Integration of vector functions

Definition
An *antiderivative* of a vector function $\mathbf{v}$ is any vector valued function $\mathbf{V}$ such that $\mathbf{V}' = \mathbf{v}$.

Remark: Antiderivatives are also called *indefinite integrals*, or *primitives*, they are denoted as $\int \mathbf{v}(t)\,dt$, that is,

$$\int \mathbf{v}(t)\,dt = \mathbf{V}(t) + \mathbf{C},$$

where $\mathbf{C}$ is a constant vector in Cartesian coordinates.

Example
Verify that $\mathbf{V} = \langle(-\cos(3t)/3 + 1), (\sin(t) - 2), (e^{2t}/2 + 2)\rangle$ is an antiderivative of $\mathbf{v} = \langle\sin(3t), \cos(t), e^{2t}\rangle$.

Solution: $\mathbf{V}' = \langle(-\cos(3t)/3 + 1)', (\sin(t) - 2)', (e^{2t}/2 + 2)'\rangle = \mathbf{v}$.

Integrals of vector functions.

Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t)\rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1\rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t)\rangle + \langle c_x, c_y, c_z\rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z\rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1\rangle = \mathbf{r}(0) = \langle 0, 0, -1\rangle + \langle c_x, c_y, c_z\rangle,$$

that is, $c_x = 1$, $c_y = 1$, $c_z = 2$.

The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2\rangle$. ◁

Example
Find the position function of a particle with acceleration $\mathbf{a}(t) = \langle 0, 0, -10\rangle$ having an initial velocity $\mathbf{v}(0) = \langle 0, 1, 1\rangle$ and initial position $\mathbf{r}(0) = \langle 1, 0, 1\rangle$.

Solution: The velocity is the antiderivative of the acceleration:

$$\mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z})\rangle,$$

where $\mathbf{v}_0 = \langle v_{0x}, v_{0y}, v_{0z}\rangle$ is fixed by the initial condition.

$$\mathbf{v}(0) = \langle 0, 1, 1\rangle = \langle v_{0x}, v_{0y}, v_{0z}\rangle$$

The velocity function is $\mathbf{v}(t) = \langle 0, 1, (-10t + 1)\rangle$.

The position is $\mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z})\rangle$, and

$$\mathbf{r}(0) = \langle 1, 0, 1\rangle = \langle r_{0x}, r_{0y}, r_{0z}\rangle,$$

The obtain that $\mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1)\rangle$.

Integrals of vector functions.

Definition
The *definite integral* of an integrable vector function $\mathbf{r}(t) = \langle \mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t)\rangle$ on the interval $[a, b]$ is given by

$$\int_a^b \mathbf{r}(t)dt = \Big\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \Big\rangle.$$

Example
Compute $\displaystyle\int_0^\pi \mathbf{r}(t)\,dt$ for the function $\mathbf{r}(t) = \langle\cos(t), \sin(t), t\rangle$.

Solution: We compute an antiderivative and we evaluate the result,

$$\mathbf{I} = \int_0^\pi \mathbf{r}(t)\,dt = \int_0^\pi \langle\cos(t), \sin(t), t\rangle\,dt.$$

Example

Compute $\int_0^{\pi} \mathbf{r}(t)\, dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

$$\mathbf{I} = \int_0^{\pi} \mathbf{r}(t)\, dt = \int_0^{\pi} \langle \cos(t), \sin(t), t \rangle dt.$$

$$\mathbf{I} = \left\langle \int_0^{\pi} \cos(t) dt, \int_0^{\pi} \sin(t) dt, \int_0^{\pi} t dt \right\rangle,$$

$$\mathbf{I} = \left\langle \sin(t)\Big|_0^{\pi}, -\cos(t)\Big|_0^{\pi}, \frac{t^2}{2}\Big|_0^{\pi} \right\rangle$$

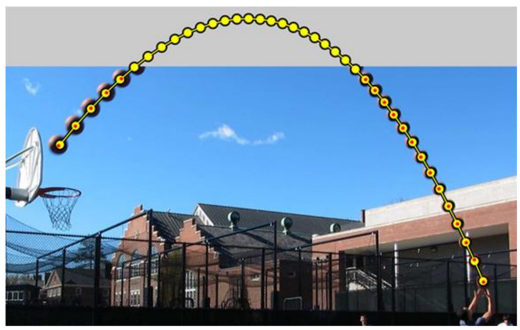
$$\mathbf{I} = \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle \quad \Rightarrow \quad \int_0^{\pi} \mathbf{r}(t)\, dt = \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle.$$

General 3-D motion

- Position: $\vec{r}(t)$; Velocity: $\vec{v}(t) = \vec{r}'(t)$; Acceleration: $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$.
- $\vec{v}(t) = \int \vec{a}(t) dt + \overrightarrow{c_1}$;
  $\vec{r}(t) = \int \vec{v}(t) dt + \overrightarrow{c_2}$;

Where $\overrightarrow{c_1}$ and $\overrightarrow{c_2}$ are constant vectors which can be determined by initial conditions.

Ideal projectile motion



Equations of a projectile motion

Remark: Projectile motion is the position of a point particle moving near the Earth surface subject to gravitational attraction.

Theorem

The motion of a particle with initial velocity $\mathbf{v}_0$ *and position* $\mathbf{r}_0$ *subject to an acceleration* $\boxed{\mathbf{a} = -g\,\mathbf{k}}$*, where* g *is a constant, is*

$$\mathbf{r}(t) = -\frac{g}{2}t^2\mathbf{k} + \mathbf{v}_0 t + \mathbf{r}_0.$$

Remarks:

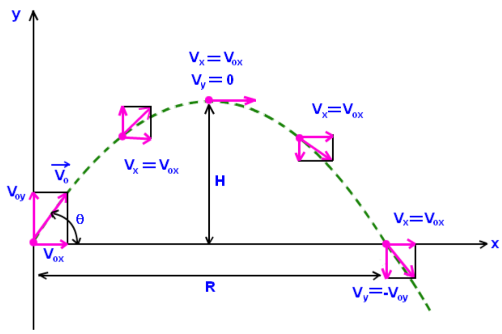
(a) The equation above in vector components is

$$\mathbf{r}(t) = \Big\langle (v_{0x}t + r_{0x}), (v_{0y}t + r_{0y}), \Big(-\frac{g}{2}t^2 + v_{0z}t + r_{0z}\Big)\Big\rangle,$$

where $\mathbf{v}_0 = \langle v_{0x}, v_{0y}, v_{0z}\rangle$ and $\mathbf{r}_0 = \langle r_{0x}, r_{0y}, r_{0z}\rangle$.

