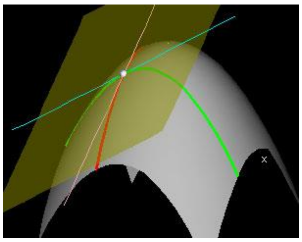
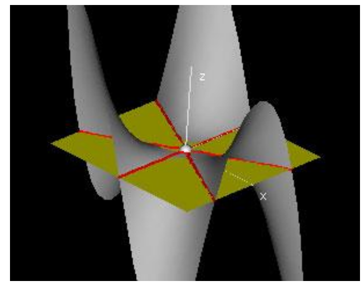
Tangent planes and linear approximation



Intuitive idea: A tangent plane is a plane which touches the surface at one point such that the plane contains any tangent line to a smooth curve on the surface through this point.



The tangent plane at a saddle point. Note that the plane touches the surface at infinitely many points.

Review: Equation of a tangent line

$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$: parametric curve in space.

Assume that all functions are differentiable.

Equation of the tangent line at $(x(t_0), y(t_0), z(t_0))$:

$$\begin{cases} x = x(t_0) + x'(t_0)t \\ y = y(t_0) + y'(t_0)t \\ z = x(t_0) + z'(t_0)t \end{cases}$$

Review: Equation of a plane in space

Equation of a plane in Cartesian coordinates

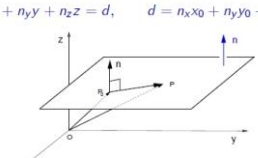
Theorem

Given any Cartesian coordinate system, the point $P = (x, y, z)$ belongs to the plane by $P_0 = (x_0, y_0, z_0)$ perpendicular to $\mathbf{n} = \langle n_x, n_y, n_z\rangle$ iff holds

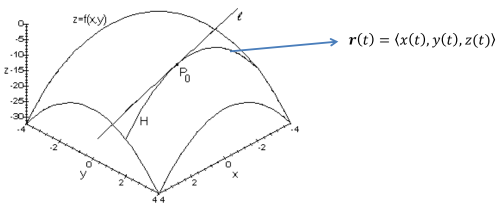
$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0.$$

Furthermore, the equation above can be written as

$$n_x x + n_y y + n_z z = d, \qquad d = n_x x_0 + n_y y_0 + n_z z_0.$$

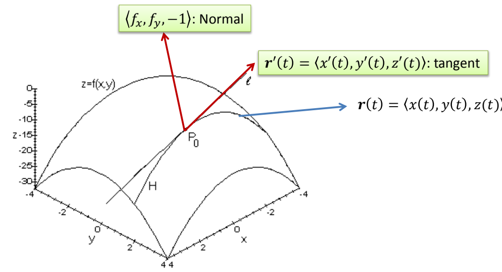


Idea



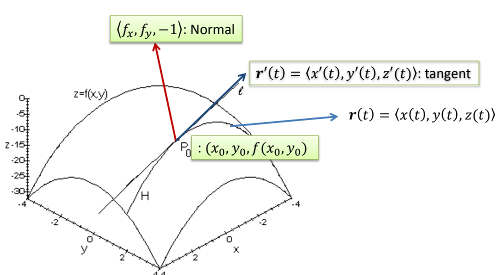
$$\Rightarrow z(t) = f(x(t), y(t)) \Rightarrow f(x(t), y(t)) - z(t) = 0$$
$$\Rightarrow f_x x'(t) + f_y y'(t) - z'(t) = 0$$
$$\Rightarrow \langle f_x, f_y, -1\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = 0$$

Tangent direction



$$\Rightarrow z(t) = f(x(t), y(t)) \Rightarrow f(x(t), y(t)) - z(t) = 0$$
$$\Rightarrow f_x x'(t) + f_y y'(t) - z'(t) = 0$$
$$\Rightarrow \langle f_x, f_y, -1\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = 0$$

ANY Tangent direction

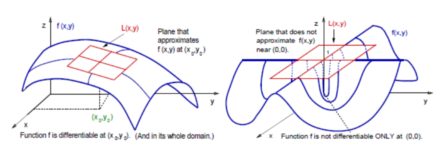


$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - \left(z - f(x_0, y_0)\right) = 0$$
$$\Rightarrow z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y) + f(x_0, y_0)$$

Z=L(x,y): Equation of the tangent plane

Review: Differentiable functions of two variables.

