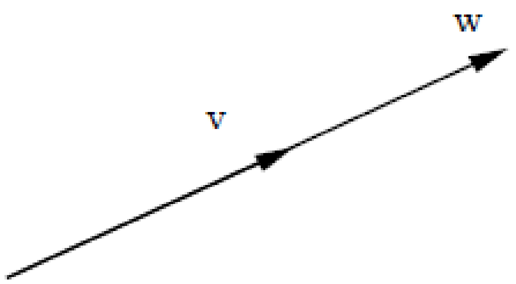
Property:

$|\mathbf{V} \times \mathbf{W}| = |\mathbf{V}||\boldsymbol{W}| \sin \boldsymbol{\theta}$ is equal to the area of the parallelogram formed by **V** and **W**.



Definition

Two vectors are *parallel* iff the angle in between them is $\theta = 0$.

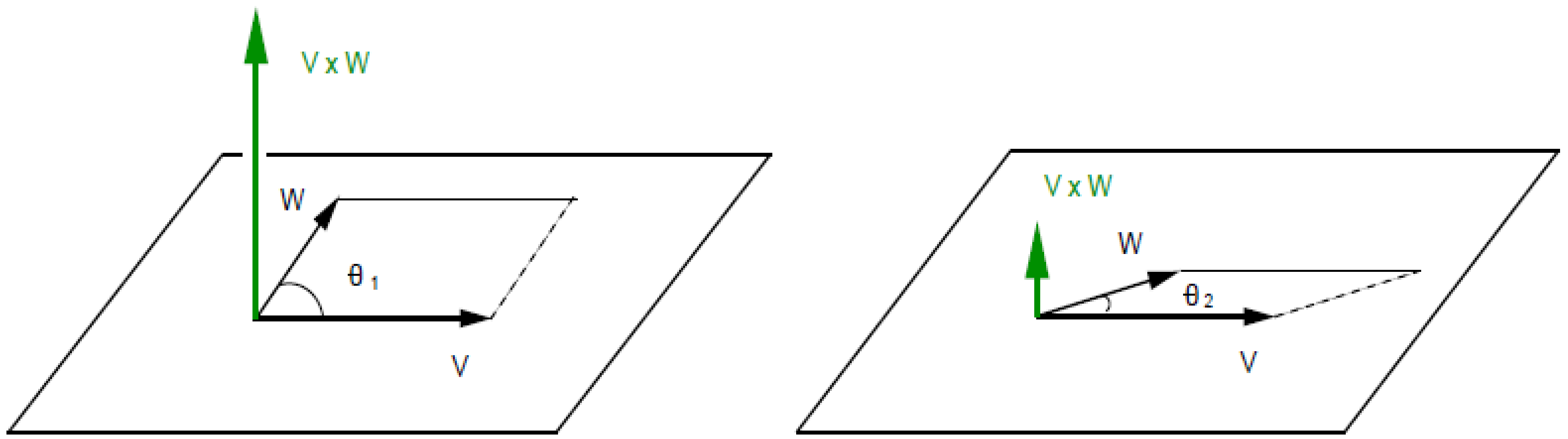


Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel iff* $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

Example

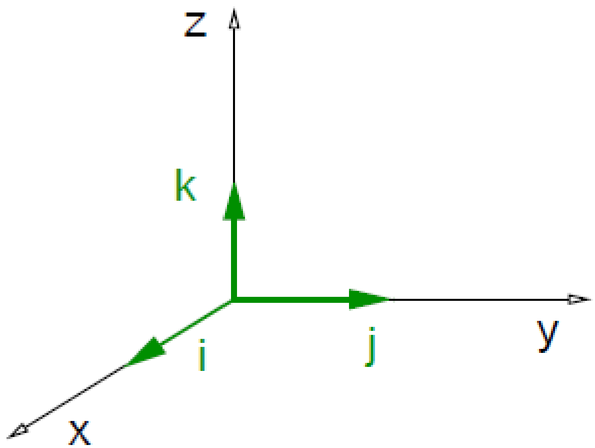
The closer the vectors **v**, **w** are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.



Example : Special and important cases:

Compute all cross products involving the vectors **i**, **j** , and **k**.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$,

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$,

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$,

Remark: (a)-(c) are trivial. (d) is proved in Appendix 8.

Example : Proof of (d) by a counter example.

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

We conclude that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. $\triangleleft$

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) \times (w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}).$$

Use the linearity property. The only non-zero terms involve $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and the symmetric analogues. The result is

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}. \qquad \square$$

Algebraic definition of cross product

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

or

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \quad \Rightarrow \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$

Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) The determinant of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

2×2 determinants are used to find 3×3 determinants.

Determinants to compute cross products.

