

# On Moufang 3-nets and groups with triality

J.I. Hall

Department of Mathematics, Michigan State University

G.P. Nagy\*

JATE Bolyai Institute (Szeged) / Universität Erlangen-Nürnberg

## Abstract

In his paper [6], S. DORO constructed a partial relationship between Moufang loops and groups with triality. We extend this relationship by showing that the following concepts are equivalent: Groups with triality and trivial centre, Moufang 3-nets, Latin square designs in which every point is the centre of an automorphism, isotopy classes of Moufang loops. Using this new approach, we also give a simple proof to a theorem of Doro.

AMS classification subject: 20N05.

## 1 Introduction

Let  $L$  be a set with binary operation  $(x, y) \mapsto x \cdot y (= xy)$ . We say that  $(L, \cdot)$  is a *loop*, if a unit element  $1 \in L$  with  $1 \cdot x = x = x \cdot 1$  exists and the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot c = d$$

have unique solutions in  $x$  and  $y$ . We denote the solutions by  $x = a \setminus b$  and  $y = d / c$ . For a loop  $L$ , the *left* and *right translation maps*

$$\lambda_x(y) = xy, \quad \rho_x(y) = yx$$

are also defined.

To any loop  $(L, \cdot)$ , one can associate a point-line incidence structure called *3-net*.  $k$ -nets are incidence structures with point set  $\mathcal{P}$  and line set  $\mathcal{L} = \mathcal{L}_1 \dot{\cup} \dots \dot{\cup} \mathcal{L}_k$  such that the line classes behave as parallel classes do: lines from different classes have

---

\*Supported by the Hungarian Eötvös State Fellowship and OTKA grants F030737, T029849.

precisely one point in common, lines from the same class are disjoint and any point is incident with precisely one element of any line class (see [16]).

Let  $(L, \cdot)$  be a given loop. One defines the associated 3-net with point set  $\mathcal{P} = L \times L$  and line classes as set of subsets of  $\mathcal{P}$ .

$$\begin{aligned}\mathcal{L}_1 &= \{\{(x, c) : x \in L\} : c \in L\}, \\ \mathcal{L}_2 &= \{\{(c, y) : y \in L\} : c \in L\}, \\ \mathcal{L}_3 &= \{\{(x, y) : x, y \in L, xy = c\} : c \in L\}.\end{aligned}$$

The elements of these classes are also called *horizontal*, *vertical* and *transversal lines*, respectively. It is well known that any 3-net can be coordinatized by a loop, and in general, isomorphic loops determine isomorphic 3-nets and isomorphic 3-nets give isotopic coordinate loops.

## 2 Basic concepts

In this paper, we deal with the class of *Moufang loops*, those loops having a weak form of associativity. Namely, they are defined by the identity

$$x \cdot (y \cdot xz) = (xy \cdot x) \cdot z. \quad (1)$$

One can show that, in a Moufang loop, the left and right inverses,  $x \setminus 1$  and  $1/x$ , of the element  $x$  coincide. Moreover, denoting this inverse by  $x^{-1}$ , one has

$$(xy)^{-1} = y^{-1}x^{-1},$$

that is, the inverse map  $J : x \mapsto x^{-1}$  is an anti-automorphism. This also implies that for Moufang loops, the identities can be dualized. (Cf. [16], [1].)

In most cases a loop equation can be expressed in the associated 3-net by a closure configuration and a closure configuration corresponds to a collineation of the 3-net. Let us fix a vertical line  $\ell : x = m$ . We define the *Bol reflection*  $\sigma_m$  with axis  $\ell$  as in Figure 1, as a permutation of the point set where points of  $\ell$  are left fixed. Similarly, we define Bol reflections with horizontal or transversal axis. In general, a Bol reflection is not a collineation. By [1, p. 120], we have the following

**Proposition 2.1 (G. Bol)** *A 3-net is coordinatized by a Moufang loop if and only if any Bol reflection is a collineation.*

It is rather difficult to determine the first appearance of Bol reflections in the literature. They appear as *automorphismes intérieurs* of 3-nets in TITS's paper [17] and are also, as we shall see, implicit in papers of G. GLAUBERMAN [9], S. DORO [6], and possibly other authors. Using the modern terminology of loop theory, Bol reflections were investigated more recently by M. FUNK and P.T. NAGY in [7].