(b) The motion occurs in a plane. We describe it with vectors in the plane $\mathbb{R}^2$. We use the coordinates x, y, only.

View as a 2-D problem: $\vec{\boldsymbol{a}} = -g\vec{\boldsymbol{j}}$



Equations of a projectile motion

Remark: Same Theorem, written in x, y coordinates in $\mathbb{R}^2$.

Theorem

The motion of a particle with initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$ *and position* $\mathbf{r}_0 = r_{0x}\mathbf{i} + r_{0y}\mathbf{j}$ *subject to the acceleration* $\boxed{\mathbf{a} = -g\,\mathbf{j}\,,}$ *where* g *is a constant, is*

$$\mathbf{r}(t) = -\frac{g}{2}\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0,$$

equivalently, $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \Big(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\Big)\mathbf{j}.$

Proof: Since $\mathbf{r}''(t) = -g\,\mathbf{j}$, then $\mathbf{r}'(t) = c_x\mathbf{i} + (-gt + c_y)\mathbf{j}$.

$$\mathbf{r}'(0) = v_{0x}\mathbf{i} + v_{0y}\mathbf{j} = c_x\mathbf{i} + c_y\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = v_{0x}\mathbf{i} + (-gt + v_{0y})\mathbf{j}.$$

One more integration, $\mathbf{r}(t) = (d_x + v_{0x}t)\mathbf{i} + (d_y + v_{0y}t - \frac{g}{2}t^2)\mathbf{j}$.

The initial condition $\mathbf{r}(0) = r_{0x}\mathbf{i} + r_{0y}\mathbf{j} = d_x\mathbf{i} + d_y\mathbf{j}$,

implies that $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \Big(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\Big)\mathbf{j}$. □

Example

Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4$ m/s, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

Solution: The projectile acceleration is $\mathbf{a} = -g\,\mathbf{j}$, with $g = 10$ m/s²

Therefore, $\mathbf{v}(t) = (-10t + v_{0y})\mathbf{j} + v_{0x}\mathbf{i}$, where

$$v_{0y} = |\mathbf{v}_0|\sin(\theta) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}, \quad v_{0x} = |\mathbf{v}_0|\cos(\theta) = 4\,\frac{1}{2} = 2.$$

Since $\mathbf{v}(t) = (-10t + 2\sqrt{3})\mathbf{j} + 2\mathbf{i}$ and $\mathbf{r}_0 = \mathbf{0}$, then

$$\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.$$

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -5\Big(\frac{x^2}{4}\Big) + 2\sqrt{3}\,\frac{x}{2} \quad\Rightarrow\quad y(x) = -\frac{5}{4}x^2 + \sqrt{3}\,x. \quad \triangleleft$$

Rmk: g = 9.8 ≈ 10 m/s²

Range, Height, Flight Time

Theorem

The the range x_r, height y_h, and the fight time t_r of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Remark: Since the initial speed $|\mathbf{v}_0|$ and the elevation angle θ determine v_{0y} and v_{0x} by the equations

$$v_{0y} = |\mathbf{v}_0|\,\sin(\theta), \qquad v_{0x} = |\mathbf{v}_0|\,\cos(\theta),$$

then holds

$$x_r = \frac{|\mathbf{v}_0|^2\,\sin(2\theta)}{g}, \quad y_h = \frac{|\mathbf{v}_0|^2\,\sin^2(\theta)}{2g}, \quad t_r = \frac{2|\mathbf{v}_0|\,\sin(\theta)}{g}.$$

Theorem

The the range x_r, height y_h, and the fight time t_r of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \qquad y_h = \frac{(v_{0y})^2}{2g}, \qquad t_r = \frac{2v_{0y}}{g}.$$

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2}\,x^2 + \frac{v_{0y}}{v_{0x}}\,x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

$$-\frac{g}{2v_{0x}}\,x_r + v_{0y} = 0 \quad\Rightarrow\quad x_r = \frac{2v_{0x}v_{0y}}{g}.$$

The maximum height occurs where $y'(x) = 0$, that is,

$$-\frac{g}{v_{0x}^2}\,x_h + \frac{v_{0y}}{v_{0x}} = 0 \quad\Rightarrow\quad x_h = \frac{v_{0x}v_{0y}}{g} \quad\Rightarrow\quad x_h = \frac{x_r}{2}.$$

(Continue:)

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2}\,x^2 + \frac{v_{0y}}{v_{0x}}\,x$, and $x_h = \frac{v_{0x}v_{0y}}{g}$.

Then, the maximum height $y_h = y(x_h)$ is

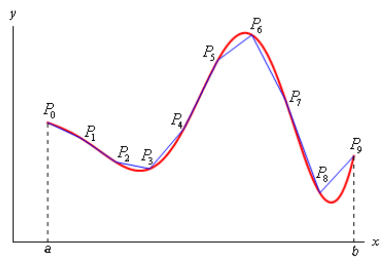
$$y_h = -\frac{g}{2v_{0x}^2}\,\frac{v_{0x}^2 v_{0y}^2}{g^2} + \frac{v_{0y}}{v_{0x}}\,\frac{v_{0x}v_{0y}}{g} = -\frac{v_{0y}^2}{2g} + \frac{v_{0y}^2}{g} \quad\Rightarrow\quad y_h = \frac{v_{0y}^2}{2g}.$$

