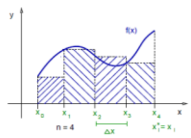
Integration with several variables.

Review: Integral of a single variable function

Definition
The definite integral of a function $f : [a,b] \to \mathbb{R}$, in the interval $[a,b]$ is the number

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x.$$

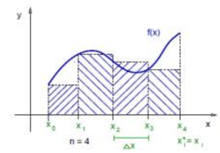
where $x_i^* \in [x_i, x_{i+1}]$ is called a sample point, while $\{x_i\}$ is a partition in $[a,b]$, $i = 0, \cdots, n$, and with $x_i = a + i\Delta x$, and $\Delta x = \frac{(b-a)}{n}$.



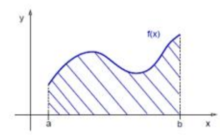
The integral as an area.

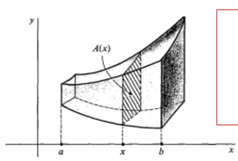
The sum $S_n = \sum_{i=0}^{n} f(x_i^*)\Delta x$ is called a Riemann sum. Then,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n.$$



The integral $\int_a^b f(x)\,dx$ is the area in between the graph of f and the horizontal axis.





Review: volume by slicing or rotation.

--- single variable

Partition "P": Divide $[a,b]$ into n subintervals:

$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

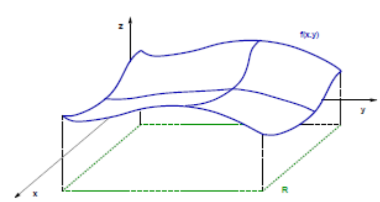
i^{th} interval: $[x_{i-1}, x_i]$, $\quad i = 1,2,\cdots n$

Riemann Sum:

$$S_P = \sum_{i=1}^{n} A(p_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} A(p_i)\Delta x_i$$

General problem: find the volume below a surface over a 2-D region.

Simple case: rectangular region.



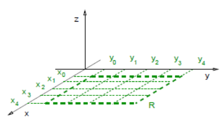
Double integrals on rectangles

Definition

The *double integral* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ in the rectangle $R = [a, b] \times [c, d]$ is the number

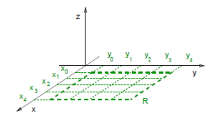
$$\iint_R f(x, y)\, dx\, dy = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y.$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$, are sample points, while $\{x_i\}$ and $\{y_j\}$, $i, j = 0, \cdots, n$ are partitions of the intervals $[a, b]$ and $[c, d]$, and $\Delta x = \dfrac{(b-a)}{n}$, $\Delta y = \dfrac{(d-c)}{n}$.

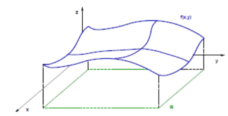


The double integral as a volume

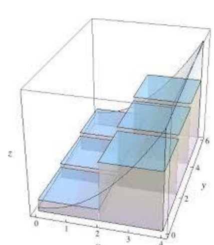
The sum $S_n = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y$ is called a Riemann sum. Then, $\iint_R f(x, y)\, dx\, dy = \lim_{n\to\infty} S_n$.

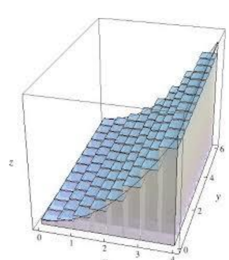


The integral $\iint_R f(x, y)\, dx\, dy$ is the volume above R and below the graph of f.

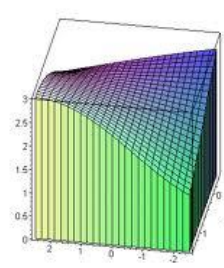


Idea of Fubini's theorem





Computing each slice by integration with single variable



Fubini Theorem on rectangular domains

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in $R = [x_0, x_1] \times [y_0, y_1]$, then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1}\Big[\int_{x_0}^{x_1} f(x,y)\,dx\Big]\,dy,$$
$$= \int_{x_0}^{x_1}\Big[\int_{y_0}^{y_1} f(x,y)\,dy\Big]\,dx.$$

Remark: Fubini's Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.

