Recall: Chain rule

Composition of two functions : $y = f(g(x))$

Chain rule : $y' = f'(g(x))g'(x)$

"Chain" expression : $y = f(u),\ u = g(x),$

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(g(x))}\underbrace{\frac{du}{dx}}_{g'(x)}$$

Power chain rule:

$$y = u^n, u = g(x)$$

$$\frac{dy}{dx} = nu^{n-1}g'(x) = ng^{n-1}(x)g'(x)$$

Exponential and logarithm chain rules:

$$y = e^u, u = g(x), \quad y' = e^u \frac{du}{dx} = e^{g(x)}g'(x)$$

$$y = \ln u, u = g(x), \quad y' = \frac{1}{u}\frac{du}{dx} = \frac{g'(x)}{g(x)}$$

Trig chain rule:

$$y = \sin u, u = g(x), \quad y' = \cos u \frac{du}{dx} = \cos u * \underline{g'(x)}$$

$$y = \cos u, u = g(x), \quad y' = -\sin u \frac{du}{dx} = -\sin u * \underline{g'(x)}$$

$$y = \tan u, u = g(x), \quad y' = \sec^2 u \frac{du}{dx} = \sec^2 u * \underline{g'(x)}$$

$$y = \cot u, u = g(x), \quad y' = -\csc^2 u \frac{du}{dx} = -\csc^2 u * \underline{g'(x)}$$

$$y = \csc u, u = g(x), \quad y' = -\csc u \cot u * \underline{g'(x)}$$

$$y = \sec u, u = g(x), \quad y' = \sec u \tan u * \underline{g'(x)}$$

Inverse trig derivatives + chain rule

1. $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d}{dx}(\arccos u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d}{dx}(\operatorname{arc}\cot u) = \frac{-1}{1+u^2}\frac{du}{dx}$
5. $\frac{d}{dx}(\operatorname{arc}\sec u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d}{dx}(\operatorname{arc}\csc u) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Derivatives of hyperbolic functions

$$\frac{d}{dt}\sinh x = \cosh x\frac{dx}{dt},\ x = x(t)$$

$$\frac{d}{dt}\cosh x = \sinh x\frac{dx}{dt}$$

$$\frac{d}{dt}\tanh x = \frac{1}{\cosh^2 x}\frac{dx}{dt} = \operatorname{sech}^2 x\frac{dx}{dt}$$

Chain rule for functions of 2, 3 variables

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$

Chain rule for change of coordinates in a line.

Theorem

If the functions $f : [x_0, x_1] \to \mathbb{R}$ and $x : [t_0, t_1] \to [x_0, x_1]$ are differentiable, then the function $\hat{f} : [t_0, t_1] \to \mathbb{R}$ given by the composition $\hat{f}(t) = f(x(t))$ is differentiable and

$$\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}(x(t))\,\frac{dx}{dt}(t).$$

Notation:

The equation above is usually written as $\frac{d\hat{f}}{dt} = \frac{df}{dx}\frac{dx}{dt}$.

Alternative notations are $\hat{f}'(t) = f'(x(t))\,x'(t)$ and $\hat{f}' = f'\,x'$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for functions defined on a curve in a plane.

Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f(\mathbf{r}(t))$ is differentiable and holds

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}(\mathbf{r}(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\mathbf{r}(t))\,\frac{dy}{dt}(t).$$

Notation:

The equation above is usually written as $\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

An alternative notation is $\hat{f}' = f_x\,x' + f_y\,y'$.

Example

Find the rate of change of the function $f(x,y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(2t)\rangle$.