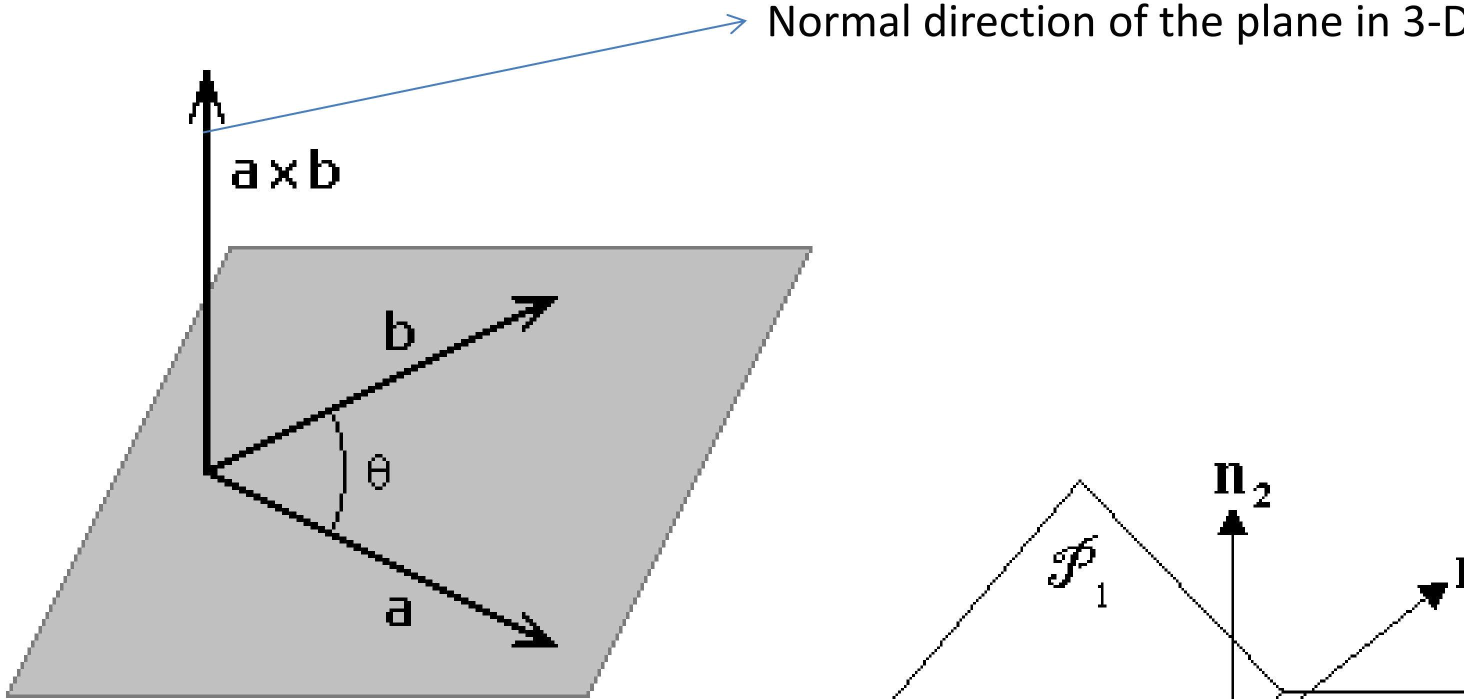
Theorem

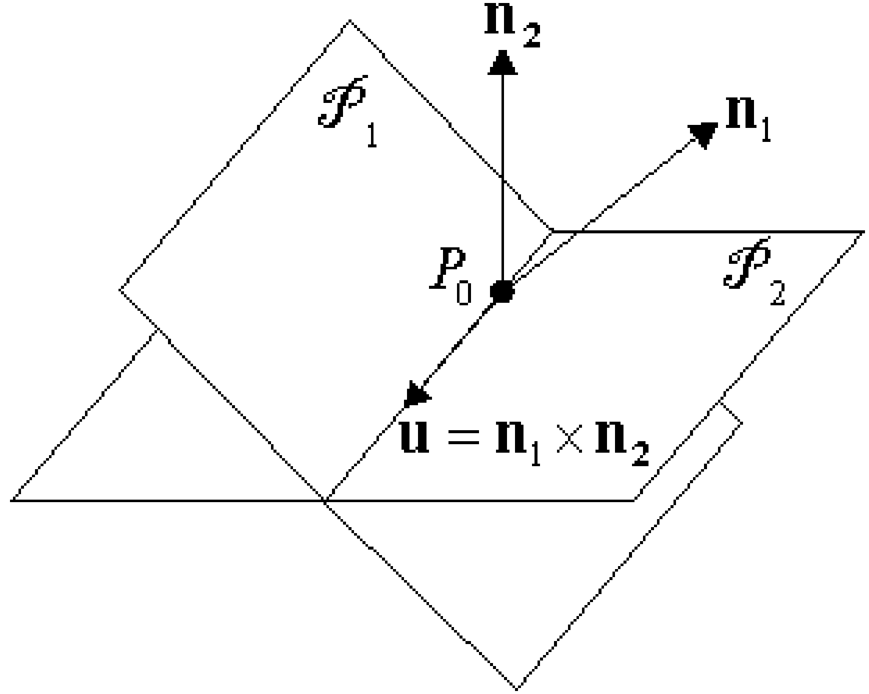
The formula to compute determinants of 3×3 matrices can be used to find the the cross product $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, as follows

