J. reine angew. Math. **669** (2012), 151—164
DOI 10.1515/CRELLE.2011.148

Journal für die reine und angewandte Mathematik

Cosmetic crossing changes of fibered knots

By *Efstratia Kalfagianni* at East Lansing

Abstract. We prove the nugatory crossing conjecture for fibered knots. We also show that if a knot K is n-adjacent to a fibered knot K', for some $n > 1$, then either the genus of K is larger than that of K' or K is isotopic to K'.

1. Introduction

An open question in classical knot theory is the question of when a crossing change on a knot changes the isotopy class of the knot. The purpose of this paper is to answer this question for fibered knots.

A crossing disc for a knot $K \subset S^3$ is an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ twice with zero algebraic intersection number. A crossing change on K can be achieved by twisting D or equivalently by performing appropriate Dehn surgery of S^3 along the crossing circle ∂D. The crossing is called nugatory if and only if ∂D bounds an embedded disc in the complement of K. This disc and D form a 2-sphere that decomposes K into a connected sum, where some of the summands may be trivial. Clearly, changing a nugatory crossing does not change the isotopy class of a knot. Problem 1.58 of [10] asks whether the converse is true (see also [15] for related conjectures), that is, if a crossing change on a knot K yields a knot isotopic to K is the crossing nugatory.

In the case that K is the trivial knot, an affirmative answer follows from work of Gabai [7]. An affirmative answer is also known in the case of 2-bridge knots [15]. In this paper we will show the following.

Theorem 1.1. *Let K be a fibered knot. A crossing change on K yields a knot isotopic to K if and only if the crossing is nugatory.*

To give a brief outline of the proof, let K be a fibered knot such that a crossing change on K gives a knot K' that is isotopic to K. The complement of K is fibered over S^1 with fiber, say S, a minimal genus Seifert surface of K. A result of Gabai implies that the crossing change from K to K' can be achieved along an arc that is properly embedded

The research was partially supported by the NSF.

on S. Equivalently, the crossing change can be achieved by twisting K along a meridian disc D of a handlebody neighborhood N of the fiber. Combining geometric properties of fibered knot complements with sutured manifold techniques, the problem reduces to the question of whether a power of a Dehn twist on the surface ∂N along the curve ∂D can be written as a single commutator in the mapping class group of the surface. A result of Kotschick implies that a product of Dehn twists of the same sign along a collection of disjoint homotopically essential curves on an orientable surface cannot be written as a single commutator in the mapping class group of the surface. Using this result, we show that the assumption that K is isotopic to K' implies that ∂D bounds a disc in the complement of K.

Theorem 1.1 says that an essential crossing change always changes the isotopy class of a fibered knot. It is natural to ask whether the crossing change produces a simpler or more complicated knot with respect to some knot complexity. A complexity function whose interplay with crossing changes has been studied using the theory of taut foliations and sutured 3-manifolds is the knot genus. Simple examples show that a single crossing change may decrease or increase the genus of a knot even if one stays within the class of fibered knots. However, there are interesting consequences if one replaces a crossing change by the more refined notion of knot adjacency [8], [9]: We recall that K is called 2-adjacent to K' if K admits a projection that contains two crossings such that changing any of them or both of them simultaneously transforms K to K'.

Theorem 1.2. *Suppose that K' is a fibered knot and that K is 2-adjacent to K'. Then either K is isotopic to K' or K has a strictly larger genus than K'.*

We organize the paper as follows: In Section 2 we summarize the mapping class group results that we need for the proof of Theorem 1.1 and in Section 3 we summarize known properties of fibered knot complements. In Section 4 we discuss a setting relating fibrations of knot complements and Heegaard splittings of S^3 from the point of view needed in the rest of the paper. In Section 5 we study nugatory crossings of fibered knots and we prove Theorem 1.1. In Section 6 we study adjacency to fibered knots and prove Theorem 1.2.

Throughout the paper we work in the PL or the smooth category.