Recalling that $x(t) = v_{0x}t$, then the flight time t_r is

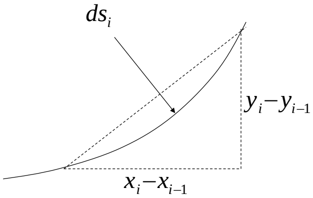
$$t_r = \frac{x_r}{v_{0x}} \quad\Rightarrow\quad t_r = \frac{2v_{0y}}{g}.$$

□

Recall: Arc length in R²

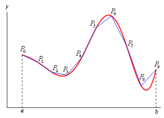


Idea:



$$ds_i \approx \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Idea:



Let $x = x(t), y = y(t)$ --- parametric

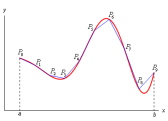
Arc length fro $t = \alpha$ to $t = \beta$

$$S = \sum ds_i \approx \sum \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sum \sqrt{\frac{(x_i - x_{i-1})^2}{(t_i - t_{i-1})^2} + \frac{(y_i - y_{i-1})^2}{(t_i - t_{i-1})^2}}\,(t_i - t_{i-1})$$

$$= \sum \sqrt{\frac{\Delta x_i^2}{\Delta t_i^2} + \frac{\Delta y_i^2}{\Delta t_i^2}}\,\Delta t_i \text{ --- Riemann sum}$$

Parametric formula:



Let $x = x(t), y = y(t)$ --- parametric

Arc length fro $t = \alpha$ to $t = \beta$

$$S = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

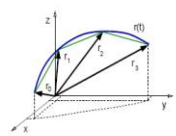
$$\boldsymbol{r}(t) = \langle x(t),\ \ y(t)\rangle,\ \ \boldsymbol{r}'(t) = \langle x'(t),\ \ y'(t)\rangle$$

$$S = \int_\alpha^\beta |\boldsymbol{r}'(t)|\,dt$$

Generalization to R^3

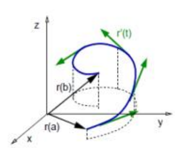
The length of a curve in space

Definition
The *length* or *arc length* of a curve in the plane or in space is the limit of the polygonal line length, as the polygonal line approximates the original curve.



Theorem
The length of a continuously differentiable curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$, *with* n=2,3, *is the number*

$$\ell_{ba} = \int_a^b |\mathbf{r}'(t)|\,dt.$$



The length of a curve in space

Recall: The length of $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is $\ell_{ba} = \displaystyle\int_a^b |\mathbf{r}'(t)|\, dt$.

- If the curve $\mathbf{r}$ is the path traveled by a particle in space, then $\mathbf{r}' = \mathbf{v}$ is the velocity of the particle.
- The length is the integral in time of the particle speed $|\mathbf{v}(t)|$.
- Therefore, the length of the curve is the distance traveled by the particle.
- In Cartesian coordinates the functions $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \qquad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Therefore the curve length is given by the expression

$$\ell_{ba} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

Example

Find the length of the curve $\mathbf{r}(t) = \langle r_0 \cos(t), r_0 \sin(t)\rangle$, for $t \in [\pi/4, 3\pi/4]$, and $r_0 > 0$.

Solution: Compute $\mathbf{r}'(t) = \langle -r_0 \sin(t), r_0 \cos(t)\rangle$. The length of the curve is given by the formula

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{\big[-r_0 \sin(t)\big]^2 + \big[r_0 \cos(t)\big]^2}\, dt$$

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{r_0^2\big(\big[-\sin(t)\big]^2 + \big[\cos(t)\big]^2\big)}\, dt = \int_{\pi/4}^{3\pi/4} r_0\, dt.$$

Hence, $\ell = \dfrac{\pi}{2}\, r_0$. (The length of quarter circle of radius r_0.)

Example

Find the length of the spiral $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$, for $t \in [0, t_0]$.

Solution: The derivative vector is

$$\mathbf{r}'(t) = \big\langle \big[-t\sin(t) + \cos(t)\big], \big[t\cos(t) + \sin(t)\big]\big\rangle,$$

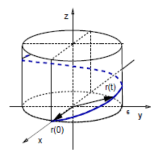
$$|\mathbf{r}'(t)|^2 = \big[t^2 \sin^2(t) + \cos^2(t) - 2t\sin(t)\cos(t)\big] + \big[t^2\cos^2(t) + \sin^2(t) + 2t\sin(t)\cos(t)\big]$$

We obtain $|\mathbf{r}'(t)|^2 = t^2 + 1$. The curve length is given by $\ell(t_0) = \displaystyle\int_0^{t_0} \sqrt{1+t^2}\, dt = \ln\big(t + \sqrt{1+t^2}\big)\Big|_0^{t_0}$.

We conclude that $\ell(t_0) = \ln\big(t_0 + \sqrt{1+t_0^2}\big)$.

Example

Find the length of the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, for $t \in [0, \pi]$.



Solution: The derivative vector is

$$\mathbf{r}'(t) = \langle -12\sin(2t), 12\cos(2t), 5\rangle,$$

$$|\mathbf{r}'(t)|^2 = 144\big[\sin^2(2t) + \cos^2(2t)\big] + 25 = 169 = (13)^2.$$

The curve length is

$$\ell = \int_0^{\pi} 13\, dt = 13\, t\big|_0^{\pi} \quad\Rightarrow\quad \ell = 13\pi.$$

The length function

Definition

The *length function* of a continuously differentiable vector function $\mathbf{r}$ is given by

$$\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau.$$

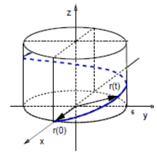
Remarks:

(a) The value $\ell(t)$ of the length function is the length along the curve $\mathbf{r}$ from t_0 to t.

(b) If the function $\mathbf{r}$ is the position of a moving particle as function of time, then the value $\ell(t)$ is the distance traveled by the particle from the time t_0 to t.

Example

Find the arc length function for the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$, starting at $t = 1$.



Solution: We have found that $|\mathbf{r}'(t)| = 13$. Therefore,

$$\ell(t) = \int_{1}^{t} 13\, d\tau \quad \Rightarrow \quad \ell(t) = 13\,(t-1).$$

Parametrizations of a curve

Remark:

A curve in space can be represented by different vector functions.