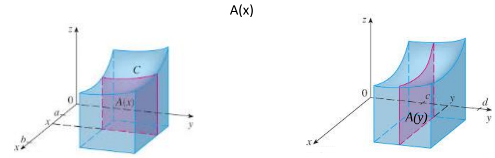
Notation: The double integral is also written as

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy.$$

Fubini's Theorem:

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1}\underbrace{\Big[\int_{x_0}^{x_1} f(x,y)\,dx\Big]}_{A(y)}\,dy$$

$$= \int_{x_0}^{x_1}\underbrace{\Big[\int_{y_0}^{y_1} f(x,y)\,dy\Big]}_{A(x)}\,dx.$$



Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: Since $x \in [0,2]$ and $y \in [1,3]$,

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3\int_0^2 (xy^2 + 2x^2y^3)\,dx\,dy$$

$$I = \int_1^3\Big[\int_0^2 (xy^2 + 2x^2y^3)\,dx\Big]\,dy.$$

We compute the interior integral in x first, keeping y constant. After that we compute the integral in y.

Solution: We compute the integral in x first, keeping y constant.

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3\Big[\int_0^2 (xy^2 + 2x^2y^3)\,dx\Big]\,dy,$$

$$I = \int_1^3\Big[\frac{y^2}{2}\Big(x^2\Big|_0^2\Big) + \frac{2y^3}{3}\Big(x^3\Big|_0^2\Big)\Big]\,dy,$$

$$I = \int_1^3\Big[2y^2 + \frac{16}{3}y^3\Big]\,dy.$$

$$= 2\,\frac{y^3}{3}\Big|_1^3 + \frac{16}{3}\,\frac{y^4}{4}\Big|_1^3.$$

$$= 2\,\frac{26}{3} + \frac{4}{3}80 = \frac{372}{3}.$$

Solution: Integrate first in y, then in x.

$$I = \iint_R f(x,y)\,dx\,dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3)dx\,dy$$

$$I = \int_0^2 \Big[\int_1^3 (xy^2 + 2x^2y^3)dy\Big]\,dx.$$

$$I = \int_0^2 \Big[\frac{x}{3}\Big(y^3\Big|_1^3\Big) + \frac{2x^2}{4}\Big(y^4\Big|_1^3\Big)\Big]\,dx.$$

$$= \int_0^2 \Big[\frac{26}{3}\,x + 40\,x^2\Big]\,dx = \frac{26}{3}\,\frac{x^2}{2}\Big|_0^2 + 40\,\frac{x^3}{3}\Big|_0^2,$$

$$I = \frac{26}{3}\,(2) + 40\,\frac{8}{3} = \frac{372}{3}.$$

A particular case of Fubini's Theorem

Corollary

If the continuous function $f : R \subset R^2 \to \mathbb{R}$ satisfies that $f(x,y) = g(x)h(y)$, then the double integral of function f in the rectangle $R = [x_0, x_1] \times [y_0, y_1]$ is given by

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} g(x)h(y)dy\,dx = \Big(\int_{x_0}^{x_1} g(x)dx\Big)\Big(\int_{y_0}^{y_1} h(y)dy\Big).$$

Remark: In the case that $f(x,y)$ is a product of two functions g, h, with $g(x)$ and $h(y)$, then the double integral of f is also a product of the integral of g times the integral of h.

Example

Compute the double integral of $f(x,y) = \dfrac{1+x^2}{1+y^2}$, in the rectangular region $R = [0,2] \times [0,1]$.