2. Commutator length and Dehn twists

2.1. Commutators in the mapping class group. Let Σ_k denote a closed oriented surface of genus k and let Γ_k denote the mapping class group of Σ_k, that is, Γ_k is the group of isotopy classes of orientation preserving homeomorphisms $\Sigma_k \to \Sigma_k$. Let $\Gamma_k' := [\Gamma_k, \Gamma_k]$ denote the commutator subgroup of Γ_k. An element $f \in \Gamma_k'$ is written as a product of commutators. The commutator length of f, denoted by $c(f)$, is the minimum number of factors needed to express f as a product of commutators. In the recent years, the growth of the commutator length of Dehn twists has been studied using methods from the theory of symplectic four-manifolds [5], [3], [12], [11]. In this paper we will need a result of D. Kotschick which we recall below.

For a simple closed curve $a \subset \Sigma_k$ let T_a denote the right-hand Dehn twist about a; then the left-hand Dehn twist about a is T_a^{-1}.

Theorem 2.1 ([12], Theorem 7). *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geqq 2$. Suppose that $a_1, \dots, a_m \subset \Sigma_k$ are homotopically essential, disjoint, simple closed curves on Σ_k. Let $f := T_{a_1} \cdot T_{a_2} \cdot \dots \cdot T_{a_m}$ denote the product of right-handed Dehn twists along $a_1, \dots, a_m$. Suppose that for some $q > 0$ we have*

$$f^q = T_{a_1}^q \cdot T_{a_2}^q \cdot \dots \cdot T_{a_m}^q \in \Gamma'_k.$$

Then we have

$$c(f^q) \geqq 1 + \frac{qm}{18k - 6}.$$

We will need the following corollary of Theorem 2.1.

Corollary 2.2. *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geqq 2$. Let $a \subset \Sigma_k$ be a simple closed curve. Suppose that there exist $g, h \in \Gamma_k$ such that*

$$T_a^q = [g, h] = ghg^{-1}h^{-1},$$

for some $q \neq 0$. Then a is homotopically trivial on Σ_k.

The proof of Theorem 2.1 given in [12] relies on the theory of Lefschetz fibrations, which, as the author points out, is sensitive to the chirality of Dehn twists. In fact, the argument of [12] breaks down if one allows f to be a product of right-handed Dehn twists and their inverses and, as the following example shows, Theorem 2.1 is not true in this case. In subsequent sections we will discuss how this situation is reflected when one tries to apply Theorem 2.1 to the study of crossing changes that do not alter the isotopy class of fibered knots (see Example 5.9).

Example 2.3 ([12], Example 9). Suppose that $a \subset \Sigma_k$ is an essential simple closed loop on a closed oriented surface of genus at least two. Let $g : \Sigma_k \to \Sigma_k$ be an orientation preserving homeomorphism such that $a \cap g(a) = \emptyset$. We will also use g to denote the mapping class of g. Set $b := g(a)$ and $f := T_a T_b^{-1}$. In the mapping class group Γ_k we have $gT_a g^{-1} = T_{g(a)}$ or equivalently $T_a = g^{-1} T_b g$. Since a, b are disjoint, we also have $T_a T_b^{-1} = T_b^{-1} T_a$. Thus

$$f^q = (T_a T_b^{-1})^q = T_a^q T_b^{-q} = (g^{-1} T_b g)^q T_b^{-q} = [g^{-1}, T_b^q],$$

for all $q > 0$. Hence we have $c(f^q) = 1$ showing that Theorem 2.1 is not true in this case.

2.2. When is T_a^q trivial? It is known that if a is homotopically essential on Σ, then no non-trivial power T_a^q $(0 \neq q \in \mathbb{Z})$ is isotopic to the identity on Σ_k. This statement is well known to researchers working on mapping class groups. It is for example asserted in [2] when the authors state that the kernel of the reduction homomorphism corresponding to an essential simple closed curve a is the free abelian group generated by Dehn twists along a. Below we include a proof that uses properties of intersection numbers stemming from Thurston's study of surface homeomorphisms [6].

Proposition 2.4. *Suppose that $T_a^q = 1$ in the mapping class group $\Gamma := \Gamma(\Sigma_k)$, $k > 0$. Then either a is homotopically trivial on Σ_k or $q = 0$.*