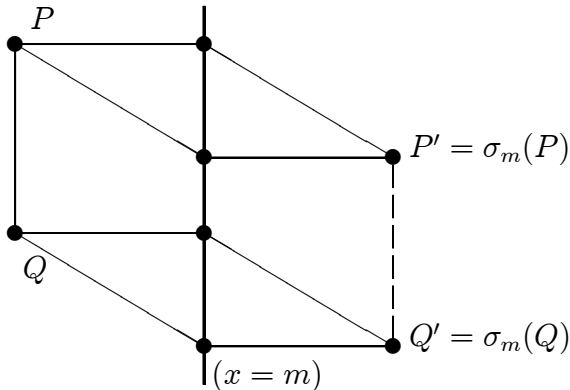


Figure 1: Bol reflection with vertical axis

Clearly, a Bol reflection leaves the points of its axis fixed and interchanges the two other directions. Conversely, a collineation of a 3-net that leaves the points of a line  $\ell$  fixed and interchanges the directions different from that of  $\ell$  *must be* a Bol reflection with axis  $\ell$ . This also implies that the set of Bol reflections is closed under conjugation in the whole collineation group of the 3-net.

For a given point-line incidence structure, it is very natural to consider its *dual*, i.e., to reformulate the axioms by interchanging the role of points and lines. At the end of this introductory section, we consider dual 3-nets.

**Definition 2.2** A Latin square design  $D$  is a pair  $D = (P, A)$  of points  $P$  and lines  $A$  (subsets of  $P$ ) with the properties:

- (i)  $P$  is the disjoint union of three parts  $R, C, E$ .
- (ii) every line  $\ell \in A$  contains exactly three points, meeting each of  $R, C, E$  exactly once.
- (iii) any pair of points from different parts belong to exactly one line.

A Latin square design is a transversal design in which each block has size 3. In general (see [12, Chap. 22]) a transversal design with blocks of size  $k$  is dual to a  $k$ -net and equivalent to a set of  $k - 2$  pairwise orthogonal Latin squares. In particular a Latin square design is equivalent to a Latin square, where we view  $R$  as indexing rows,  $C$  as indexing columns, and  $E$  as indexing entries, so that the entry in row  $x$  and column  $y$  is  $z$  if and only if the line  $\langle x, y, z \rangle$  belongs to the line set  $A$ . (We think of  $\langle x, y, z \rangle$  as an ordered triple with  $x \in R, y \in C, z \in E$ .) Switching the roles of  $R, C$ , and  $E$  gives new Latin squares (referred to as being *conjugate* to the original one). The standard reference on Latin squares is [4].

We are mainly interested in automorphisms of the Latin square design  $D$ . For  $x \in P$ , say  $x \in R$ , consider the partial permutation  $\tau_x$  that exchanges the sets  $C$

and  $E$  via

$$\tau_x(y) = z, \tau_x(z) = y$$

if and only if  $\langle x, y, z \rangle \in A$ . The question is whether  $\tau_x$  can be extended to a permutation on all  $P$  (by defining it on  $R$ ) so that  $\tau_x$  is in the automorphism group  $Aut(D)$ .

If  $\tau_x$  has two extensions  $\alpha$  and  $\beta$  to  $Aut(D)$ , then  $\alpha\beta$  and  $\alpha\beta^{-1}$  are trivial on  $C \cup E$  and so on  $P$ . That is,  $\alpha = \beta^{-1} = \beta$ . Therefore,  $\tau_x$  has at most one such extension, and if it exists then it has order 2. Thus we may, without confusion, call this extension  $\tau_x$  as well. It is a *central automorphism* of  $L$  with *center*  $x$ .