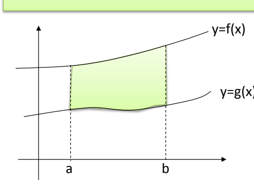
Solution: $I = \iint_R f(x,y)\,dx\,dy = \int_0^2 \int_0^1 \frac{1+x^2}{1+y^2}dy\,dx,$

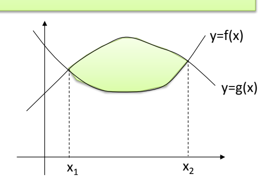
$$I = \Big[\int_0^2 (1+x^2)dx\Big]\Big[\int_0^1 \frac{1}{1+y^2}dy\Big],$$

$$I = \Big(x\Big|_0^2 + \frac{1}{3}x^3\Big|_0^2\Big)\Big(\arctan(y)\Big|_0^1\Big) = \Big(2 + \frac{8}{3}\Big)\,\frac{\pi}{4} = \frac{14}{3}\,\frac{\pi}{4}$$

We conclude $\iint_R f(x,y)\,dx\,dy = \dfrac{7}{6}\,\pi.$

Review: Area between curves





$$Area = \int_a^b [\,f(x) - g(x)]dx$$

where $[a,b]$ is given.

$$Area = \int_{x_1}^{x_2} [g(x) - f(x)]dx$$

where x_1 and x_2 are not given.
Need to solve $f(x) = g(x)$ for x_1 and x_2.

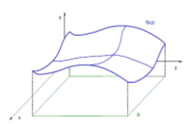
Review: Fubini's Theorem on rectangular domains

Theorem
If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx, \\ = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Remark: Fubini result says that double integrals can be computed doing two one-variable integrals.

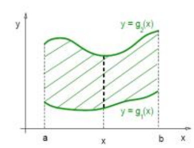
Remark: On a rectangle is simple to switch the order of integration in double integrals of continuous functions.

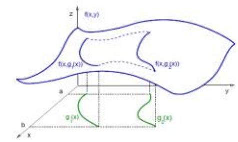


Fubini's Theorem on Type I domains, $y(x)$

Theorem
If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type I):
If $D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a,b]$, then

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

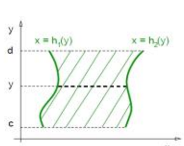


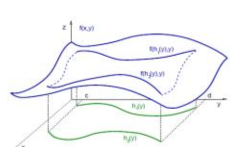


Fubini's Theorem on Type II domains, $x(y)$

Theorem
If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type II):
If $D = \{(x,y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c,d]\}$, with h_1, h_2 continuous functions on $[c,d]$, then

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$





Summary: Fubini's Theorem on non-rectangular domains

Theorem
If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold:

(a) *(Type I) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a,b]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) *(Type II) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c,d]\}$, with h_1, h_2 continuous functions on $[c,d]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

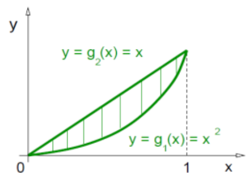
Solution:

This is a Type I domain, with lower boundary

$$y = g_1(x) = x^2,$$

and upper boundary

$$y = g_2(x) = x.$$



(Conti.)

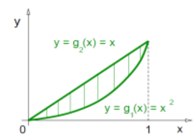
Solution: $I = \iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

with $g_1(x) = x^2$ and $g_2(x) = x$, we obtain

$$I = \iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x^2 + y^2)dy\,dx,$$

$$I = \int_0^1 \left[x^2\left(y\Big|_{x^2}^{x}\right) + \left(\frac{y^3}{3}\Big|_{x^2}^{x}\right)\right] dx.$$

$$= \int_0^1 \left[x^2(x - x^2) + \frac{1}{3}(x^3 - x^6)\right] dx.$$



(Conti.)

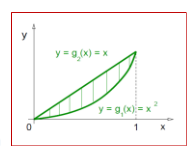
Solution: Recall: $I = \int_0^1 \left[x^2(x - x^2) + \frac{1}{3}(x^3 - x^6)\right] dx.$

$$I = \int_0^1 \left[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6\right] dx = \left[\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^4}{12} - \frac{x^7}{21}\right]\Big|_0^1$$

$$I = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{(3)(5)(7)}.$$

We conclude: $\iint_D f(x,y)\,dx\,dy = \frac{3}{35}.$ ◁

Rmk: For some problems, they can be solved either as Type I or Type II.