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

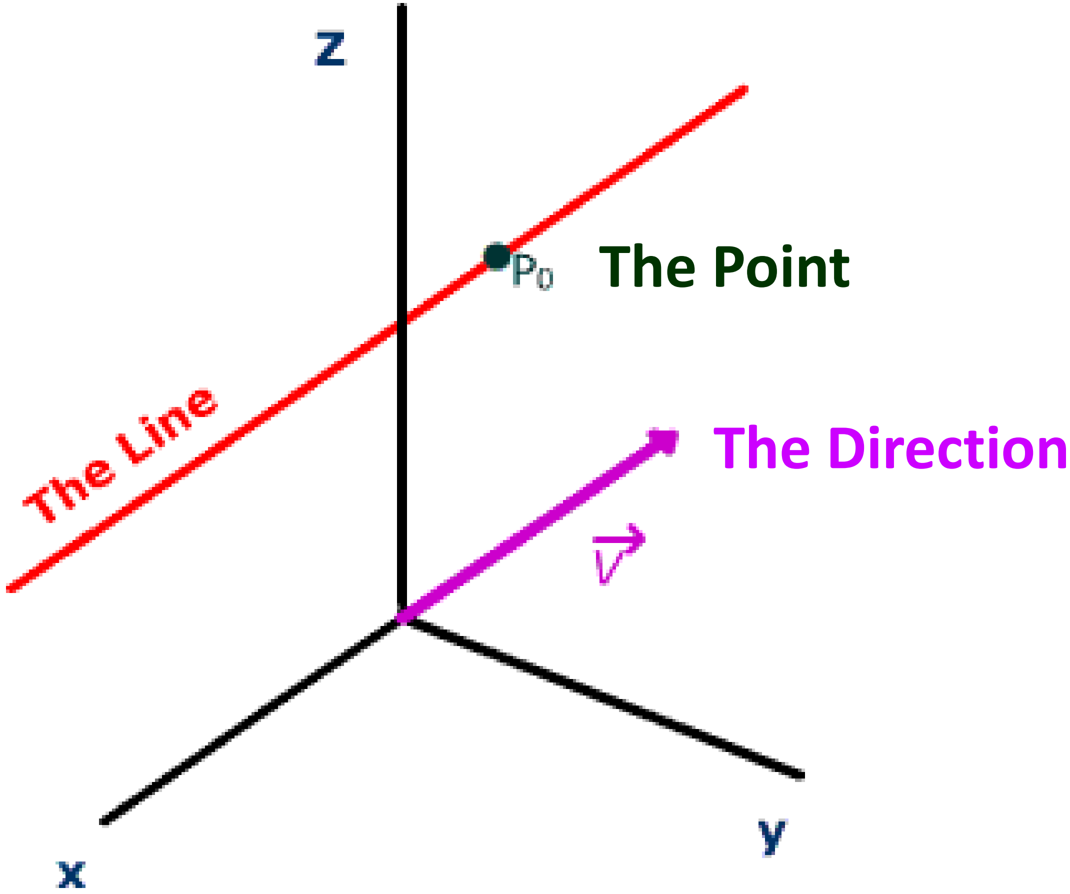
Proof: Exercise

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$


Normal direction of the plane in 3-D
a×b
b
θ
a


n2
n1
P1
P2
P0
u = n1 × n2

Equation of a line in 3-D

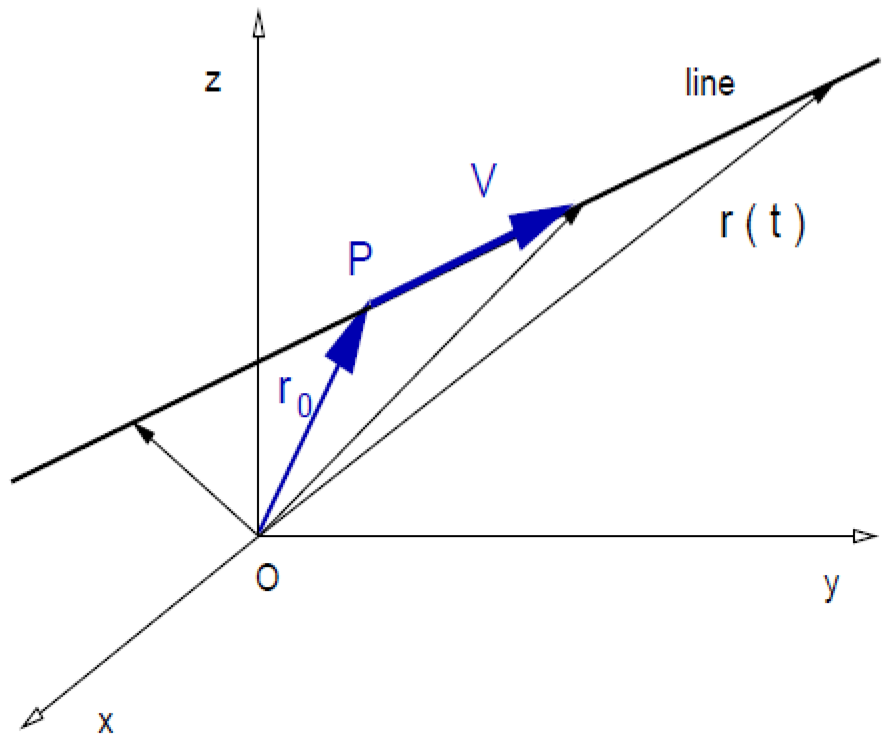


Vector equation of a line in space

Definition
The *vector equation of the line* by the point P parallel to the vector $\mathbf{v}$ is the set of terminal points of the vectors

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}, \quad t \in \mathbb{R},$$

where $\mathbf{r}_0 = \overrightarrow{OP}$ and O is the origin of Cartesian coordinates in $\mathbb{R}^3$.



Remark: A line can refer to both the set of vectors $\mathbf{r}(t)$ and the set of terminal points of these vectors.

Example

Find the vector equation of the line by the point $P = (1, -2, 1)$ parallel to the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

Solution:

First, we construct the vector $\mathbf{r}_0 = \overrightarrow{OP} = \langle 1, -2, 1 \rangle$.