Solution: The rate of change of f along the curve $\mathbf{r}(t)$ is the derivative of $\hat{f}(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$. We do not need to compute $\hat{f}(t) = f(\mathbf{r}(t))$. Instead, the chain rule implies

$$\hat{f}'(t) = f_x(\mathbf{r})\, x' + f_y(\mathbf{r})\, y' = 2x\, x' + 6y^2\, y'.$$

Since $x(t) = \sin(t)$ and $y(t) = \cos(2t)$,

$$\hat{f}'(t) = 2\sin(t)\,\cos(t) + 6\cos^2(2t)\,\big[-2\sin(2t)\big].$$

The result is $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. ◁

Idea of the proof:

Denote $z = f(x,y)$ and $z_0 = f(x_0, y_0)$

$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

Recall the definition of differentiability:

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

$$\Rightarrow \frac{\Delta z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

$$\overset{\Delta t \to 0}{\Longrightarrow} \boxed{\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},} \quad \text{where } x = x(t), y = y(t)$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the change of coordinate functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f(x(t,s), y(t,s))$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

Remark: We denote by $f(x,y)$ the function values in the coordinates (x,y), while we denote by $\hat{f}(t,s)$ are the function values in the coordinates (t,s).

Example

Given the function $f(x,y) = x^2 + 3y^2$, in Cartesian coordinates (x,y), find the derivatives of f in polar coordinates (r,θ).

Solution: The relation between Cartesian and polar coordinates is

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta).$$

The function f in polar coordinates is $\hat{f}(r,\theta) = f(x(r,\theta), y(r,\theta))$.
The chain rule says $\hat{f}_r = f_x\, x_r + f_y\, y_r$ and $\hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta$, hence

$$\hat{f}_r = 2x\cos(\theta) + 6y\sin(\theta) \;\Rightarrow\; \hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta).$$

$$\hat{f}_\theta = -2xr\sin(\theta) + 6yr\cos(\theta),$$

$$\hat{f}_\theta = -2r^2\cos(\theta)\sin(\theta) + 6r^2\cos(\theta)\sin(\theta). \quad ◁$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

Chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f(\mathbf{r}(t))$ is differentiable and holds

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t\rangle$.

Solution: Recall: We do not need to compute

$$\hat{f}(t) = f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

to find $\hat{f}'$. We only need to use the chain rule formula,

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

$$\hat{f}' = -2x\sin(t) + 3y^2\cos(t) + 4z^3(3).$$

$$\hat{f}' = -2\cos(t)\sin(t) + 3\sin^2(t)\cos(t) + 4(3)(3^3)t^3. \quad \lhd$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the surface given by functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, and $z(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f\big(x(t,s), y(t,s), z(t,s)\big)$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s)$ the function values at the surface point with coordinates (t,s).

Example

Given the function $f(x,y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x,y), find its derivatives on the surface given by $x(t,s) = t + s, \quad y(t,s) = t^2 + s^2, \quad z(t,s) = t - s.$

Solution: Recall: We do not need to compute the function

$$\hat{f}(t,s) = f(x(t,s), y(t,s), z(t,s))$$

to obtain the derivatives of f along the surface $x(t,s)$, $y(t,s)$ and $z(t,s)$, which are given by $\hat{f}_t$ and $\hat{f}_s$. We just use the chain rule,

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \quad \hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

$$\hat{f}_t = 2(t+s) + 6(t^2+s^2)(2t) + 4(t-s),$$

$$\hat{f}_s = 2(t+s) + 6(t^2+s^2)(2s) - 4(t-s). \quad \lhd$$

Recall:

Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f(\mathbf{r}(t))$ is differentiable and holds

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}(\mathbf{r}(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\mathbf{r}(t))\,\frac{dy}{dt}(t).$$

Notation:

The equation above is usually written as $\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

An alternative notation is $\hat{f}' = f_x\, x' + f_y\, y'$.

Recall

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the change of coordinate functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t, s)$ and $y(t, s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t, s) = f(x(t, s), y(t, s))$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

Remark: We denote by $f(x, y)$ the function values in the coordinates (x, y), while we denote by $\hat{f}(t, s)$ are the function values in the coordinates (t, s).

Recall

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

Chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f(\mathbf{r}(t))$ is differentiable and holds

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Recall

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the surface given by functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t, s)$ and $y(t, s)$, and $z(t, s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x, y, z)$ the function values in the coordinates (x, y, z), while we denote by $\hat{f}(t, s)$ the function values at the surface point with coordinates (t, s).