Previous graph

Example

Find the integral of $f(x,y) = x^2 + y^2$ on the domain $D = \{(x,y) \in \mathbb{R}^2 \ : \ y \leqslant x \leqslant \sqrt{y}, \quad 0 \leqslant y \leqslant 1\}$.

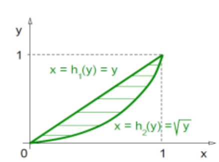
Solution:

This is a Type II domain, with left boundary

$$x = h_1(y) = y,$$

and right boundary

$$x = h_2(y) = \sqrt{y}.$$



Remark:

This domain is both Type I and Type II: $y = x^2 \Leftrightarrow x = \sqrt{y}$.

(Conti.)

Solution: $I = \iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$
with $h_1(y) = y$ and $h_2(y) = \sqrt{y}$, we obtain

$$I = \int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\,dx\,dy,$$

$$I = \int_0^1 \Big[\Big(\frac{x^3}{3}\Big|_y^{\sqrt{y}}\Big) + y^2\Big(x\Big|_y^{\sqrt{y}}\Big)\Big]\,dy,$$

$$I = \int_0^1 \Big[\frac{1}{3}(y^{3/2} - y^3) + y^2(y^{1/2} - y)\Big]\,dy.$$

(Conti.)

Solution: $I = \int_0^1 \Big[\frac{1}{3}(y^{3/2} - y^3) + y^2(y^{1/2} - y)\Big]\,dy.$

$$I = \int_0^1 \Big[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\Big]\,dy,$$

$$I = \Big[\frac{1}{3}\frac{2}{5}y^{5/2} - \frac{1}{3}\frac{y^4}{4} + \frac{2}{7}y^{7/2} - \frac{y^4}{4}\Big]\Big|_0^1,$$

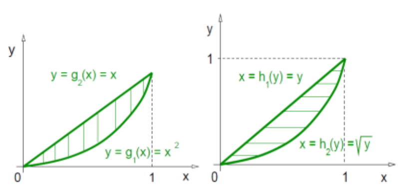
$$I = \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{(3)(5)(7)}.$$

We conclude $\iint_D f(x,y)\,dx\,dy = \frac{3}{35}.$

Domains Type I and Type II

Summary: We have shown that a double integral of a function f on the domain D given in the pictures below holds,

$$\iint_D f(x,y)dx\,dy = \int_0^1 \int_{x^2}^{x} f(x,y)dy\,dx = \int_0^1 \int_y^{\sqrt{y}} f(x,y)dx\,dy.$$



Example

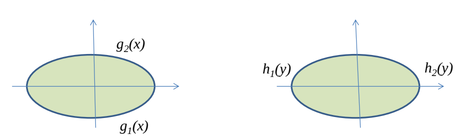
Find the limits of integration of $\iint_D f(x,y)\,dxdy$ in the domain $D = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leqslant 1\}$ when D is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
So, the boundary as Type I is given by

$$y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).$$

The boundary as Type II is given by

$$x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y), \quad x = 3\sqrt{1 - \frac{y^2}{4}} = h_2(y). \quad \triangleleft$$

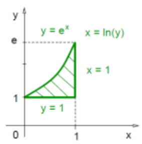


Example

Reverse the order of integration in $\int_0^1 \int_1^{e^x} dy\,dx$.

Solution:

This integral is written as Type I, since we first integrate on vertical intervals $[1, e^x]$, with boundaries $y = e^x$, $y = 1$, while $x \in [0, 1]$.

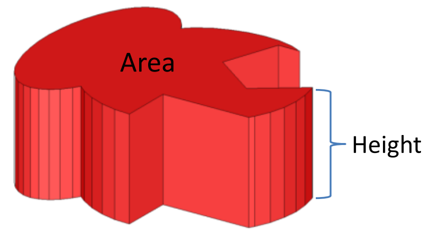


Invert the first equation and from the figure we get the left and right boundaries:

$$x = \ln(y), \quad x = 1, \quad \text{with} \quad y \in [1, e].$$

Therefore, we conclude that $\int_0^1 \int_1^{e^x} dy\,dx = \int_1^e \int_{\ln(y)}^1 dx\,dy.$ ◁