Then, the formula $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ implies

$$\mathbf{r}(t) = \langle 1, -2, 1 \rangle + t\,\langle 1, 2, 3 \rangle.$$

$\triangleleft$

Definition

The *parametric equations of a line* by $P = (x_0, y_0, z_0)$ tangent to $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ are given by

$$x(t) = x_0 + t\, v_x\,,$$
$$y(t) = y_0 + t\, v_y\,,$$
$$z(t) = z_0 + t\, v_z\,.$$

Remark: It is simple to obtain the parametric equations f from he vector equation, and vice-versa, noticing the relation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$$
$$\langle x(t), y(t), z(t) \rangle = \langle x_0, y_0, z_0 \rangle + t\,\langle v_x, v_y, v_z \rangle$$
$$\langle x(t), y(t), z(t) \rangle = \langle (x_0 + t\, v_x), (y_0 + t\, v_y), (z_0 + t\, v_z) \rangle.$$

Parametric equation of a line in space

Example

Find the parametric equations of the line with vector equation

$$\mathbf{r}(t) = \langle 1, -2, 1 \rangle + t \, \langle 1, 2, 3 \rangle.$$

Solution: Rewrite the vector equation in vector components,

$$\langle x(t), y(t), z(t) \rangle = \langle (1 + t), (-2 + 2t), (1 + 3t) \rangle.$$

We conclude that

$$x(t) = 1 + t,$$
$$y(t) = -2 + 2t,$$
$$z(t) = 1 + 3t.$$

Example

Find both the vector equation and the parametric equation of the line containing the points $P = (1, 2, -3)$ and $Q = (3, -2, 1)$.

Solution: A vector tangent to the line is $\mathbf{v} = \overrightarrow{PQ}$, given by

$$\mathbf{v} = \langle (3-1), (-2-2), (1+3) \rangle \quad \Rightarrow \quad \mathbf{v} = \langle 2, -4, 4 \rangle.$$

Either P or Q can be used in the vector equation of the line.

If we choose P, then

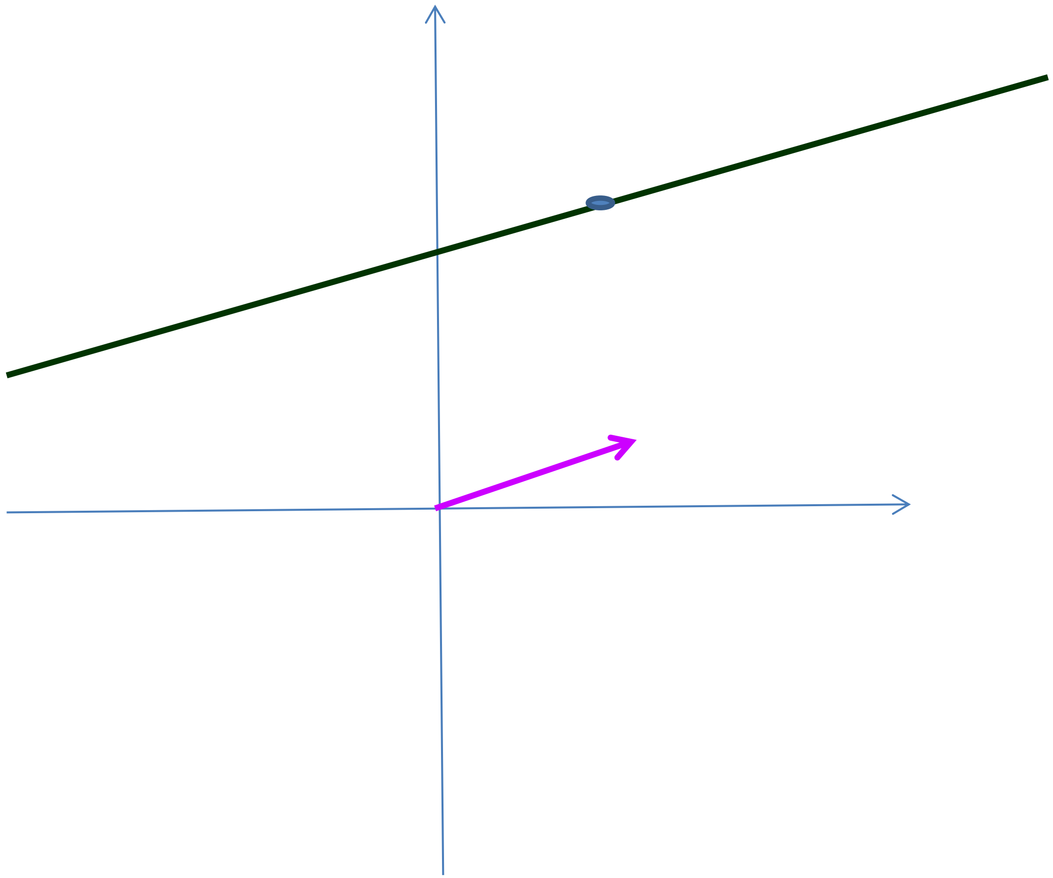
$$\mathbf{r}(t) = \langle 1, 2, -3 \rangle + t \, \langle 2, -4, 4 \rangle.$$

If we choose Q, then

$$\tilde{\mathbf{r}}(s) = \langle 3, -2, 1 \rangle + s \, \langle 2, -4, 4 \rangle.$$

Remark: t and s are different; $t = s + 1$, and $\mathbf{r}(s+1) = \tilde{\mathbf{r}}(s)$.