A 3-D example

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + 2z^2$
Find its partial derivatives on the surface given by:

$x(t,s) = t + s, \quad y(t,s) = t^2 + s^2, \quad z(t,s) = t - s.$

Solution: Recall: We do not need to compute the function

$$\hat{f}(t,s) = f(x(t,s), y(t,s), z(t,s))$$

to obtain the derivatives of f along the surface $x(t,s)$, $y(t,s)$ and $z(t,s)$, which are given by $\hat{f}_t$ and $\hat{f}_s$. We just use the chain rule,

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \qquad \hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

$$\hat{f}_t = 2(t+s) + 6(t^2+s^2)(2t) + 4(t-s),$$

$$\hat{f}_s = 2(t+s) + 6(t^2+s^2)(2s) - 4(t-s). \qquad \triangleleft$$

Why is it a surface?

Given the surface in parametric form by the equations

$$x(t,s) = t + s, \qquad y(t,s) = t^2 + s^2, \qquad z(t,s) = t - s,$$

express that surface as an equation for x, y and z.

Solution: Invert the equations for x and z and obtain t and s,

$$\frac{x+z}{2} = t, \qquad \frac{x-z}{2} = s.$$

We introduce these t and s into the equation for y,

$$y = \left(\frac{x+z}{2}\right)^2 + \left(\frac{x-z}{2}\right)^2 = \frac{(x^2+z^2+2xz) + (x^2+z^2-2xz)}{4}$$

hence, $y = \dfrac{x^2}{2} + \dfrac{z^2}{2}$, a circular paraboloid along the y axis. $\triangleleft$

General idea

- $\vec{r}(t) = \langle x(t), y(t) \rangle$
  represents a curve in plane R^2
- $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
  represents a curve in space R^3
- $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$
  represents a surface in space R^3

- $\vec{r}(s,t) = \langle x(s,t), y(s,t) \rangle$?
Represents change of variables (coordinates) in R^2

One more theorem

Chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ *and the change of coordinate functions* $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ *are differentiable, with* $x(t,s,r)$, $y(t,s,r)$, *and* $z(t,s,r)$, *then the function* $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ *given by the composition* $\hat{f}(t,s,r) = f(x(t,s,r), y(t,s,r), z(t,s,r))$ *is differentiable and*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s$$
$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s,r)$ the function values in the coordinates (t,s,r).

Chain rule for change of coordinates in space.

Example

Given the function $f(x,y,z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates, find its r-derivative in spherical coordinates (r,θ,ϕ),

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: Recall: We do not need to compute the function

$$\hat{f}(r,\theta,\phi) = f\big(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\big).$$

to obtain the r-derivative of f. We just use the chain rule,

$$\hat{f}_r = f_x\,x_r + f_y\,y_r + f_z\,z_r = 2x\,x_r + 6y\,y_r + 2z\,z_r$$

$$\hat{f}_r = 2r\cos^2(\phi)\sin^2(\theta) + 6r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta)$$

$$\hat{f}_r = 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).$$

We conclude that $\hat{f}_r = 2r + 4r\sin^2(\phi)\sin^2(\theta)$. ◁

Chain rule for higher dimensions

Case-1. $f = f(x,y)$: $R^2 \to R$,
$x = x(t): R \to R$, $\quad y = y(t): R \to R$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Case-2. $f = f(x,y): R^2 \to R$,
$x = x(t,s): R^2 \to R, y = y(t,s): R^2 \to R$

$$\frac{\partial f}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

NO need for "^" notation

A formula for implicit differentiation

Theorem

If the differentiable function with values $F(x,y)$ defines implicitly the function values $y(x)$ by the equation $F(x,y) = 0$, and if the function $F_y \neq 0$, then y is differentiable and

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Proof: *Let* $x = x, y = y(x)$

(treating x as the parameter in the chain rule),

then $$\frac{dF}{dx} = F_x \underbrace{\frac{dx}{dx}}_{=1} + F_y\frac{dy}{dx} = 0. \blacksquare$$

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x,y) = 0$, where $F(x,y) = x\,e^y + \cos(x-y)$.