Completely similar remarks are valid for central automorphisms  $\tau_y$  and  $\tau_z$  with centers  $y \in C$  and  $z \in E$ .

**Proposition 2.3** *For an arbitrary Latin square design  $D$ ,*

$$\{\tau_p \mid p \in P, \tau_p \in Aut(D)\}$$

*is a normal set of elements of order 2 in  $Aut(D)$ . If  $p$  and  $q$  belong to different parts of  $P$  and  $\tau_p, \tau_q \in Aut(D)$ , then  $\tau_p\tau_q$  has order 3.*

*Proof.* Clearly, for any central automorphism  $\tau_p$  and  $\alpha \in Aut(D)$ , we have  $\alpha\tau_p\alpha^{-1} = \tau_{\alpha(p)}$ , hence the first statement follows. Let us assume  $p \in R, q \in C$  with  $r \in E$  being the third point on the line through  $p, q$ . Then,  $\tau_p, \tau_q \in Aut(D)$  implies  $\tau_r = \tau_p\tau_q\tau_p = \tau_q\tau_p\tau_q$ , hence

$$(\tau_p\tau_q)^3 = (\tau_p\tau_q\tau_p)(\tau_q\tau_p\tau_q) = \tau_r^2 = 1.$$

□

As a consequence, if  $D$  has central automorphisms with centers from two different parts of  $P$ , then in fact the set of central automorphisms is a single conjugacy class of  $Aut(D)$ .

Of particular interest is the case where  $\tau_p \in Aut(D)$ , for every point  $p$  of  $P$ .

**Proposition 2.4** *If every point  $p$  is the center of a central automorphism of  $D$ , then  $D$  is a dual Moufang 3-net.*

*Proof.* By duality, central automorphisms of Latin square designs correspond precisely to Bol reflections of 3-nets. The statement follows from Proposition 2.1.

### 3 Moufang 3-nets and groups with triality

Let  $G$  be a group. We use the usual notations for group elements  $x, y \in G$ :  $x^y = y^{-1}xy$  and  $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$ . Let  $\alpha$  be an automorphism of  $G$ , then  $\alpha(x)$  will be denoted by  $x^\alpha$  as well, and  $[x, \alpha] = x^{-1}x^\alpha$ . The element  $\alpha^y \in \text{Aut}(G)$  maps  $x$  to  $x^{y^{-1}\alpha y}$ .

We have the following definition due to Doro [6].

**Definition 3.1** *The pair  $(G, S)$  is called a group with triality, if  $G$  is a group,  $S \leq \text{Aut}(G)$ ,  $S = \langle \sigma, \rho; \sigma^2 = \rho^3 = (\sigma\rho)^2 = 1 \rangle \cong S_3$ , and for all  $g \in G$  the triality identity*

$$[g, \sigma] [g, \sigma]^\rho [g, \sigma]^{\rho^2} = 1$$

*holds.*

The principle of triality had been introduced by Cartan [3] in 1938 as a property of orthogonal groups in dimension 8, and these examples motivated Tits [17]. Doro was the first to define the concept of an abstract group with triality, away from any context of a given geometric or algebraic object.

In the following,  $(G, S)$  stands for a group  $G$  with automorphism group  $S$  isomorphic to  $S_3$ . Let  $\sigma$  and  $\rho$  be the given elements of  $S$ , and let the three involutions of  $S$  be  $\sigma_1 = \sigma$ ,  $\sigma_2 = \sigma\rho$  and  $\sigma_3 = \rho\sigma = \sigma\rho^2$ . Finally, we need a notation for the conjugacy class

$$\mathcal{C}_i = \sigma_i^G.$$

The following lemma gives a more conceptual reformulation of Doro's triality. (It is similar to Lemma 3.2 of [11], attributed by Liebeck to Richard Parker.)