Recall: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Theorem

If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

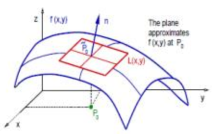
The tangent plane to the graph of a function

Theorem

The plane tangent to the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point (x_0, y_0) is given by

$$z = L(x,y) = f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + f(x_0,y_0).$$

Proof



Since at (x_0, y_0) the function L satisfies that

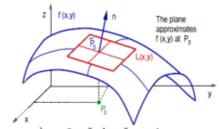
$$L(x_0,y_0) = f(x_0,y_0).$$

then the plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0,y_0))$.

We only need to find its normal vector $\mathbf{n}$.

The tangent plane to the graph of a function.

The vector $\mathbf{n}$ normal to the plane $L(x,y)$ is a vector perpendicular to the surface $z = f(x,y)$ at $P_0 = (x_0, y_0)$.



This surface is the level surface $F(x,y,z) = 0$ of the function $F(x,y,z) = f(x,y) - z$. A vector normal to this level surface is its gradient ∇F. That is, $\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$.

Therefore, the normal to the tangent plane $L(x,y)$ at the point P_0 is $\mathbf{n} = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$. Recall that the plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0,y_0))$. The equation for the plane is

$$f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - f(x_0,y_0)) = 0. \quad \square$$

Example

Show that $f(x,y) = \arctan(x + 2y)$ is differentiable and find the plane tangent to $f(x,y)$ at $(1,0)$.

Solution: The partial derivatives of f are given by

$$f_x(x,y) = \frac{1}{1 + (x+2y)^2}, \quad f_y(x,y) = \frac{2}{1 + (x+2y)^2}.$$

These functions are continuous in $\mathbb{R}^2$, so $f(x,y)$ is differentiable at every point in $\mathbb{R}^2$. The plane $L(x,y)$ at $(1,0)$ is given by

$$L(x,y) = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0),$$

where $f(1,0) = \arctan(1) = \pi/4$, $f_x(1,0) = 1/2$, $f_y(1,0) = 1$.

Then, $\boxed{L(x,y) = \frac{1}{2}(x-1) + y + \frac{\pi}{4}.}$ $\triangleleft$

The linear approximation of a differentiable function

Definition

The *linear approximation* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $(x_0, y_0) \in D$ is the plane

$$L(x,y) = f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + f(x_0,y_0).$$

Example

Find the linear approximation of $f = \sqrt{17 - x^2 - 4y^2}$ at $(2,1)$.

Solution: $L(x,y) = f_x(2,1)(x-2) + f_y(2,1)(y-1) + f(2,1)$.

We need three numbers: $f(2,1)$, $f_x(2,1)$, and $f_y(2,1)$.

These are: $f(2,1) = 3$, $f_x(2,1) = -2/3$, and $f_y(2,1) = -4/3$.

Then the plane is given by $L(x,y) = -\frac{2}{3}(x-2) - \frac{4}{3}(y-1) + 3.$ $\triangleleft$

Bounds for the error of a linear approximation

Theorem

Assume that the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has first and second partial derivatives continuous on an open set containing a rectangular region $R \subset D$ centered at the point (x_0, y_0).
If $M \in \mathbb{R}$ is the upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in R, then the error $E(x,y) = f(x,y) - L(x,y)$ satisfies the inequality

$$|E(x,y)| \leqslant \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2,$$

where $L(x,y)$ is the linearization of f at (x_0, y_0), that is,

$$L(x,y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Bounds for the error of a linear approximation

Example

Find an upper bound for the error in the linear approximation of $f(x,y) = x^2 + y^2$ at the point $(1,2)$ over the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \;:\; |x - 1| < 0.1, \quad |y - 2| < 0.1\}$$

Solution: The second derivatives of f are $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$. Therefore, we can take $M = 2$.