Lines in 2-D as special cases

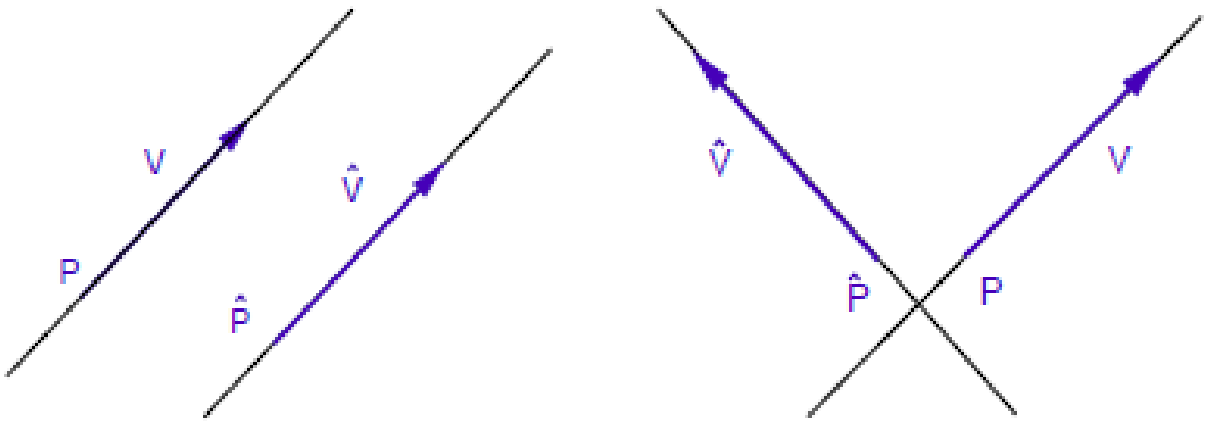


Velocity: $\vec{v} = \langle v_1, v_2, v_3 \rangle$; **Speed:** $|\vec{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

Parallel lines, perpendicular lines, intersections

Definition

The lines $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ and $\hat{\mathbf{r}}(t) = \hat{\mathbf{r}}_0 + t\,\hat{\mathbf{v}}$ are *parallel* iff their tangent vectors $\mathbf{v}$ and $\hat{\mathbf{v}}$ are parallel; they are *perpendicular* iff $\mathbf{v}$ and $\hat{\mathbf{v}}$ are perpendicular; and the lines *intersect* iff they have a common point.



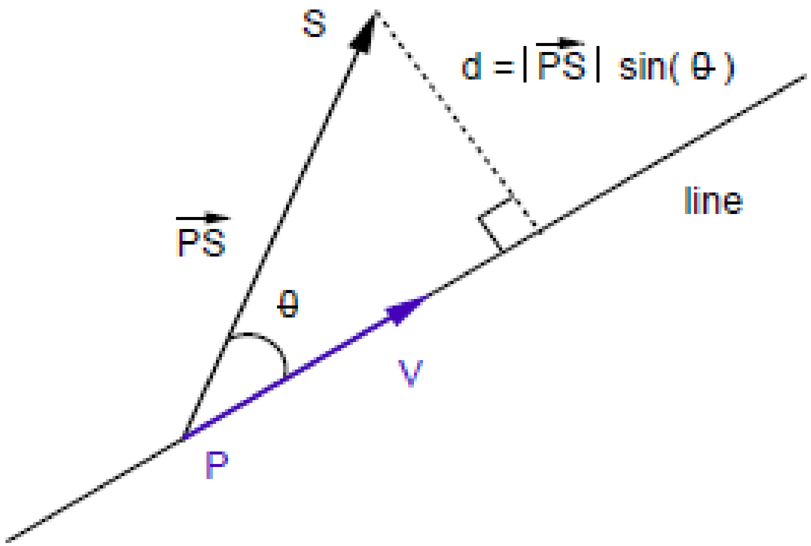
Remark: Perpendicular lines in space may not intersect.
Non-parallel lines in space may not intersect.

Distance from a point to a line in space

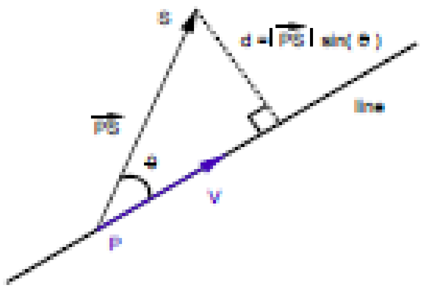
Theorem

The distance from a point S in space to a line through the point P with tangent vector $\mathbf{v}$ *is given by*

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$



Distance from a point to a line in space



Proof.

The distance from the point S to the line passing by the point P with tangent vector $\mathbf{v}$ is given by

$$d = |\overrightarrow{PS}| \sin(\theta).$$

Recalling that $|\overrightarrow{PS} \times \mathbf{v}| = |\overrightarrow{PS}|\,|\mathbf{v}| \sin(\theta)$, we conclude that