**Lemma 3.2** *The pair  $(G, S)$  is a group with triality if and only if for all  $\tau_i \in \mathcal{C}_i$ ,  $(i, j \in \{1, 2, 3\}, i \neq j)$ ,  $(\tau_i\tau_j)^3 = 1$ . In this case,  $(G, \langle \tau_i, \tau_j \rangle)$  is a group with triality, as well.*

*Proof.* The condition of the first statement claims something about the conjugacy classes  $\mathcal{C}_i$ , which do not change if we replace  $S$  by  $\langle \tau_i, \tau_j \rangle$ . This means that the first statement implies the second one.

For the first statement, it suffices to investigate the case  $i = 1, j = 3, \tau_1 = \sigma^g$  and  $\tau_3 = \sigma\rho^2$ , with arbitrary  $g \in G$ . Then,

$$\begin{aligned} 1 = (\sigma^g(\sigma\rho^2))^3 &\iff 1 = \sigma^g(\sigma\rho^2) \cdot \sigma^g(\sigma\rho^2) \cdot \sigma^g(\sigma\rho^2) \\ &\iff 1 = [g, \sigma]\rho^2 \cdot [g, \sigma]\rho^2 \cdot [g, \sigma]\rho^2 \\ &\iff 1 = [g, \sigma] \cdot \rho^{-1}[g, \sigma]\rho \cdot \rho[g, \sigma]\rho^2 \\ &\quad = [g, \sigma] [g, \sigma]^\rho [g, \sigma]^{\rho^2}, \end{aligned}$$

for all  $g \in G$ , as required. □

The next lemma already fore-shadows the relation between Moufang 3-nets and groups with triality.

**Lemma 3.3** *Let  $P$  be a point of the Moufang 3-net  $\mathcal{N}$  and denote by  $\ell_1, \ell_2$  and  $\ell_3$  the three lines through  $P$  with corresponding Bol reflections  $\sigma_1, \sigma_2, \sigma_3$ . Then the collineation group  $S = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong S_3$  acts faithfully on the set  $\{\ell_1, \ell_2, \ell_3\}$ . This action is equivalent to the induced action of  $S$  on the parallel classes of  $\mathcal{N}$ .*

*Proof.* As we already said, the conjugate of a Bol reflection is a Bol reflection again with the corresponding axis. Thus, we have  $\sigma_1\sigma_2\sigma_1 = \sigma_3 = \sigma_2\sigma_1\sigma_2$ , which proves the first statement. The rest is trivial.  $\square$

Using these lemmas, we can prove two key propositions.

**Proposition 3.4** *Let  $\mathcal{N}$  be a Moufang 3-net and  $M$  be the group of collineations generated by all the Bol reflection. Let  $M_0 \leq M$  be direction preserving subgroup of  $M$ . Let us fix an arbitrary point  $P$  of  $\mathcal{N}$  and denote by  $S$  the group generated by the Bol reflections with axes through  $P$ . Then  $M_0 \triangleleft M$ ,  $M = M_0S$ , and the pair  $(M_0, S)$  is a group with triality.*

*Proof.*  $M_0 \triangleleft M = M_0S$  is obvious. Thus  $S$  is a group of automorphism of  $M_0$  by conjugation. By Lemma 3.2, it is sufficient to show  $\langle \sigma_1^g, \sigma_2^h \rangle \cong S_3$  for all  $g, h \in M_0$ , where  $\sigma_1$  and  $\sigma_2$  are the reflections on two different lines through  $P$ . Since  $g, h$  preserve the directions, the axes of  $\sigma_1^g$  and  $\sigma_2^h$  intersect in a point  $P'$ , hence by Lemma 3.3,  $\langle \sigma_1^g, \sigma_2^h \rangle \cong S_3$ .  $\square$

The converse of the proposition is true as well.