Solution:

The partial derivatives of function F are

$$F_x = e^y - \sin(x-y), \qquad F_y = x\,e^y + \sin(x-y).$$

Therefore, the Theorem above implies

$$y'(x) = \frac{[\sin(x-y) - e^y]}{[x\,e^y + \sin(x-y)]}.$$

◁

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x, y) = 0$, where $F(x, y) = x\, e^y + \cos(x - y)$.

Solution: We now use the old method.

Since $F(x, y(x)) = x\, e^y + \cos(x - y) = 0$, then differentiating on both sides we get

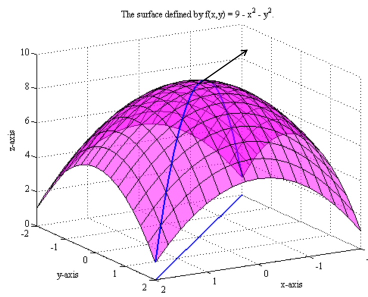
$$e^y + x\, y'\, e^y - \sin(x - y) - \sin(x - y)(-y') = 0.$$

Reordering terms,

$$y' \left[x\, e^y + \sin(x - y)\right] = \sin(x - y) - e^y.$$

We conclude that: $y'(x) = \dfrac{[\sin(x - y) - e^y]}{[x\, e^y + \sin(x - y)]}$. ◁

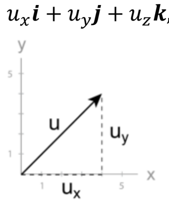
Directional derivatives and gradient vectors

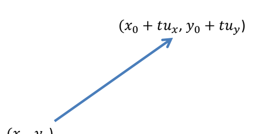


Unit directions in 2-D and 3-D

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j}, \qquad |\boldsymbol{u}| = \sqrt{{u_x}^2 + {u_y}^2} = 1$$

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}, \qquad |\boldsymbol{u}| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} = 1$$





Change along the direction $\boldsymbol{u}$

Directional derivative of functions of two variables.

Remark: The directional derivative generalizes the partial derivatives to any direction.

Definition

The *directional derivative* of the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y\rangle$ is given by the limit

$$\left(D_{\mathbf{u}} f\right)_{P_0} = \lim_{t \to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right].$$

Remarks: The line by $\mathbf{r}_0 = \langle x_0, y_0\rangle$ tangent to $\mathbf{u}$ is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$.

(a) Equivalently, $\left(D_{\mathbf{u}} f\right)_{P_0} = \lim_{t \to 0} \frac{1}{t}\left[f(\mathbf{r}(t)) - f(\mathbf{r}(0))\right]$.

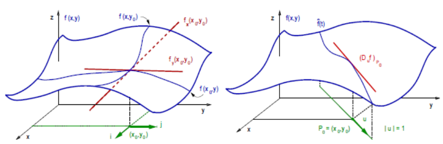
(b) If $\hat{f}(t) = f(\mathbf{r}(t))$, then holds $\left(D_{\mathbf{u}} f\right)_{P_0} = \hat{f}'(0)$.

Directional derivatives generalize partial derivatives

Example

The partial derivatives f_x and f_y are particular cases of directional derivatives $\left(D_{\mathbf{u}}f\right)_{P_0} = \lim_{t\to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right]$:

- $\mathbf{u} = \langle 1, 0\rangle = \mathbf{i}$, then $\left(D_{\mathbf{i}}f\right)_{P_0} = f_x(x_0, y_0)$.
- $\mathbf{u} = \langle 0, 1\rangle = \mathbf{j}$, then $\left(D_{\mathbf{j}}f\right)_{P_0} = f_y(x_0, y_0)$.



Example

Find the derivative of $f(x, y) = x^2 + y^2$ at $P_0 = (1, 0)$ in the direction of $\theta = \pi/3$ counterclockwise from the x-axis.

Solution: A unit vector in the direction of θ is $\mathbf{u} = \langle \cos(\theta), \sin(\theta)\rangle$. For $\theta = \pi/3$ we get $\mathbf{u} = \langle 1/2, \sqrt{3}/2\rangle$.