Calculating the area as the volume



Idea: Volume = Area * Height. If Height=1, then Volume=Area

Areas of a region on a plane

Definition

The *area* of a closed, bounded region R on a plane is given by

$$A = \iint_R dx\,dy.$$

Remark:

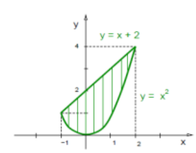
- To compute the area of a region R we integrate the function $f(x, y) = 1$ on that region R.
- The area of a region R is computed as the volume of a 3-dimensional region with base R and height equal to 1.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2],\ y \in [x^2, x + 2]\}$.

Solution: We express the region R as an integral Type I, integrating first on vertical directions:

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx.$$



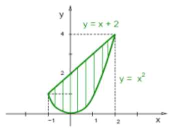
$$A = \int_{-1}^{2} \Big(y\Big|_{x^2}^{x+2} \Big)\,dx = \int_{-1}^{2} (x + 2 - x^2)\,dx = \Big(\frac{x^2}{2} + 2x - \frac{x^3}{3}\Big)\Big|_{-1}^{2}.$$

$$A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3 \quad \Rightarrow \quad A = \frac{9}{2}. \quad ◁$$

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \; : \; x \in [-1, 2], \; y \in [x^2, x + 2]\}$.

Solution: We express the region R as an integral Type I, integrating first on vertical directions:



$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$ Rmk

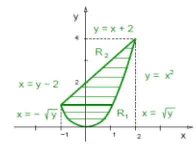
$$A = \int_{-1}^{2} \Big(y\Big|_{x^2}^{x+2} \Big) dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \Big(\frac{x^2}{2} + 2x - \frac{x^3}{3} \Big)\Big|_{-1}^{2}.$$

$$A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3 \quad \Rightarrow \quad A = \frac{9}{2}. \quad \triangleleft$$

Rmk.: This part is the set-up we learned in Calculus I.

(Continue)

Solution: We express the region R as an integral Type II, integrating first on horizontal directions:



$$A = \iint_{R_1} dx\, dy + \iint_{R_2} dx\, dy.$$

$$A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

$$A = \int_0 2\sqrt{y}\, dy + \int_1 (\sqrt{y} - y + 2)\, dy$$

$$A = 2\Big(\frac{2}{3} y^{3/2}\Big)\Big|_0^1 + \Big(\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y\Big)\Big|_1^4$$

$$A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} + 8 - 2 = 9/2.$$

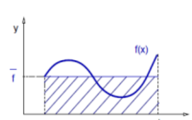
Average value of a function

Review: The average of a single variable function.

Definition

The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b - a)} \int_a^b f(x)\, dx.$$



Definition

The *average* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on the region R with area $A(R)$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{A(R)} \iint_R f(x, y)\, dx\, dy.$$

Example

Find the average of $f(x, y) = xy$ on the region $R = \{(x, y) \in \mathbb{R}^2 \; : \; x \in [0, 2], \; y \in [0, 3]\}$.

Solution: The area of the rectangle R is $A(R) = 6$.

We only need to compute $I = \iint_R f(x, y)\, dx\, dy$.

$$I = \int_0^2 \int_0^3 xy\, dy\, dx = \int_0^2 x \Big(\frac{y^2}{2}\Big|_0^3 \Big) dx = \int_0^2 \frac{9}{2} x\, dx.$$

$$I = \frac{9}{2} \Big(\frac{x^2}{2}\Big|_0^2 \Big) \quad \Rightarrow \quad I = 9.$$

Since $\overline{f} = I / A(R) = 9/6$, we get $\overline{f} = 3/2$.