**Proposition 3.5** *Let  $(G, S)$  be a group with triality. The following construction determines a Moufang 3-net  $\mathcal{N}(G, S)$ . Let the three line classes be the conjugacy classes  $\mathcal{C}_1, \mathcal{C}_2$  and  $\mathcal{C}_3$ . By definition, three mutually non-parallel lines  $\tau_i \in \mathcal{C}_i$  ( $i = 1, 2, 3$ ) intersect in a common point if and only if*

$$\langle \tau_1, \tau_2, \tau_3 \rangle \cong S_3.$$

*Moreover, if  $G_1 = [G, S]S = \langle \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ , then the group  $M(\mathcal{N})$  generated by the Bol reflections of  $\mathcal{N}$  is*

$$M(\mathcal{N}) \cong G_1/Z(G_1).$$

*Remark.* From the point of view of dual 3-nets, that is, Latin square designs, we have the point set being the union of the three classes  $\mathcal{C}_i$  with lines consisting of the intersection of an  $S_3$  subgroup with each of the three classes.

*Proof.* By the definition, parallel lines do not intersect. When formulating the triality identity as in Lemma 3.2, we see that two non-parallel lines have a point in common such that there is precisely one line from the third parallel class incident with this point. This shows that  $\mathcal{N}(G, S)$  is a 3-net indeed.

The Moufang property follows from the construction immediately, since one can naturally associate an involutorial collineation to any line. This involution interchanges the two other parallel classes and fixes the points on its line, that is, it normalizes the  $S_3$  subgroups containing itself.

Finally, since a Bol reflection acts on the line set in the same way that the associated  $\mathcal{C}_i$ -element acts on the set  $\cup \mathcal{C}_j$  by the conjugation, we have the isomorphism  $M(\mathcal{N}) \cong G_1/Z(G_1)$ .  $\square$

**Theorem 3.6** *The following concepts are equivalent:*

- (i) *Groups  $(G, S)$  with triality and  $Z(GS) = \{1\}$ .*
- (ii) *Latin square designs in which  $\tau_p$  extends to a central automorphism for every point  $p$ .*
- (iii) *Moufang 3-nets.*
- (iv) *Isotopy classes of Moufang loops.*

*Proof.* (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) follows from Propositions 3.4, 3.5 and 2.4, (iii)  $\Leftrightarrow$  (iv) is well known.  $\square$

The relation between Moufang loops and groups with triality was first shown by Doro in [6]. Doro's construction has the disadvantage of being quite complicated, particularly if the nucleus of the loop is not trivial. In our procedure the geometric approach is simple and intuitive, and the nucleus need not be singled out for examination.

We give one example of an interesting result of Doro's [6, Corollary 1.4] for which the geometric reasoning is simpler.

**Proposition 3.7** *Let  $(G, S)$  be a finite group with triality and suppose that  $\rho \in \text{InnAut}(G)$ . Then all elements  $[g, \sigma]$  have order 3 and  $[G, S]$  is a 3-group. Moreover, the coordinate loop of the 3-net  $\mathcal{N}(G, S)$  is centrally nilpotent.*

*Proof.* Put  $H = GS$ . If  $\rho \in \text{InnAut}(G)$ , then there exists elements  $r \in G$  and  $c \in C_H(G)$  such that  $\rho = rc$ . This implies

$$\rho = [\rho, \sigma] = [r, \sigma]c',$$

with  $c' \in C_H(G)$ , since  $C_H(G) \triangleleft H$ . This means that

$$\rho[\sigma, r] = \rho^{-1}\sigma\rho \cdot r^{-1}\sigma r \in C_H(G).$$

Put  $\tau_1 = \sigma^r$  and  $\tau_2 = \sigma_2 = \sigma^\rho$ . By Lemma 3.2,  $\bar{S} = \langle \tau_1, \tau_2 \rangle \cong S_3$  and  $(G, \bar{S})$  is a group with triality. The conjugacy classes  $\mathcal{C}_i$  do not change. However, an important

change is the fact that  $\bar{\rho} = \tau_1\tau_2$  centralizes  $G$  and the triality identity becomes  $[g, \tau_1]^3 = 1$ . This is equivalent with the identity  $[g, \sigma]^3 = 1$ .