The line containing the vector $\mathbf{r}_0 = \langle 1, 0\rangle$ and tangent to $\mathbf{u}$ is

$$\mathbf{r}(t) = \langle 1, 0\rangle + \frac{t}{2}\langle 1, \sqrt{3}\rangle \quad\Rightarrow\quad x(t) = 1 + \frac{t}{2}, \quad y(t) = \frac{\sqrt{3}\,t}{2}.$$

Hence $\hat{f}(t) = f(x(t), y(t))$ is given by

$$\hat{f}(t) = \left(1 + \frac{t}{2}\right)^2 + \frac{3t^2}{4} \quad\Rightarrow\quad \hat{f}(t) = 1 + t + t^2.$$

Since $(D_{\mathbf{u}}f)_{P_0} = \hat{f}'(0)$, and $\hat{f}'(t) = 1 + 2t$, then $(D_{\mathbf{u}}f)_{P_0} = 1$. $\triangleleft$

Remark: The condition $|\mathbf{u}| = 1$ in the line $\mathbf{r}(t) = \langle x_0, y_0\rangle + \mathbf{u}\,t$ implies that the parameter t is the distance between the points $(x(t), y(t)) = (x_0 + u_x t, y_0 + u_y t)$ and (x_0, y_0).

In other words: The arc length function of the line is $\ell = t$.

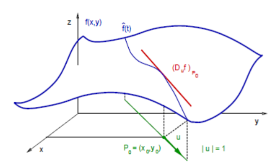
Proof:

$$d = |\langle x - x_0, y - y_0\rangle| = |\langle u_x t, u_y t\rangle| = |t|\,|\mathbf{u}|, \quad\Rightarrow\quad d = |t|.$$

Remark: The directional derivative $\left(D_{\mathbf{u}}f\right)_{P_0}$ is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through P_0.

Directional derivative and partial derivatives

Remark: The directional derivative $\left(D_{\mathbf{u}}f\right)_{P_0}$ is the derivative of f along the line $\mathbf{r}(t) = \langle x_0, y_0\rangle + \mathbf{u}\,t$.



Theorem

If the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0)$ and $\mathbf{u} = \langle u_x, u_y\rangle$ is a unit vector, then

$$\left(D_{\mathbf{u}}f\right)_{P_0} = f_x(x_0, y_0)\,u_x + f_y(x_0, y_0)\,u_y.$$

Proof:
The line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \langle u_x, u_y \rangle\, t$ has parametric equations: $x(t) = x_0 + u_x t$ and $y(t) = y_0 + u_y t$;

Denote f evaluated along the line as $\hat{f}(t) = f(x(t), y(t))$.

Now, on the one hand, $\hat{f}'(0) = (D_{\mathbf{u}} f)_{P_0}$, since

$$\hat{f}'(0) = \lim_{t\to 0} \frac{1}{t}\big[\hat{f}(t) - \hat{f}(0)\big]$$

$$\hat{f}'(0) = \lim_{t\to 0} \frac{1}{t}\big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\big] = D_{\mathbf{u}} f(x_0, y_0).$$

On the other hand, the chain rule implies:

$$\hat{f}'(0) = f_x(x_0, y_0)\, x'(0) + f_y(x_0, y_0)\, y'(0).$$

Therefore, $(D_{\mathbf{u}} f)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y$. □

Example
Compute the directional derivative of $f(x, y) = \sin(x + 3y)$ at the point $P_0 = (4, 3)$ in the direction of vector $\mathbf{v} = \langle 1, 2 \rangle$.

Solution: We need to find a unit vector in the direction of $\mathbf{v}$.

Such vector is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \dfrac{1}{\sqrt{5}} \langle 1, 2 \rangle$.

We now use the formula $(D_{\mathbf{u}} f)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y$.

That is, $(D_{\mathbf{u}} f)_{P_0} = \cos(x_0 + 3y_0)(1/\sqrt{5}) + 3\cos(x_0 + 3y_0)(2/\sqrt{5})$.