Example

The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

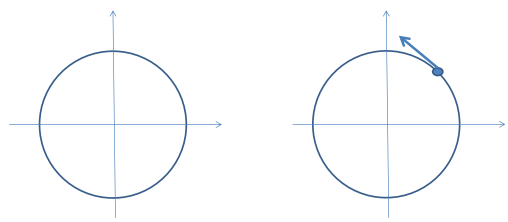
- $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$;
- $\mathbf{r}_2(t) = \langle \cos(5t), \sin(5t) \rangle$;
- $\mathbf{r}_3(t) = \langle \cos(e^t), \sin(e^t) \rangle$.

Remark:

The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.

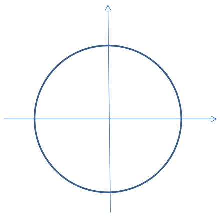
Observe the difference

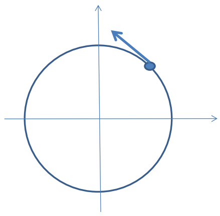
$x^2 + y^2 = r^2$ $\qquad$ $v(\theta) = \langle r\cos\theta, r\sin\theta \rangle$



Observe the difference

$x^2 + y^2 = r^2$ $\qquad$ $v(\theta) = \langle r\cos\theta\,, r\sin\theta\,\rangle$





$\theta = cos^{-1}\frac{x}{r}$, the solution is not unique

Exceptions: special cases.

Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4$ m/s, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

Solution: The projectile acceleration is $\mathbf{a} = -g\mathbf{j}$, with $g = 10$ m/s. Therefore, $\mathbf{v}(t) = (-10t + v_{0y})\mathbf{j} + v_{0x}\mathbf{i}$, where

$$v_{0y} = |\mathbf{v}_0|\sin(\theta) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}, \qquad v_{0x} = |\mathbf{v}_0|\cos(\theta) = 4\,\frac{1}{2} = 2.$$

Since $\mathbf{v}(t) = (-10t + 2\sqrt{3})\mathbf{j} + 2\mathbf{i}$ and $\mathbf{r}_0 = \mathbf{0}$, then

$$\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.$$

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t,$ the trajectory is

$$y(x) = -5\left(\frac{x^2}{4}\right) + 2\sqrt{3}\,\frac{x}{2} \quad\Rightarrow\quad y(x) = -\frac{5}{4}x^2 + \sqrt{3}\,x. \quad \triangleleft$$

Parametrizations of a curve

Remarks:

- If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion. The time t.
- Another preferred parameter useful to describe a moving particle is the distance traveled by the particle. The length ℓ.
- The latter parameter is defined for every curve, either the curve represents motion or not.
- A common problem when describing motion is the following: Given a vector function parametrized by the time t, switch the curve parameter to the curve length ℓ.
- This is called the curve length parametrization.

The length parametrization of a curve

Problem:
Given vector function $\mathbf{r}$ in terms of a parameter t, find the arc length parametrization of that curve.

Solution:

(a) With the function values $\mathbf{r}(t)$ compute the arc length function $\ell(t)$, starting at some $t = t_0$.

(b) Invert the function values $\ell(t)$ to find the function values $t(\ell)$.

(c) Example: If $\ell(t) = 3e^{t/2}$, then $t(\ell) = 2\ln(\ell/3)$.

(d) Compute the composition function $\hat{\mathbf{r}}(\ell) = \mathbf{r}(t(\ell))$. That is, replace t by $t(\ell)$ in the function values $\mathbf{r}(t)$.

Remark: The function values $\hat{\mathbf{r}}(\ell)$ are the parametrization of the function values $\mathbf{r}(t)$ using the curve length as the new parameter.

The length parametrization of a curve

Example

Find the curve length parametrization of the vector function $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ starting at $t = 1$.

Solution: First find the derivative function:

$$\mathbf{r}'(t) = \langle -4\sin(t), 4\cos(t), 3\rangle.$$

Hence, $|\mathbf{r}'(t)|^2 = 4^2\sin^2(t) + 4^2\cos^2(t) + 3^2 = 16 + 9 = 5^2$.

Find the arc length function: $\ell(t) = \int_1^t 5\,d\tau \Rightarrow \ell(t) = 5(t-1)$.

Invert the equation above: $t = \frac{\ell}{5} + 1$, that is, $t = \frac{(\ell+5)}{5}$.

So, $\hat{\mathbf{r}}(\ell) = \Big\langle 4\cos\Big[\frac{(\ell+5)}{5}\Big], 4\sin\Big[\frac{(\ell+5)}{5}\Big], \frac{3(\ell+5)}{5}\Big\rangle$.

Why do we need length parametrization?

Theorem

If the continuously differentiable curve $\mathbf{r}$ *has length parametrization values* $\hat{\mathbf{r}}(\ell)$, *then* $\mathbf{u}(\ell) = \frac{d\hat{\mathbf{r}}}{d\ell}$ *is a unit vector tangent to the curve.*

Proof:

Given the function values $\mathbf{r}(t)$, let $\hat{\mathbf{r}}(\ell)$ be the reparametrization of $\mathbf{r}$ with the curve length function $\ell(t) = \int_{t_0}^t |\mathbf{r}'(\tau)|\,d\tau$.

Notice that $\frac{d\ell}{dt} = |\mathbf{r}'(t)|$ and $\frac{dt}{d\ell} = \frac{1}{|\mathbf{r}'(t)|}$.

Therefore, $\mathbf{u}(\ell) = \frac{d\hat{\mathbf{r}}(\ell)}{d\ell} = \frac{d\mathbf{r}(t)}{dt}\frac{dt}{d\ell} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

We conclude that $|\mathbf{u}(\ell)| = 1$. □

Example

Find a unit vector tangent to the curve given by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ for $t \geqslant 0$.