Then the formula $|E(x,y)| \leqslant \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2$, implies

$$|E(x,y)| \leqslant \left(|x - 1| + |y - 2|\right)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x,y)| < 0.04$. ◁

Since $f(1,2) = 5$, the % relative error is $100\,\dfrac{E(x,y)}{f(1,2)} \leqslant 0.8\%$.

Review: Differential of functions of one variable.

Definition

The *differential at* $x_0 \in D$ of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is the linear function

$$df(x) = L(x) - f(x_0).$$

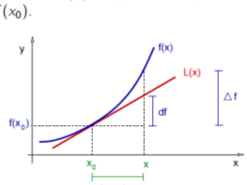
Remark: The linear approximation of $f(x)$ at x_0 is the line given by $L(x) = f'(x_0)\,(x - x_0) + f(x_0)$.

Therefore

$$df(x) = f'(x_0)\,(x - x_0).$$

Denoting $dx = x - x_0$,

$$df = f'(x_0)\,dx.$$



Differential of functions of more than one variable

Definition

The *differential at* $(x_0, y_0) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the linear function

$$df(x,y) = L(x,y) - f(x_0, y_0).$$

Remark: The linear approximation of $f(x,y)$ at (x_0, y_0) is the plane $L(x,y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$.

Therefore $df(x,y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

Denoting $dx = x - x_0$ and $dy = (y - y_0)$ we obtain the usual expression

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.$$

Therefore, df and L are similar concepts: The linear approximation of a differentiable function f.

Example

Compute the df of the function $f(x, y) = \ln(1 + x^2 + y^2)$ at the point $(1, 1)$. Evaluate this df for $dx = 0.1$, $dy = 0.2$.

Solution: The differential of f at (x_0, y_0) is given by

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

The partial derivatives f_x and f_y are given by

$$f_x(x, y) = \frac{2x}{1 + x^2 + y^2}, \qquad f_y(x, y) = \frac{2y}{1 + x^2 + y^2}.$$

Therefore, $f_x(1, 1) = \frac{2}{3} = f_y(1, 1)$. Then $df = \frac{2}{3}\,dx + \frac{2}{3}\,dy$.
Evaluating this differential at $dx = 0.1$ and $dy = 0.2$ we obtain

$$df = \frac{2}{3}\frac{1}{10} + \frac{2}{3}\frac{2}{10} = \frac{2}{3}\frac{3}{10} \quad \Rightarrow \quad df = \frac{1}{5}. \qquad \triangleleft$$

Example

The total differential of the function:
$z = \ln(xy) + x^2 + y$, is

$$dz = (\frac{1}{x} + 2x)dx + (\frac{1}{y} + 1)dy$$

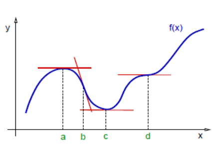
If x changes from 1 to 1.05 and y changes from 2 to 1.98, then the values of dz and Δz are:

$$dz = (\frac{1}{1} + 2(1))(.05) + (\frac{1}{2} + 1)(-.02) = .1200$$

$$\Delta z = \log(1.05)(1.98) + (1.05)^2 + 1.98 - (log(1)(2) + 1^2 + 2) = .1212$$

Review: Local extrema for functions of one variable

Recall: Main results on local extrema for $f(x)$:



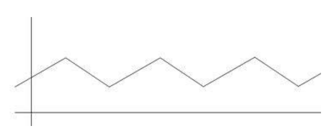
at	f	f'	f''
a	max.	0	< 0
b	infl.	$\neq 0$	$\pm 0 \mp$
c	min.	0	> 0
d	infl.	$= 0$	$\pm 0 \mp$

Remarks: Assume that f is twice continuously differentiable.

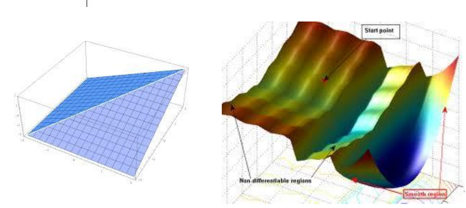
- If x_0 is local maximum or minimum of f, then $f'(x_0) = 0$.
- If $f'(x_0) = 0$, then x_0 is a critical point of f, that is, x_0 is a maximum or a minimum or an inflection point.
- The second derivative test determines whether a critical point is a maximum, minimum or an inflection point.