Proof. Suppose that the curve a is not homotopically trivial on Σ and that $T_a^q = 1$ in the mapping class group $\Gamma(\Sigma)$. We will argue that $q = 0$. First suppose that a is a non-separating loop on Σ. Then, we can find an embedded loop b that intersects a exactly once. Orient a, b so that the algebraic intersection of a, b is 1, that is, $\langle a, b \rangle = 1$. In $H_1(\Sigma)$ we have

$$T_a^q(b) = b + q\langle a, b\rangle a = b + qa.$$

Thus we have $\langle T_a^q(b), b\rangle = \langle b, b\rangle + q\langle a, b\rangle$, which, since $T_a^q(b) = b$, gives $q = 0$ as desired. If a is separating, we appeal to the geometric intersection number. For b, a simple closed loop on Σ, let $i(a, b)$ denote the intersection number, i.e., the minimal number of intersections in the isotopy classes of a and b. Since we assumed that a is homotopically essential on Σ, we can find b so that $i(a, b) \neq 0$. By [6], Exposé 4, we have the following:

$$i\big(T_a^q(b), b\big) = |q|\big(i(a, b)\big)^2.$$

Since $T_a^q = 1$, we have $0 = i(b, b) = i\big(T_a^q(b), b\big)$. Thus we obtain $|q|\big(i(a, b)\big)^2 = 0$, which implies that $q = 0$. □

Notation. To simplify our notation, throughout the paper we will use $\Sigma := \Sigma_k$ to denote an oriented surface of any genus $k \geqq 0$ and $\Gamma := \Gamma_k$ to denote the mapping class group of Σ. Also, as we have done in this section, we will use the same symbol to denote a homeomorphism of Σ and its class in Γ.

3. Uniqueness properties of knot fibrations

Here we summarize some known properties of fibered knots that we need in subsequent sections. For details and proofs the reader is referred to [4], Section 5, and [16]. Suppose that K is a fibered knot and let S be a minimum genus Seifert surface for K. Let $\eta(K)$ denote a tubular neighborhood of K. Then the complement $\overline{S^3 \backslash \eta(K)}$ admits a fibration over S^1 with fiber S. More specifically, it is shown that the complement $\overline{S^3 \backslash \eta(K)}$ cut along S is homeomorphic to $S \times [-1, 1]$. Thus, there is an orientation preserving homeomorphism $h : S \to S$ such that $\overline{S^3 \backslash \eta(K)}$ is obtained from $S \times [-1, 1]$ by identifying $S \times \{-1\}$ with $S \times \{1\}$ so that $(x, -1) = \big(h(x), 1\big)$. The map h is called the monodromy of the fibration. We write

$$\overline{S^3 \backslash \eta(K)} = S \times J/h,$$

where $J := [-1, 1]$. We need the following:

Proposition 3.1. (a) *Let $M := \overline{S^3 \backslash \eta(K)} = S \times J/h$ be an oriented, fibered knot complement and set $S_1 := S \times \{1\} = S \times \{-1\}$. Given a minimum genus Seifert surface S_2, with $\partial S_2 = \partial S_1$, there exists an orientation preserving homeomorphism of M that is fixed on ∂M and brings S_2 to the fiber S_1. In fact, such a homeomorphism is isotopic to the identity on M by an isotopy relatively to the boundary ∂M.*

(b) *Let $M := S \times J/h$ and $M' := S' \times J/h'$ be fibered, oriented knot complements. Then there exists an orientation preserving homeomorphism $F : M \to M'$ with*

$$F(\partial S \times \{j\}) = \partial S' \times \{j\} \quad (j \in J)$$

if and only if there exists an orientation preserving surface homeomorphism

$$f : (S, \partial S) \to (S', \partial S')$$

such that fhf^{-1} *and* h' *are equal up to isotopy on* S'.

4. Splittings of fibered knot complements

Given a fibration of a knot complement $M := \overline{S^3 \backslash \eta(K)} = S \times J/h$, set

$$N_1 := S \times [0,1], \quad N_2 := S \times [-1,0], \quad E := \partial S \times (0,1) \quad \text{and} \quad E' := \partial S \times (-1,0).$$