Solution: Reparametrize the curve using the arc length.
Recall: $|\mathbf{r}'(t)| = 5$, and $\ell(t) = 5t$, so $t = \ell/5$. We get

$$\hat{\mathbf{r}}(\ell) = \langle 4\cos(\ell/5), 4\sin(\ell/5), 3\ell/5\rangle.$$

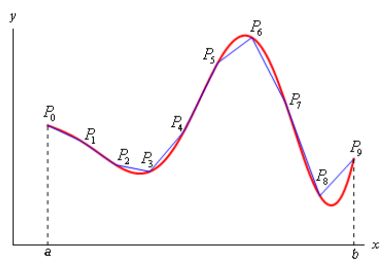
Therefore, a unit tangent vector is

$$\mathbf{u}(\ell) = \frac{d\hat{\mathbf{r}}}{d\ell} \quad\Rightarrow\quad \mathbf{u}(\ell) = \Big\langle -\frac{4}{5}\sin(\ell/5), \frac{4}{5}\cos(\ell/5), \frac{3}{5}\Big\rangle. \quad \lhd$$

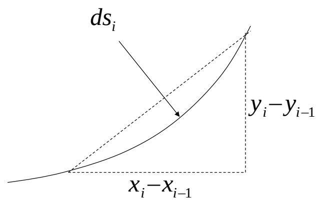
We can verify that this is a unit vector, since

$$|\mathbf{u}(\ell)|^2 = \Big(\frac{4}{5}\Big)^2\big[\sin^2(\ell/5) + \cos^2(\ell/5)\big] + \Big(\frac{3}{5}\Big)^2 \quad\Rightarrow\quad |\mathbf{u}(\ell)| = 1.$$

Recall: Arc length in R^2

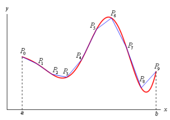


Idea:



$$ds_i \approx \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Idea:



Let $x = x(t), y = y(t)$ --- parametric

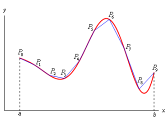
Arc length fro $t = \alpha$ to $t = \beta$

$$S = \sum ds_i \approx \sum \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sum \sqrt{\frac{(x_i - x_{i-1})^2}{(t_i - t_{i-1})^2} + \frac{(y_i - y_{i-1})^2}{(t_i - t_{i-1})^2}}\,(t_i - t_{i-1})$$

$$= \sum \sqrt{\frac{\Delta x_i^2}{\Delta t_i^2} + \frac{\Delta y_i^2}{\Delta t_i^2}}\,\Delta t_i \text{ --- Riemann sum}$$

Parametric formula:



Let $x = x(t), y = y(t)$ --- parametric

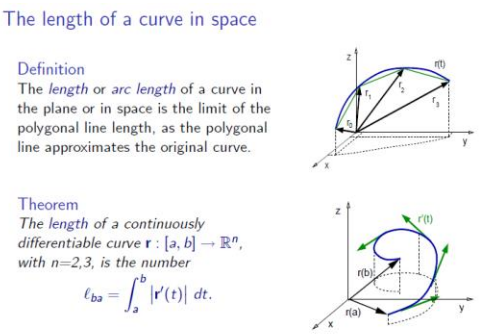
Arc length fro $t = \alpha$ to $t = \beta$

$$S = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$\boldsymbol{r}(t) = \langle x(t), \quad y(t) \rangle, \quad \boldsymbol{r}'(t) = \langle x'(t), \quad y'(t) \rangle$$

$$S = \int_{\alpha}^{\beta} |\boldsymbol{r}'(t)|\,dt$$

Generalization to R^3



The length of a curve in space

Recall: The length of $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is $\ell_{ba} = \int_a^b |\mathbf{r}'(t)|\, dt$.

- If the curve $\mathbf{r}$ is the path traveled by a particle in space, then $\mathbf{r}' = \mathbf{v}$ is the velocity of the particle.
- The length is the integral in time of the particle speed $|\mathbf{v}(t)|$.
- Therefore, the length of the curve is the distance traveled by the particle.
- In Cartesian coordinates the functions $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \qquad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Therefore the curve length is given by the expression

$$\ell_{ba} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

4-19-2013

Example

Find the length of the curve $\mathbf{r}(t) = \langle r_0 \cos(t), r_0 \sin(t)\rangle$, for $t \in [\pi/4, 3\pi/4]$, and $r_0 > 0$.

Solution: Compute $\mathbf{r}'(t) = \langle -r_0 \sin(t), r_0 \cos(t)\rangle$. The length of the curve is given by the formula

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{\big[-r_0 \sin(t)\big]^2 + \big[r_0 \cos(t)\big]^2}\, dt$$

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{r_0^2\big(\big[-\sin(t)\big]^2 + \big[\cos(t)\big]^2\big)}\, dt = \int_{\pi/4}^{3\pi/4} r_0\, dt.$$

Hence, $\ell = \frac{\pi}{2} r_0$. (The length of quarter circle of radius r_0.)

Example

Find the length of the spiral $\mathbf{r}(t) = \langle t \cos(t), t \sin(t)\rangle$, for $t \in [0, t_0]$.

Solution: The derivative vector is

$$\mathbf{r}'(t) = \big\langle \big[-t \sin(t) + \cos(t)\big], \big[t \cos(t) + \sin(t)\big]\big\rangle,$$

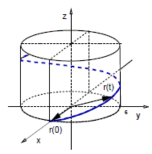
$$|\mathbf{r}'(t)|^2 = \big[t^2 \sin^2(t) + \cos^2(t) - 2t \sin(t)\cos(t)\big] + \big[t^2 \cos^2(t) + \sin^2(t) + 2t \sin(t)\cos(t)\big]$$

We obtain $|\mathbf{r}'(t)|^2 = t^2 + 1$. The curve length is given by $\ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2}\, dt = \ln\big(t + \sqrt{1 + t^2}\big)\Big|_0^{t_0}$.

We conclude that $\ell(t_0) = \ln\big(t_0 + \sqrt{1 + t_0^2}\big)$.

Example

Find the length of the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, for $t \in [0, \pi]$.