Non-differentiable functions

1-D



2-D

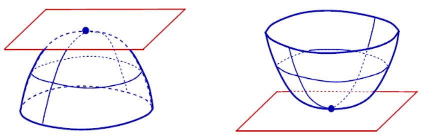


Definition of local extrema for functions of two variables

Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local maximum* at the point $(a, b) \in D$ iff holds that $f(a, b) \geqslant f(x, y)$ for every point (x, y) in a neighborhood of (a, b).

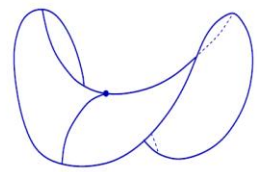
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local minimum* at the point $(a, b) \in D$ iff holds that $f(a, b) \leqslant f(x, y)$ for every point (x, y) in a neighborhood of (a, b).



Definition of local extrema for functions of two variables

Definition

A differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *saddle point* at an interior point $(a, b) \in D$ iff in every open disk in D centered at (a, b) there always exist points (x, y) where $f(a, b) < f(x, y)$ and other points (x, y) where $f(a, b) > f(x, y)$.



Characterization of local extrema

Theorem (First Derivative Test)

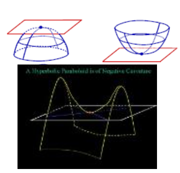
If a differentiable function f has a local maximum or minimum at (a, b) then holds $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$.

Remark: The tangent plane at a local extremum is horizontal, since its normal vector is $\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 0, 0, -1\rangle$.

Definition

The interior point $(a, b) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a *critical point* of f iff $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$.

Remark:
Critical points include local maxima, local minima, and saddle points.



Example

Find the critical points of the function $f(x, y) = -x^2 - y^2$.

Solution: The critical points are the points where ∇f vanishes. Since $\nabla f = \langle -2x, -2y\rangle$, the only solution to $\nabla f = \langle 0, 0\rangle$ is $x = 0$, $y = 0$. That is, $(a, b) = (0, 0)$. $\triangleleft$

Remark: Since $f(x, y) \leqslant 0$ for all $(x, y) \in \mathbb{R}^2$ and $f(0, 0) = 0$, then the point $(0, 0)$ must be a local maximum of f.

Example

Find the critical points of the function $f(x, y) = x^2 - y^2$.

Solution: Since $\nabla f = \langle 2x, -2y\rangle$, the only solution to $\nabla f = \langle 0, 0\rangle$ is $x = 0$, $y = 0$. That is, we again obtain $(a, b) = (0, 0)$. $\triangleleft$

Theorem (Second derivative test)

Let (a,b) be a critical point of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, that is, $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$. Assume that f has continuous second derivatives in an open disk in D with center in (a,b) and denote

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2.$$

Then, the following statements hold:

- *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
- *If $D < 0$, then $f(a,b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

Notation: The number D is called the *discriminant* of f at (a,b).

Example

Find the local extrema of $f(x,y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

Solution: We first find the critical points:

$$\nabla f = \langle -2x, 2y\rangle \quad \Rightarrow \quad (\nabla f)\big|_{(a,b)} = \langle 0,0\rangle \ \text{ iff } \ (a,b) = (0,0).$$

The only critical point is $(a,b) = (0,0)$.

We need to compute $D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2$.

Since $f_{xx}(0,0) = -2$, $f_{yy}(0,0) = 2$, and $f_{xy}(0,0) = 0$, we get

$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0,0).$$ ◁

Example

Is the point $(a,b) = (0,0)$ a local extrema of $f(x,y) = y^2x^2$?

Solution: We first verify that $(0,0)$ is a critical point of f:

$$\nabla f(x,y) = \langle 2xy^2, 2yx^2\rangle, \quad \Rightarrow \quad (\nabla f)\big|_{(0,0)} = \langle 0,0\rangle,$$

therefore, $(0,0)$ is a critical point.

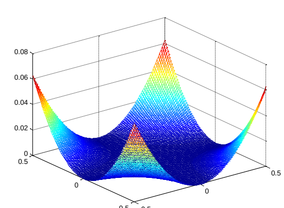
Remark: The whole axes $x = 0$ and $y = 0$ are critical points of f.