We have $\partial N_1 = (S \times \{0\}) \cup E \cup (S \times \{1\})$. Similarly, $\partial N_2 = (S \times \{-1\}) \cup E' \cup (S \times \{0\})$. We will assume that $K := \partial S \times \left\{\frac{1}{2}\right\}$ on ∂N_1. Define $g : \partial N_1 \to \partial N_1$ by

$$(1) \qquad g(x,0) = (x,0), \qquad \text{for } x \in S,$$

$$(2) \qquad g(x,t) = (x,t), \qquad \text{for } x \in \partial S \text{ and } 0 < t < 1,$$

$$(3) \qquad g(x,1) = \bigl(h(x),1\bigr), \quad \text{for } x \in S.$$

Consider the homeomorphism $rg : \partial N_1 \to \partial N_2$, where $r : N_1 \to N_2$ is defined by $(x,t) \to (x,-t)$. We obtain a Heegaard splitting

$$(4) \qquad S^3 = N_1 \cup_{rg} N_2 := N_1 \sqcup N_2/\{y \sim rg(y) \mid y \in \partial N_1\}$$

such that K lies on the Heegaard surface. Next, we push K on $S \times \left\{\frac{1}{2}\right\}$ slightly in the interior of N_1 and then we take $A(K)$ to be an annulus neighborhood of K on $S \times \left\{\frac{1}{2}\right\}$. Then we remove a tubular neighborhood of K, say $\eta(K) := A(K) \times \left(\left\{\frac{1}{2}\right\} - \varepsilon, \left\{\frac{1}{2}\right\} + \varepsilon\right)$, from $\operatorname{int}(N_1)$ and we set $H_1 := \overline{N_1 \backslash \eta(K)}$. The decomposition

$$(5) \qquad M = H_1 \cup_{rg} N_2 := H_1 \sqcup N_2/\{y \sim rg(y) \mid y \in \partial N_1\}$$

is called the HN-splitting corresponding to the fibration of M. The HN-surface of this decomposition is $Q := \partial N_1 \sqcup \partial N_2/\{y \sim rg_1(y) \mid y \in \partial N_1\}$. Now set $N := N_1 = S \times [0,1]$ and identify N_2 with $(-N)$ via r^{-1}, where $(-N)$ denotes N with the opposite orientation. Also set $H := \overline{N \backslash \eta(K)}$ and $\Sigma := \partial N_1$ and let $i : N \to (-N)$ denote the orientation reversing involution.

Definition 4.1. The pair (Σ, g) is called the HN-model associated to the fibration $M = S \times J/h$. Note that, by (1)–(4), g is the identity on $\Sigma \backslash (S \times \{1\})$.

Definition 4.2. Let K be a fibered knot with $M := \overline{S^3 \backslash \eta(K)} = S \times J/h$ and let H, N and Σ be as above. Also let $g_1 : \Sigma \to \Sigma$ be an orientation preserving homeomorphism. The pair (Σ, g) is called an HN-model for M if there is an orientation-preserving homeomorphism $\Phi : M \to H \cup_{g_1} (-N)$ such that $\Phi \mid \partial\eta(K) = \mathrm{id}$. Here,

$$H \cup_{g_1} (-N) := H \sqcup N/\{y \sim ig_1(y) \mid y \in \Sigma\}.$$

The surface $\Sigma \sqcup \Sigma/\{y \sim ig_1(y) \mid y \in \Sigma\}$ will be called the HN-surface of the decomposition $H \cup_{g_1} (-N)$.

The next lemma reformulates part (b) of Proposition 3.1 in terms of the models of the two fibrations.

Lemma 4.3. *Let $M := S \times J/h$ and $M' := S' \times J/h'$ be fibered, oriented knot complements in S^3 and let (Σ, g), (Σ, g') denote the models corresponding to the fibration of M, M', respectively. There exists an orientation-preserving homeomorphism $F : M \to M'$ with $F(\partial S \times \{j\}) = \partial S' \times \{j\}$ $(j \in J)$ if and only if there is an orientation-preserving homeomorphism $f : \Sigma \to \Sigma$ such that in the mapping class group $\Gamma = \Gamma(\Sigma)$ we have*

$$g' = fgf^{-1}.$$

Proof. By Proposition 3.1, there exists an orientation-preserving homeomorphism $F : M \to M'$ with $F(\partial S \times \{j\}) = \partial S' \times \{j\}$ $(j \in J)$ if and only if there exists an orientation-preserving surface homeomorphism $f : (S, \partial S) \to (S', \partial S')$ such that fhf^{-1} and h' are equal up to isotopy on S'. Now g is constructed out of h as in (1)–(4); in a similar fashion g' is constructed out of h'. Set $I := [0, 1]$. We may extend f to a homeomorphism of pairs $\big(S \times I, \partial(S \times I)\big) \to \big(S' \times I, \partial(S' \times I)\big)$ by defining $f(x, t) = \big(f(x), t\big)$. By our construction of the HN-splittings corresponding to fibrations, this extension is considered as a map $(N, \Sigma) \to (N, \Sigma)$. Since g is the identity on $\Sigma \backslash (S \times \{1\})$ and g' is the identity on $\Sigma \backslash (S' \times \{1\})$, we have $g' = fgf^{-1}$ up to isotopy on Σ. □