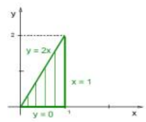
Example

Find the integral of $\rho(x,y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

Solution: We need to compute

$$M = \iint_R \rho(x,y)\, dxdy.$$

Remark: If ρ is the mass density, then M is the total mass.



$$M = \int_0^1 \int_0^{2x} (x+y)\, dy\, dx = \int_0^1 \Big[x\Big(y\Big|_0^{2x}\Big) + \Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\Big]\, dx.$$

$$M = \int_0^1 \big[2x^2 + 2x^2\big]\, dx = 4\,\frac{x^3}{3}\Big|_0^1 \quad \Rightarrow \quad M = \frac{4}{3}. \qquad \triangleleft$$

Example

Given the function $\rho(x,y) = x + y$, the number M computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

$$\bar{r}_x = \frac{1}{M}\int_R x\rho(x,y)\, dy\, dx, \quad \bar{r}_y = \frac{1}{M}\iint_R y\rho(x,y)\, dy\, dx.$$

Remark: $\mathbf{r} = \langle \bar{r}_x, \bar{r}_y \rangle$ is the center of mass of the body.

Solution: Recall: $M = \frac{4}{3}$. We need to compute

$$\bar{r}_x = \frac{1}{M}\int_0^1 \int_0^{2x} (x+y)x\, dydx = \frac{3}{4}\int_0^1 \Big[x^2\Big(y\Big|_0^{2x}\Big) + x\Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\Big]\, dx$$

$$\bar{r}_x = \frac{3}{4}\int_0^1 \big[2x^3 + 2x^3\big]\, dx = \frac{3}{4}\, x^4\Big|_0^1 \quad \Rightarrow \quad \bar{r}_x = \frac{3}{4}.$$

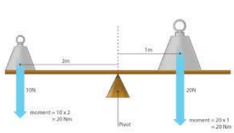
(Continue.)

$$\bar{r}_y = \frac{1}{M}\int_0^1 \int_0^{2x} (x+y)y\, dydx = \frac{3}{4}\int_0^1 \Big[x\Big(\frac{y^2}{2}\Big|_0^{2x}\Big) + \Big(\frac{y^3}{3}\Big|_0^{2x}\Big)\Big]\, dx$$

$$\bar{r}_y = \frac{3}{4}\int_0^1 \Big[2x^3 + \frac{8}{3}x^3\Big]\, dx = \frac{3}{4}\Big[2\Big(\frac{x^4}{4}\Big|_0^1\Big) + \frac{8}{3}\Big(\frac{x^4}{4}\Big|_0^1\Big)\Big]$$

$$\bar{r}_y = \frac{3}{4}\Big[\frac{1}{2} + \frac{2}{3}\Big] = \frac{3}{4}\,\frac{7}{6} \Rightarrow \bar{r}_y = \frac{7}{8}. \qquad \triangleleft$$

$\Rightarrow \langle \bar{r}_x, \bar{r}_y \rangle = \langle \frac{3}{4}, \frac{7}{8} \rangle$ --- Center of the mass.



Bacterium population.

If $f(x,y) = \frac{10{,}000e^y}{1+|x|/2}$ represents the population density of a certain bacterium on the *xy*-plane where *x* and *y* are measured in centimeters, find the total population of bacteria whithin the reactange: $-5 \le x \le 5, -2 \le y \le 0$.

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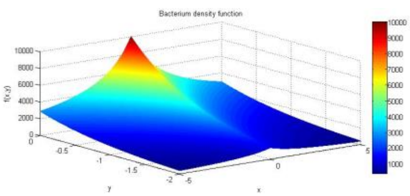

Solution: $Total\ population = \iint density\ dA$

$$= \int_{-5}^{5}\int_{-2}^{0} \frac{10{,}000e^y}{1+|x|/2}\,dydx = \int_{-5}^{5} \frac{10{,}000}{1+\frac{|x|}{2}}(1-e^{-2})dx$$

$$= 10^4(1-e^{-2})\left[\int_{-5}^{0} \frac{1}{1-x/2}dx + \int_{0}^{5} \frac{1}{1+x/2}dx\right]$$

$$= 10^4(1-e^{-2})\left[-2ln\left|1-\frac{x}{2}\right|\Big|_{-5}^{0} + 2ln\left|1+\frac{x}{2}\right|\Big|_{0}^{5}\right]$$

$$= 4 * 10^4(1-e^{-2}) * \ln\frac{7}{2} \cong 43329$$