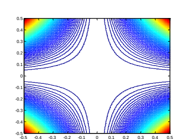
We need to compute $D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2$.

Since $f_{xx}(x,y) = 2y^2$, $f_{yy}(x,y) = 2x^2$, and $f_{xy}(x,y) = 4xy$,

we obtain $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, and $f_{xy}(0,0) = 0$,

hence $D = 0$ and the test is inconclusive. ◁





Is the point $(a,b) = (0,0)$ a local extrema of $f(x,y) = y^2x^2$?

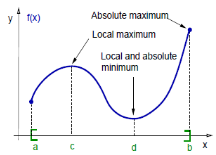
Solution: Since $f(x,y) = x^2y^2 \geqslant 0$ for all (x,y), and $f(0,0) = 0$, then $(0,0)$ is a local minimum. (Also a global minimum.) ◁

This is confirmed in the graph of f.

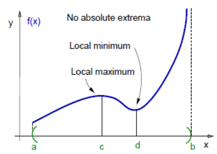
Absolute extrema of a function in a domain

Definition
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute maximum at the point $(a, b) \in D$ iff $f(a, b) \geqslant f(x, y)$ for all $(x, y) \in D$.
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute minimum at the point $(a, b) \in D$ iff $f(a, b) \leqslant f(x, y)$ for all $(x, y) \in D$.

Remark: Local extrema need not be the absolute extrema.

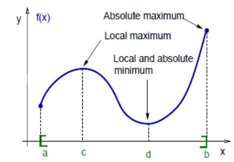


Remark: Absolute extrema may not be defined on open intervals.



Review: Functions of one variable

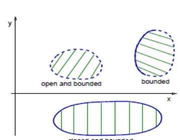
Theorem
Every continuous function on a closed interval, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, with $a < b \in \mathbb{R}$, always has absolute extrema.

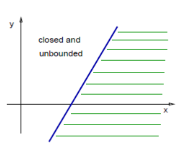


Recall:
- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.
- The set $[a, b]$ is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.

Recall: On open and closed sets in $\mathbb{R}^n$

Definition
A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





Theorem
If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in a closed and bounded set D, then f has an absolute maximum and an absolute minimum in D.

Absolute extrema on closed and bounded sets

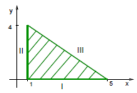
Problem:
Find the absolute extrema of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ in a closed and bounded set D.

Solution:
(1) Find every critical point of f in the interior of D and evaluate f at these points.
(2) Find the boundary points of D where f has local extrema, and evaluate f at these points.
(3) Look at the list of values for f found in the previous two steps.

If $f(x_0, y_0)$ is the biggest (smallest) value of f in the list above, then (x_0, y_0) is the absolute maximum (minimum) of f in D.

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:

(1) We find all critical points in the interior of the domain:

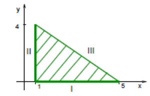
$$\nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad (x_0, y_0) = (-2, 1).$$

Since $(-2, 1)$ does not belong to the domain, we discard it.

(2) Three segments form the boundary of D:
Boundary I: The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.
We look for critical point on the interior of Boundary I:
Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$.
No critical points in the interior of Boundary I.

Example (continue)

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

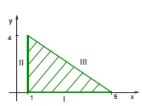


Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$.
We look for critical point on the interior of Boundary II:
Since $g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y$, so $g' = 3 \neq 0$.
No critical points in the interior of Boundary II.

Boundary III: The segment $y = -x + 5$, $x \in [1, 5]$.
We look for critical point on the interior of Boundary III:
Since $g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5)$.
We obtain $g(x) = -x^2 + 2x + 13$, hence $g'(x) = -2x + 2 = 0$ implies $x = 1$. So, $y = 4$, and we selected the point $(1, 4)$, which was already in our list. No critical points in the interior of III.

Example (continue)

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:

(3) Our list of values is:

$$f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2.$$

We conclude:

(a) Absolute maximum at $(1, 4)$,

(b) Absolute minimum at $(5, 0)$.

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