Solution: The derivative vector is

$$\mathbf{r}'(t) = \langle -12\sin(2t), 12\cos(2t), 5\rangle,$$

$$|\mathbf{r}'(t)|^2 = 144\big[\sin^2(2t) + \cos^2(2t)\big] + 25 = 169 = (13)^2.$$

The curve length is

$$\ell = \int_0^{\pi} 13\, dt = 13\, t\big|_0^{\pi} \quad \Rightarrow \quad \ell = 13\pi.$$

The length function

Definition

The *length function* of a continuously differentiable vector function $\mathbf{r}$ is given by

$$\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau.$$

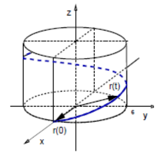
Remarks:

(a) The value $\ell(t)$ of the length function is the length along the curve $\mathbf{r}$ from t_0 to t.

(b) If the function $\mathbf{r}$ is the position of a moving particle as function of time, then the value $\ell(t)$ is the distance traveled by the particle from the time t_0 to t.

Example

Find the arc length function for the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$, starting at $t = 1$.



Solution: We have found that $|\mathbf{r}'(t)| = 13$. Therefore,

$$\ell(t) = \int_{1}^{t} 13\, d\tau \quad \Rightarrow \quad \ell(t) = 13\,(t-1).$$

Parametrizations of a curve

Remark:

A curve in space can be represented by different vector functions.

Example

The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

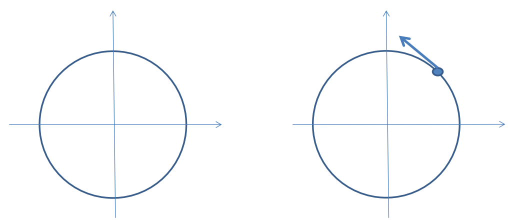
- $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$;
- $\mathbf{r}_2(t) = \langle \cos(5t), \sin(5t) \rangle$;
- $\mathbf{r}_3(t) = \langle \cos(e^t), \sin(e^t) \rangle$.

Remark:

The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.

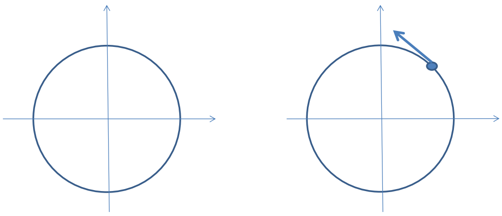
Observe the difference

$x^2 + y^2 = r^2$ $\qquad$ $v(\theta) = \langle r\cos\theta, r\sin\theta \rangle$



Observe the difference

$x^2 + y^2 = r^2$ $\quad$ $v(\theta) = \langle r\cos\theta, r\sin\theta \rangle$



$\theta = cos^{-1}\frac{x}{r}$, the solution is not unique

Exceptions: special cases.

Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4$ m/s, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

Solution: The projectile acceleration is $\mathbf{a} = -g\mathbf{j}$, with $g = 10$ m/s. Therefore, $\mathbf{v}(t) = (-10t + v_{0y})\mathbf{j} + v_{0x}\mathbf{i}$, where

$$v_{0y} = |\mathbf{v}_0|\sin(\theta) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}, \quad v_{0x} = |\mathbf{v}_0|\cos(\theta) = 4\,\frac{1}{2} = 2.$$

Since $\mathbf{v}(t) = (-10t + 2\sqrt{3})\mathbf{j} + 2\mathbf{i}$ and $\mathbf{r}_0 = \mathbf{0}$, then

$$\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.$$

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -5\left(\frac{x^2}{4}\right) + 2\sqrt{3}\,\frac{x}{2} \quad \Rightarrow \quad y(x) = -\frac{5}{4}x^2 + \sqrt{3}\,x. \quad \triangleleft$$

Parametrizations of a curve

Remarks:

- If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion. The time t.
- Another preferred parameter useful to describe a moving particle is the distance traveled by the particle. The length ℓ.
- The latter parameter is defined for every curve, either the curve represents motion or not.
- A common problem when describing motion is the following: Given a vector function parametrized by the time t, switch the curve parameter to the curve length ℓ.
- This is called the curve length parametrization.

The length parametrization of a curve

Problem:
Given vector function $\mathbf{r}$ in terms of a parameter t, find the arc length parametrization of that curve.

Solution:

(a) With the function values $\mathbf{r}(t)$ compute the arc length function $\ell(t)$, starting at some $t = t_0$.

(b) Invert the function values $\ell(t)$ to find the function values $t(\ell)$.

(c) Example: If $\ell(t) = 3e^{t/2}$, then $t(\ell) = 2\ln(\ell/3)$.

(d) Compute the composition function $\hat{\mathbf{r}}(\ell) = \mathbf{r}(t(\ell))$. That is, replace t by $t(\ell)$ in the function values $\mathbf{r}(t)$.

Remark: The function values $\hat{\mathbf{r}}(\ell)$ are the parametrization of the function values $\mathbf{r}(t)$ using the curve length as the new parameter.

The length parametrization of a curve

Example

Find the curve length parametrization of the vector function $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ starting at $t = 1$.

Solution: First find the derivative function:

$$\mathbf{r}'(t) = \langle -4\sin(t), 4\cos(t), 3\rangle.$$

Hence, $|\mathbf{r}'(t)|^2 = 4^2\sin^2(t) + 4^2\cos^2(t) + 3^2 = 16 + 9 = 5^2$.

Find the arc length function: $\ell(t) = \int_1^t 5\,d\tau \Rightarrow \ell(t) = 5(t-1)$.

Invert the equation above: $t = \dfrac{\ell}{5} + 1$, that is, $t = \dfrac{(\ell+5)}{5}$.

So, $\hat{\mathbf{r}}(\ell) = \Big\langle 4\cos\Big[\dfrac{(\ell+5)}{5}\Big], 4\sin\Big[\dfrac{(\ell+5)}{5}\Big], \dfrac{3(\ell+5)}{5}\Big\rangle$.

Why do we need length parametrization?