Equivalently, $(D_{\mathbf{u}} f)_{P_0} = (7/\sqrt{5}) \cos(x_0 + 3y_0)$.

Then , $(D_{\mathbf{u}} f)_{P_0} = (7/\sqrt{5}) \cos(13)$. ◁

Directional derivative of functions of three variables

Definition
The *directional derivative* of the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0, z_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is given by the limit

$$(D_{\mathbf{u}} f)_{P_0} = \lim_{t\to 0} \frac{1}{t}\big[f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0)\big].$$

Theorem
If the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0, z_0)$ and $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is a unit vector, then

$$(D_{\mathbf{u}} f)_{P_0} = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0)\, u_z.$$

Example
Find $(D_{\mathbf{u}} f)_{P_0}$ for $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at the point $P_0 = (3, 2, 1)$ along the direction given by $\mathbf{v} = \langle 2, 1, 1 \rangle$.

Solution: We first find a unit vector along $\mathbf{v}$,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle.$$

Then, $(D_{\mathbf{u}} f)$ is given by $(D_{\mathbf{u}} f) = (2x)u_x + (4y)u_y + (6z)u_z$.

We conclude, $(D_{\mathbf{u}} f)_{P_0} = (6)\dfrac{2}{\sqrt{6}} + (8)\dfrac{1}{\sqrt{6}} + (6)\dfrac{1}{\sqrt{6}}$,

that is, $(D_{\mathbf{u}} f)_{P_0} = \dfrac{26}{\sqrt{6}}$.

View as a dot product of 2 vectors

$$D_u f = f_x u_x + f_y u_y$$

$$= \langle f_x, f_y \rangle \cdot \langle u_x, u_y \rangle$$

∇f (Gradient vector) $\quad$ u

The gradient vector and directional derivatives

Remark: The directional derivative of a function can be written in terms of a dot product.

(a) In the case of 2 variable functions: $D_\mathbf{u} f = f_x u_x + f_y u_y$

$$D_\mathbf{u} f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y \rangle.$$

(b) In the case of 3 variable functions: $D_\mathbf{u} f = f_x u_x + f_y u_y + f_z u_z$,

$$D_\mathbf{u} f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y, f_z \rangle.$$

The gradient vector and directional derivatives

Definition
The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at any point $(x, y) \in D$ is the vector $\nabla f = \langle f_x, f_y \rangle$.

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at any point $(x, y, z) \in D$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$.

Notation:

- For two variable functions: $\nabla f = f_x\, \mathbf{i} + f_y\, \mathbf{j}$.
- For two variable functions: $\nabla f = f_x\, \mathbf{i} + f_y\, \mathbf{j} + f_z\, \mathbf{k}$.

Theorem
If $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, is a differentiable function and $\mathbf{u}$ is a unit vector, then,

$$D_\mathbf{u} f = (\nabla f) \cdot \mathbf{u}.$$

Example
Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

Solution: The gradient is $\nabla f = \langle f_x, f_y \rangle$, that is, $\nabla f = \langle 2x, 2y \rangle$. ◁

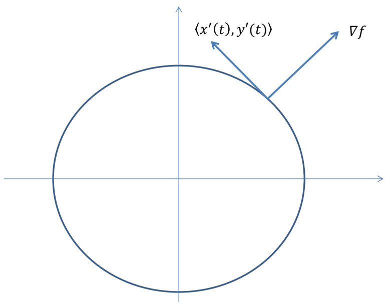
Recall contour curves:

$$f(x, y) = x^2 + y^2 = const.$$

$$\rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\rightarrow \nabla f \cdot \langle x'(t), y'(t) \rangle = 0$$

Gradient vector $\quad$ Tangent vector



Example

Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

Solution: The gradient is $\nabla f = \langle f_x, f_y\rangle$, that is, $\nabla f = \langle 2x, 2y\rangle$. ◁

Remark:
$\nabla f = 2\mathbf{r}$,
with
$\mathbf{r} = \langle x, y\rangle$.