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

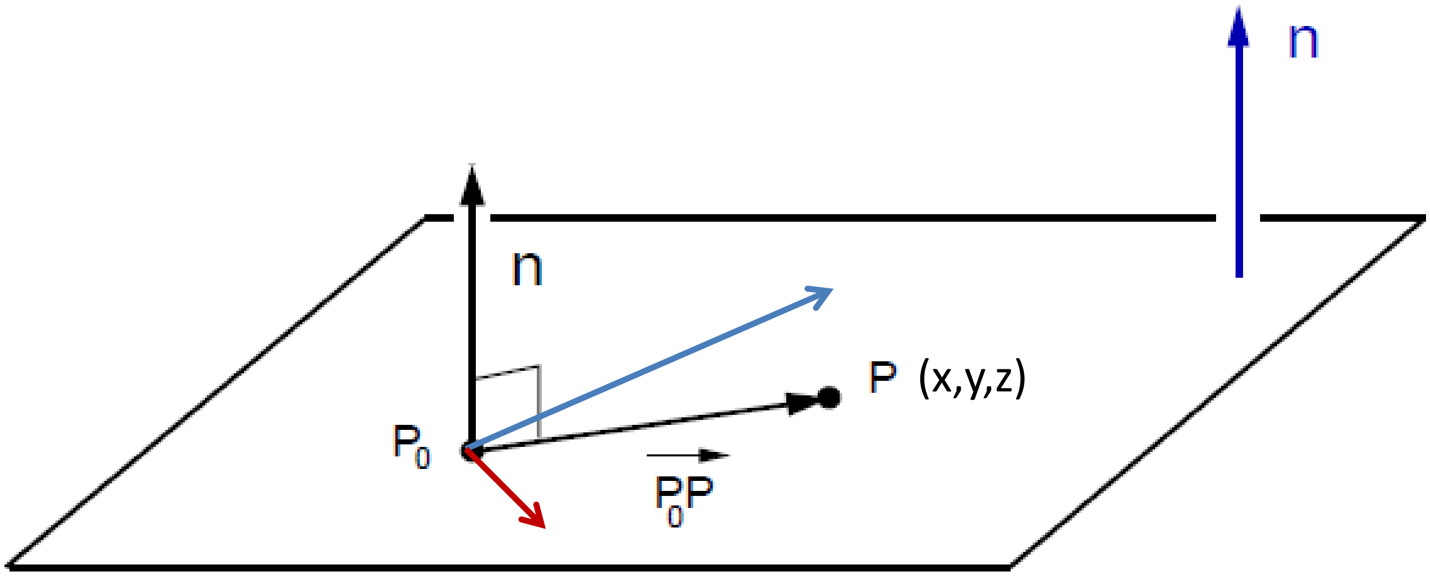
□

A point an a vector determine a plane.

Definition

The *plane* by a point P_0 perpendicular to a non-zero vector $\mathbf{n}$, called the *normal vector*, is the set of points P solution of the equation

$$(\overrightarrow{P_0 P}) \cdot \mathbf{n} = 0.$$



Equation of a plane in Cartesian coordinates

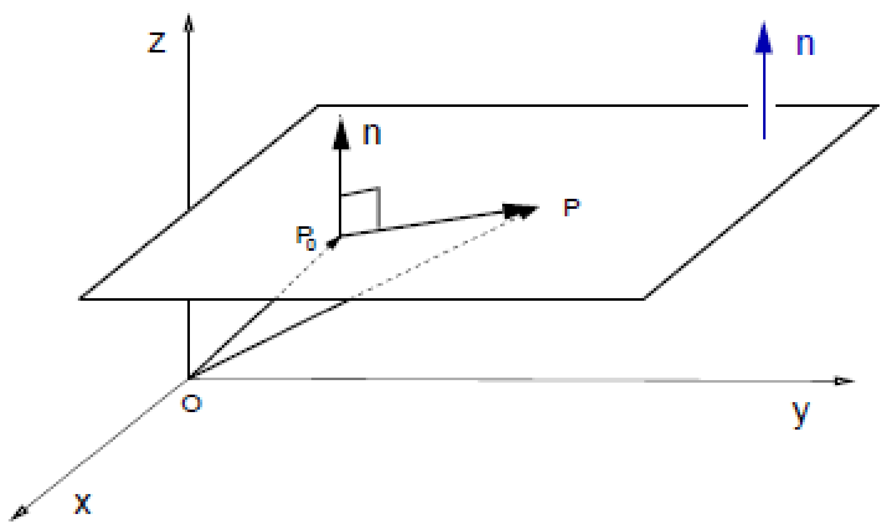
Theorem

Given any Cartesian coordinate system, the point $P = (x, y, z)$ *belongs to the plane by* $P_0 = (x_0, y_0, z_0)$ *perpendicular to* $\mathbf{n} = \langle n_x, n_y, n_z \rangle$ *iff holds*

$$(\overrightarrow{P_0 P}) \cdot \mathbf{n} = (x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0.$$

Furthermore, the equation above can be written as

$$n_x x + n_y y + n_z z = d, \qquad d = n_x x_0 + n_y y_0 + n_z z_0.$$



Example

Find the equation of a plane containing $P_0 = (1, 2, 3)$ and perpendicular to $\mathbf{n} = \langle 1, -1, 2 \rangle$.