Theorem

If the continuously differentiable curve $\mathbf{r}$ *has length parametrization values* $\hat{\mathbf{r}}(\ell)$, *then* $\mathbf{u}(\ell) = \dfrac{d\hat{\mathbf{r}}}{d\ell}$ *is a unit vector tangent to the curve.*

Proof:

Given the function values $\mathbf{r}(t)$, let $\hat{\mathbf{r}}(\ell)$ be the reparametrization of $\mathbf{r}$ with the curve length function $\ell(t) = \int_{t_0}^t |\mathbf{r}'(\tau)|\,d\tau$.

Notice that $\dfrac{d\ell}{dt} = |\mathbf{r}'(t)|$ and $\dfrac{dt}{d\ell} = \dfrac{1}{|\mathbf{r}'(t)|}$.

Therefore, $\mathbf{u}(\ell) = \dfrac{d\hat{\mathbf{r}}(\ell)}{d\ell} = \dfrac{d\mathbf{r}(t)}{dt}\dfrac{dt}{d\ell} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

We conclude that $|\mathbf{u}(\ell)| = 1$. □

Example

Find a unit vector tangent to the curve given by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ for $t \geqslant 0$.

Solution: Reparametrize the curve using the arc length. Recall: $|\mathbf{r}'(t)| = 5$, and $\ell(t) = 5t$, so $t = \ell/5$. We get

$$\hat{\mathbf{r}}(\ell) = \langle 4\cos(\ell/5), 4\sin(\ell/5), 3\ell/5\rangle.$$

Therefore, a unit tangent vector is

$$\mathbf{u}(\ell) = \frac{d\hat{\mathbf{r}}}{d\ell} \quad\Rightarrow\quad \mathbf{u}(\ell) = \Big\langle -\frac{4}{5}\sin(\ell/5), \frac{4}{5}\cos(\ell/5), \frac{3}{5}\Big\rangle. \quad \triangleleft$$

We can verify that this is a unit vector, since

$$|\mathbf{u}(\ell)|^2 = \Big(\frac{4}{5}\Big)^2\big[\sin^2(\ell/5) + \cos^2(\ell/5)\big] + \Big(\frac{3}{5}\Big)^2 \quad\Rightarrow\quad |\mathbf{u}(\ell)| = 1.$$

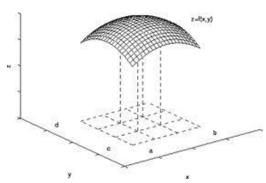
Scalar functions of several variables

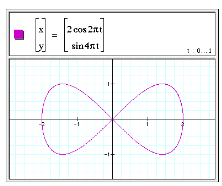
Definition

A *scalar function of n variables* is a function $f : D \subset \mathbb{R}^n \to R \subset \mathbb{R}$, where $n \in \mathbb{N}$, the set D is called the *domain* of the function, and the set R is called the *range* of the function.

Remark:

Comparison between $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$.





Functions of several variables

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.
- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of this room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.
- Another example of a scalar function of three variables is the speed of the air in the room.
- An example of a vector-valued function of three variables, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$, is the velocity of the air in the room.

◁

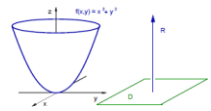
Example

Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is defined.

Solution:

The function $f(x, y) = x^2 + y^2$ is defined for all points $(x, y) \in \mathbb{R}^2$, therefore, $D = \mathbb{R}^2$.

Since $f(x, y) = x^2 + y^2 \geqslant 0$ for all $(x, y) \in D$, then the range space is $R = [0, \infty)$. ◁

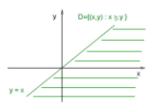


Example

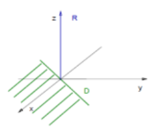
Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = \sqrt{x - y}$ is defined.

Solution:

The function f is defined for points $(x, y) \in \mathbb{R}^2$ such that $x - y \geqslant 0$. So, $D = \{(x, y) \in \mathbb{R}^2 \;:\; x \geqslant y\}$.



Since $f(x, y) = \sqrt{x - y} \geqslant 0$ for all $(x, y) \in D$, the range space is $R = [0, \infty)$. ◁

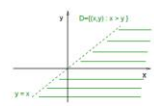


Example

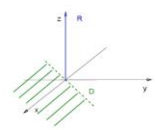
Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = 1/\sqrt{x - y}$ is defined.

Solution:

The function f is defined for points $(x, y) \in \mathbb{R}^2$ such that $x - y > 0$. So, $D = \{(x, y) \in \mathbb{R}^2 \;:\; x > y\}$.



Since $f(x, y) = 1/\sqrt{x - y} \geqslant 0$ for all $(x, y) \in D$, the range space is $R = (0, \infty)$. ◁



On open and closed sets in $\mathbb{R}^n$

Remark: We first generalize from $\mathbb{R}^3$ to $\mathbb{R}^n$ the definition of a ball of radius r centered at $\hat{P}_c$.

Definition

An *open ball* of radius $r > 0$ centered at $\hat{P}_c = (\hat{x}_1, \cdots, \hat{x}_n)$ is the set in $\mathbb{R}^n$, with $n \in \mathbb{N}$, given by

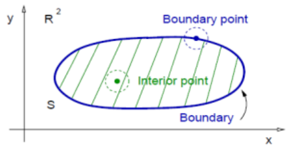
$$B_r(\hat{P}_c) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \ : \ (x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2 < r^2\}.$$

Remark: An open ball $B_r(\hat{P}_c)$ contains the points *inside* a sphere of radius r centered at $\hat{P}_c$ *without* the points of the sphere.

On open and closed sets in $\mathbb{R}^n$

Definition

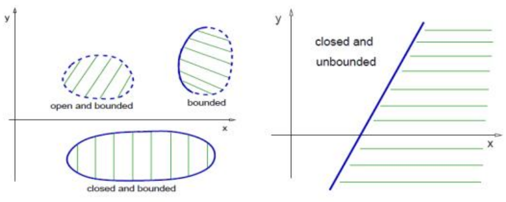
A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called an *interior point* iff there is a ball $B_r(P) \subset S$. A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called a *boundary point* iff every ball $B_r(P)$ contains points in S and points outside S. The *boundary* of a set S is the set of all boundary points of S.