Therefore, by Lemma 3.2, if  $\tau_1, \tau_2 \in \sigma^H$ , then  $(\tau_1\tau_2)^3 = 1$ . By Glauberman's  $Z^*$ -theorem [8, p. 71] (indeed, in this special and elementary case by [10]) the group  $\langle \sigma^H \rangle$  has a normal 3-subgroup of index 2. Its subgroup  $[G, S]$  is then a 3-group as well. Therefore, the order of the 3-net  $\mathcal{N}(G, S)$  and so the order of the coordinate loop are powers of 3. In particular the loop is a finite Moufang 3-loop, and so by [9, Theorem 4] is centrally nilpotent.  $\square$

## 4 On the classification of finite simple Moufang loops

The classification of finite simple Moufang loops is based on the classification of finite simple groups with triality. Using the results of the previous section, the classification can be done in the following steps.

**Proposition 4.1** *Let  $\varphi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$  be map between two 3-nets, which preserves incidence and directions.*

- (i) *Let us suppose that  $\varphi(P_1) = P_2$  hold for the points  $P_1, P_2$ . Then,  $\varphi$  defines a homomorphism  $\bar{\varphi} : L_1 \rightarrow L_2$  in a natural way, where  $L_i$  is the coordinate loop of the 3-net  $\mathcal{N}_i$  with origin  $P_i$ . Conversely, the loop homomorphism  $\bar{\varphi} : L_1 \rightarrow L_2$  uniquely defines a  $\varphi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$  collineation.*
- (ii) *Let us now suppose that the 3-nets  $\mathcal{N}_i$  ( $i = 1, 2$ ) are of Moufang type and  $\varphi$  is a collineation onto. Let us denote by  $(M_i, S)$  the group with triality, which corresponds to the 3-net  $\mathcal{N}_i$ . Then, the maps  $\sigma_\ell \mapsto \sigma_{\varphi(\ell)}$  induce a  $\tilde{\varphi} : M_1 \rightarrow M_2$  surjective  $S$ -homomorphism, where  $\sigma_\ell$  is the Bol reflection belonging to the line  $\ell$  of  $\mathcal{N}_1$ . Conversely, an  $S$ -homomorphism  $M_1 \rightarrow M_2$  defines a direction preserving collineation between the 3-nets  $\mathcal{N}(M_1, S)$  and  $\mathcal{N}(M_2, S)$ .*

*Proof.* The first part of statement (i) follows from the geometric definition of the loop operation in a coordinate loop; the second part is trivial. For the (ii) statement, it is sufficient to see that a relation of the  $\sigma_\ell$ 's corresponds to a closure configuration of the 3-net, and the  $\varphi$ -image of the configuration induces the relation on the  $\sigma_{\varphi(\ell)}$ 's. The converse follows from Proposition 3.5.  $\square$

In the sense of the proposition above, we can speak of *simple 3-nets*, that is, of 3-nets having only trivial homomorphisms. The next proposition follows immediately.

**Proposition 4.2** *If  $L$  is a simple Moufang loop, then the associated 3-net  $\mathcal{N}$  is simple as well. That is, the group  $(M_0, S)$  with triality determined by  $\mathcal{N}$  is a  $S$ -simple.  $\square$*



The next step is the following proposition of Doro.

**Proposition 4.3 (Doro)** *Let  $L$  be a non-commutative finite simple Moufang loop. If  $L$  is associative, then  $M_0 \cong L \times L \times L$  and  $M \cong L \wr S_3$ . If  $L$  is non-associative, then  $M_0$  is a finite simple group.*

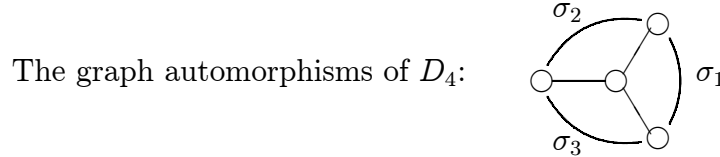
*Proof.* Cf. [6, Proposition 1]. □

The classification of the finite simple Moufang loops was achieved by Liebeck [11]. The main step of Liebeck's classification is the following theorem ([11, Proposition and Theorem 4.1]), the proof of which uses the classification of finite simple groups. In the proof, Proposition 3.7 plays an important role, since it shows that the elements of  $S$  are outer automorphisms.