Solution: The point $P = (x, y, z)$ belongs to the plane above iff $(\overrightarrow{P_0 P}) \cdot \mathbf{n} = 0$, that is,

$$\langle (x-1), (y-2), (z-3) \rangle \cdot \langle 1, -1, 2 \rangle = 0.$$

Computing the dot product above we get

$$(x-1) - (y-2) + 2(z-3) = 0.$$

The equation of the plane can be also written as

$$x - y + 2z = 5.$$

◁

Example:

Find the equation of a plane which passes the point (-2,2,1) and perpendicular to the line:

$$\langle x(t),\quad y(t),\quad z(t)\rangle = \langle 1+t,\ 3t,\ 2-5t\rangle$$

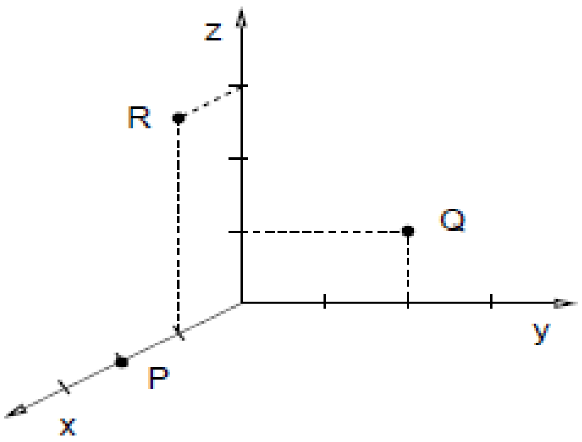
Solution: The direction of the line, <1,3,-5> is also the normal direction of the plane. This implies that the equations is:

$$\boldsymbol{(x+2)+3(y-2)-5(z-1)=0}$$

$$\xrightarrow{\textbf{yields}}\quad \boldsymbol{x+3y-5z=-1}$$

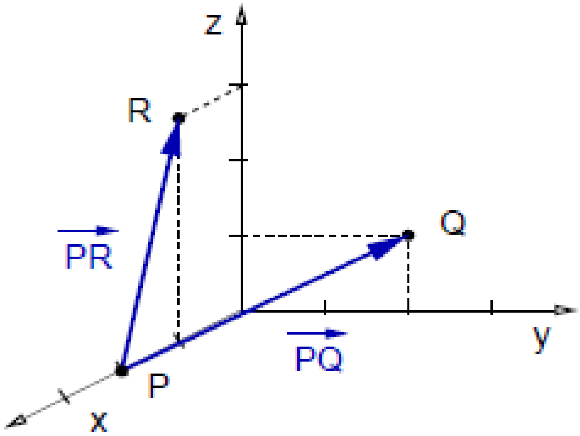
Example

Find the equation of the plane containing the points $P = (2, 0, 0)$, $Q = (0, 2, 1)$, $R = (1, 0, 3)$.



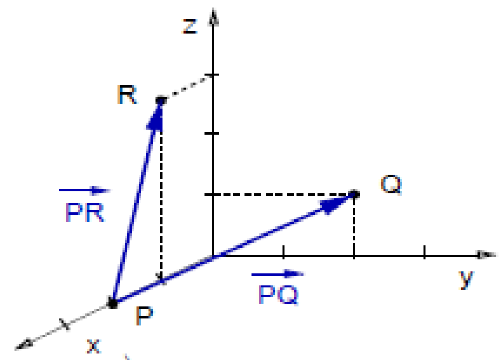
Solution:

Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$.



Solution:

Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1\rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3\rangle$.



Find a vector $\mathbf{n}$ perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
One way is using the cross product: $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. That is,

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -1 & 0 & 3 \end{vmatrix} = (6-0)\,\mathbf{i} - (-6+1)\,\mathbf{j} + (0+2)\,\mathbf{k}.$$

The result is: $\mathbf{n} = \langle 6, 5, 2\rangle$. Choose any point on the plane, say $P = (2, 0, 0)$. Then, the equation of the plane is:

$$6(x-2) + 5y + 2z = 0.$$

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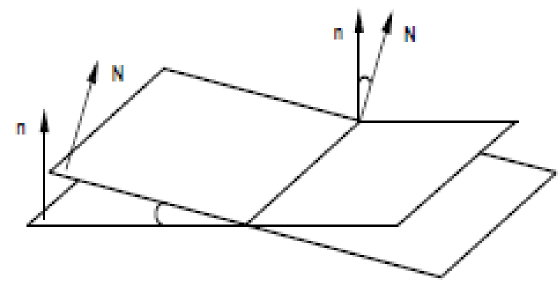
The line of intersection of two planes.

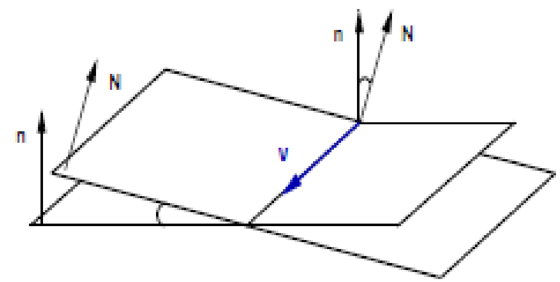
Example

Find a vector tangent to the line of intersection of the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.



Solution:

We need to find a vector perpendicular to both normal vectors $\mathbf{n} = \langle 2, 1, -3 \rangle$ and $\mathbf{N} = \langle -1, 2, -1 \rangle$.



We choose $\mathbf{v} = \mathbf{N} \times \mathbf{n}$. That is,

$$\mathbf{v} = \mathbf{N} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = (-6 + 1)\,\mathbf{i} - (3 + 2)\,\mathbf{j} + (-1 - 4)\,\mathbf{k}$$

Result: $\mathbf{v} = \langle -5, -5, -5 \rangle$. A simpler choice is $\mathbf{v} = \langle 1, 1, 1 \rangle$. $\triangleleft$

Example (Continue) :

Find the parametric equation of the intersection line.