On open and closed sets in $\mathbb{R}^n$

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.



Example

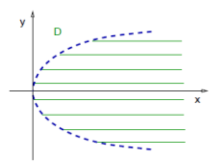
Find and describe the maximum domain of the function $f(x, y) = \ln(x - y^2)$.

Solution:

The maximum domain of f is

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x > y^2\}.$$

D is an open, unbounded set. $\lhd$



The graph of a function of two variables is a surface in $\mathbb{R}^3$

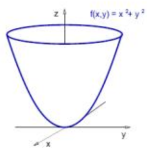
Definition

The *graph* of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points (x, y, z) in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$. The graph of a function f is also called the surface $z = f(x, y)$.

Example

Draw the graph of $f(x, y) = x^2 + y^2$.

Solution: The graph of f is the surface $z = x^2 + y^2$. This is a paraboloid along the z axis. ◁



Level curves, contour curves

Definition

The *contour curves* of a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ are the curves in $\mathbb{R}^3$ given by the equation

$$f(x, y) = k, \qquad z = k, \qquad (x, y) \in D, \qquad k \in R.$$

The *level curves* of the function f are the curves in the domain $D \subset \mathbb{R}^2$ given by the equation

$$f(x, y) = k, \qquad (x, y) \in D, \qquad k \in R.$$

Remark: Contour curves are the intersection of the graph of f with horizontal planes $z = k$.

Remark: Level curves are the vertical translation of contour curves to the function domain.

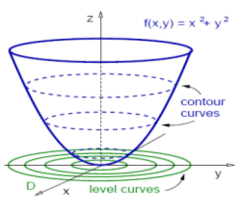
Example

Find and draw few level curves and contour curves for the function $f(x, y) = x^2 + y^2$.

Solution:

The level curves are solutions of the equation $x^2 + y^2 = k$ with $k \geqslant 0$. These curves are circles of radius $\sqrt{k}$ in $D = \mathbb{R}^2$.

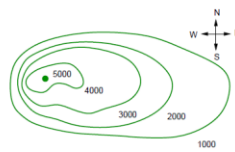
The contour curves are the circles $\{(x, y, z) : x^2 + y^2 = k, \ z = k\}$.



Level curves, contour curves

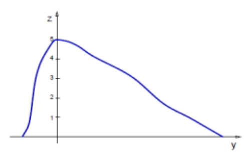
Example

Given the topographic map in the figure, which way do you choose to the summit?

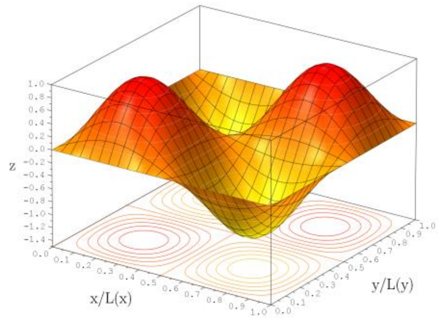


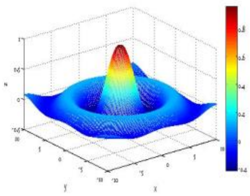
Solution:

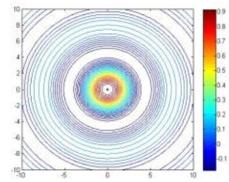
From the east side. ◁

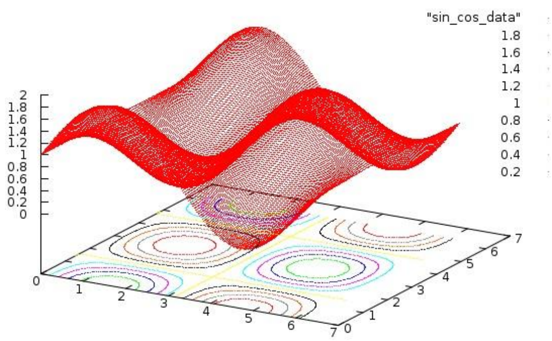


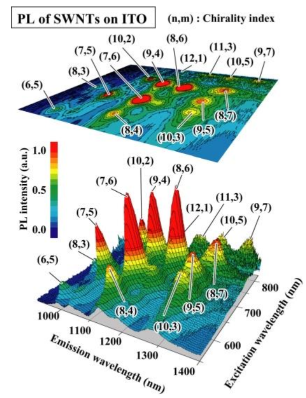
New graphing tools by computer software.

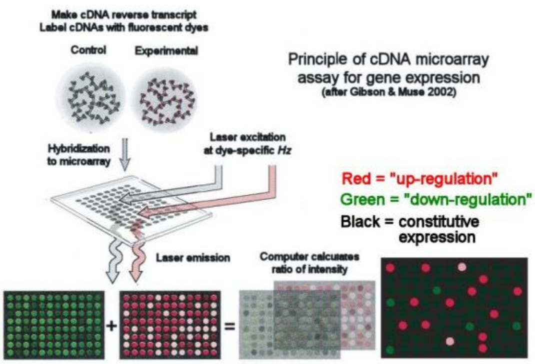












Scalar functions of three variables

Definition
The *graph* of a scalar function of three variables, $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$, is the set of points in $\mathbb{R}^4$ of the form $(x, y, z, f(x, y, z))$ for every $(x, y, z) \in D$.

Remark:
The graph a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ requires four space dimensions. We cannot picture such graph.

Definition
The *level surfaces* of a function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$ are the surfaces in the domain $D \subset \mathbb{R}^3$ of f solutions of the equation $f(x, y, z) = k$, where $k \in R$ is a constant in the range of f.

Example
Draw one level surface of the function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$
$f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

Solution:

The domain of f is $D = \mathbb{R}^3$, the range is $R = (0, \infty)$. For $k > 0$ the level surfaces $f(x, y, z) = k$ are

$$x^2 + y^2 + z^2 = \frac{1}{k},$$

spheres radius $R = \dfrac{1}{\sqrt{k}}$. ◁