**Theorem 4.4 (Liebeck [11])** *a) The only finite simple groups with triality are the simple groups  $D_4(q)$ .*

*b) Let  $S = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong S_3$  be a group of outer automorphisms of  $G = D_4(q)$  such that the pair  $(G, S)$  is a group with triality. Then  $S$  consists of graph automorphisms of  $D_4(q)$ .* □

From this result one obtains the classification of finite simple Moufang loops immediately if one uses Proposition 4.2 and 4.3.



Knowing the group  $M_0 = D_4(q) = P\Omega_8^+(q)$  and that  $S$  is the group of graph automorphisms (the finite counterpart to Cartan's triality [3]), we implicitly have the Moufang 3-net and its finite simple coordinate loop. Following L.J. PAIGE [15], this loop can be constructed explicitly, as well.

## References

- [1] R.H. BRUCK, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Heidelberg-New York, 1958.
- [2] R.H. BRUCK, What is a loop?, in *Studies in Modern Algebra*, A.A. Albert (ed.), MAA Studies in Mathematics, 1963, pp. 59–99.
- [3] E. CARTAN, *Leçons sur la Théorie des Spineurs*, Hermann et Cie., Paris, 1938.

- [4] J. DÉNES and A.D. KEEDWELL, *Latin Squares and their Applications*, Academic Press, New York and London, 1974.
- [5] M. DETTWEILER, Einfache Moufang Loops, Master Thesis, University of Erlangen, 1995.
- [6] S. DORO, Simple Moufang loops, *Math. Proc. Camb. Philos. Soc.* **83** (1978), 377-392.
- [7] M. FUNK and P.T. NAGY, On collineation groups generated by Bol reflections, *J. Geom.* **41** (1993), 63-78.
- [8] G. GLAUBERMAN, Central elements in core-free groups, *J. Alg.* **4** (1966), 404-420.
- [9] G. GLAUBERMAN, On loops of odd order II, *J. Alg.* **8** (1968), 393-414.
- [10] J.I. HALL, On the order of Hall triple systems, *J. Combin. Th. (Ser. A)* **29** (1980), 261-262.
- [11] M.W. LIEBECK, The classification of finite simple Moufang loops, *Math. Proc. Camb. Philos. Soc.* **102** (1987), 33-47.
- [12] J.H. VAN LINT and R.M. WILSON, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.
- [13] G.P. NAGY, Collineation groups of the smallest Bol 3-nets, *Note di Matematica*, **14**(1) (1994), 115-128.
- [14] G.P. NAGY, Group invariants of certain Burn loop classes, *Bull. Belg. Math. Soc.*, **5** (1998), 403-415.
- [15] L.J. PAIGE, A class of simple Moufang loops, *Proc. Amer. Math. Soc.* **7** (1956), 471-482.
- [16] H.O. PFLUGFELDER, *Quasigroups and Loops: Introduction*, Heldermann Verlag, Berlin, 1990.
- [17] J. TITS, Sur la trialité et les algèbres d'octaves, *Acad. Roy. Belg. Bull. Cl. Sci.* **44**(5) (1958), 332-350.

Addresses of the authors:

Jonathan I. Hall  
Department of Mathematics  
Michigan State University  
East Lansing, Michigan 48824, U.S.A.  
email: [jhall@math.msu.edu](mailto:jhall@math.msu.edu)

Gábor P. Nagy  
JATE Bolyai Institute  
Aradi vértanúk tere 1  
H-6720 Szeged (Hungary)  
email: [nagy@math.u-szeged.hu](mailto:nagy@math.u-szeged.hu)

and Mathematisches Institut der  
Universität Erlangen-Nürnberg  
Bismarkstr. 1 $\frac{1}{2}$   
D-91054 Erlangen (Germany)