Solution: We have the direction of the line <1,1,1>. So we just need a point on the line (any point),

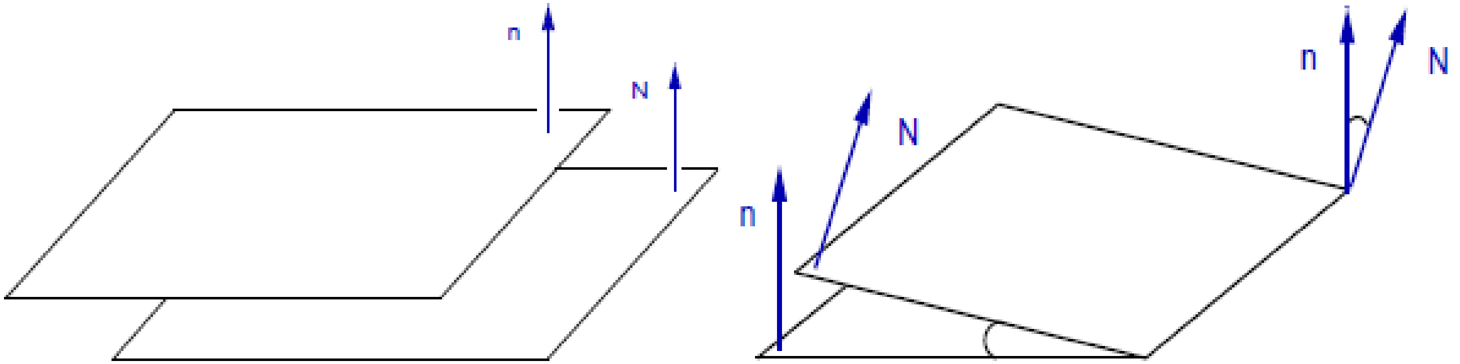
$$2x + y - 3z = 2 \text{ and } -x + 2y - z = 1.$$

$$Let\ x = 0, \quad \begin{cases} y - 3z = 2 \\ 2y - z = 1 \end{cases} \rightarrow \begin{cases} y = 1/5 \\ z = -3/5 \end{cases}$$

Parametric equation: $\begin{cases} x(t) = t \\ y(t) = 0.2 + t \\ z(t) = -0.6 + t \end{cases}$

Definition

Two planes are *parallel* if their normal vectors are parallel. The *angle* between two non-parallel planes is the smaller angle between their normal vectors.



Parallel planes and angle between planes

Example

Find the angle between the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.

Solution: We need to find the angle between the normal vectors $\mathbf{n} = \langle 2, 1, -3 \rangle$ and $\mathbf{N} = \langle -1, 2, -1 \rangle$.

We use the dot product: $\boxed{\cos(\theta) = \dfrac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}|\,|\mathbf{N}|}.}$

The numbers we need are:

$$\mathbf{n} \cdot \mathbf{N} = -2 + 2 + 3 = 3,$$

$$|\mathbf{n}| = \sqrt{4 + 1 + 9} = \sqrt{14}, \qquad |\mathbf{N}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

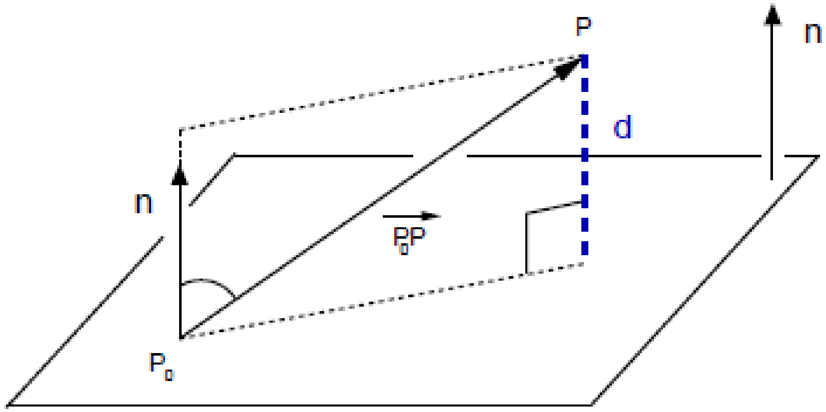
Therefore, $\cos(\theta) = 3/\sqrt{84}$. We conclude that $\theta = \cos^{-1} \dfrac{3}{\sqrt{84}}$

Distance formula from a point to a plane

Theorem

The distance d from a point P to a plane containing P_0 with normal vector $\mathbf{n}$ *is the shortest distance from P to any point in the plane, and is given by the expression*

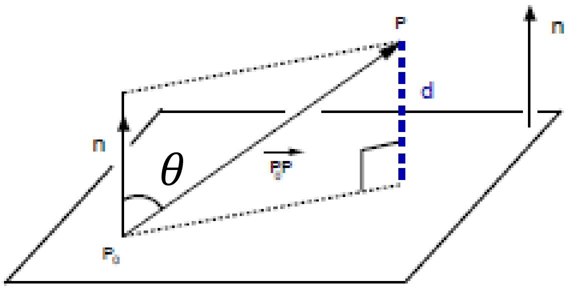
$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



Distance formula from a point to a plane

Proof: It is simple to obtain the distance formula

$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



From the picture above, and denoting θ is the angle between $\overrightarrow{P_0P}$ and $\mathbf{n}$, we see that

$$d = \left||\overrightarrow{P_0P}|\cos(\theta)\right| = \left|\frac{(\overrightarrow{P_0P}) \cdot \mathbf{n}}{|\mathbf{n}|}\right|.$$

□