Let Q denote the HN-surface of the splitting associated to a fibration $\overline{S^3 \backslash \eta(K)} = S \times J/h$. By construction, we have a surface $S_1 \subset S \times \left\{\frac{1}{2}\right\}$ that is disjoint from Q. Furthermore, S_1 and $S \times \left\{\frac{1}{2}\right\}$ differ by an annulus. We will think of this HN-surface as sitting in the original fibration $\overline{S^3 \backslash \eta(K)} = S \times J/h$ and S_1 is a fiber surface of the fibration.

Lemma 4.4. *Let $M' := \overline{S^3 \backslash \eta(K')} = S' \times J/h'$ be an oriented fibered knot complement. Let (Σ, g') denote the HN-model associated to the fibration with Q the corresponding HN-surface of M' sitting in the fibration so that $S_1' := S' \times \left\{\frac{1}{2}\right\}$ is disjoint from it. Let (Σ, g'') be a second HN-model of M' and let Q' denote the corresponding HN-surface. Suppose that there exists an orientation-preserving homeomorphism $F : M' \to M'$ with $F \mid \partial M' = \mathrm{id}$, and such that*

$$F(Q) = Q' \quad \text{and} \quad F(S' \times x) = S' \times x, \quad \text{for all } x \in J. \tag{6}$$

Then there is an orientation-preserving homeomorphism $f : \Sigma \to \Sigma$ such that in the mapping class group $\Gamma = \Gamma(\Sigma)$ we have

$$g'' = fg'f^{-1}. \tag{7}$$

Proof. The existence of the homeomorphism in (6) implies that Q' is the HN-surface corresponding to a fibration of M' with fiber S_1'. We will now discuss a model of this fibration: If we let f_1 denote the restriction of F on the fiber S_1', then the monodromy of our second fibration should be a conjugate of h' by f_1 (Proposition 3.1). That is the monodromy of the fibration in which Q' is the corresponding HN-surface is $h_1 := f_1 h' f_1^{-1}$ (where, recall, the equality is understood up to isotopy on the fiber). Following the process described in (1)–(4), we can identify M' with $H \cup_{g_1} (-N)$ where (Σ, g_1) is the model corresponding to the fibration with monodromy h_1. By Lemma 4.3, there is an orientation preserving homeomorphism $f : \Sigma \to \Sigma$ such that in the mapping class group $\Gamma = \Gamma(\Sigma)$ we have

$$g_1 = fg'f^{-1}.$$

Now $M' = H \cup_{g_1} (-N) = H \cup_{g''} (-N)$, with $Q' = \Sigma \cup_{g_1} \Sigma = \Sigma \cup_{g''} \Sigma$ being the HN-surface in both splittings. This defines a homeomorphism $m : H \cup_{g_1} (-N) \to H \cup_{g''} (-N)$ with $m \,|\, \partial M' = \mathrm{id}$, $m \,|\, S_1' = \mathrm{id}$ and $m(Q') = Q'$. Let m_1 denote the restriction of m on the surface $\Sigma \subset \partial H$ and let m_2 denote the restriction of m on $\Sigma = \partial N$. Clearly, we have $g'' = m_1^{-1} g_1 m_2$. Let $R := M' \backslash S_1' \cong S_1' \times J$. Now m gives rise to a homeomorphism $m : R \to R$ such that: (i) $m(S_1' \times x) = S_1' \times x$, for all $x \in J$; (ii) $m \,|\, \partial R = \mathrm{id}$; and (iii) $m(Q') = Q'$. Now m can be isotopied to the identity on R by an isotopy that is level preserving ([16], Lemma 3.5). Such an isotopy will preserve Q'. It follows that m_1, m_2 are isotopic to the identity on Σ. Since, as discussed earlier, $g'' = m_1^{-1} g_1 m_2$, we infer that $g'' = g_1 = fg'f^{-1}$ up to isotopy in Σ. □

5. Crossing changes and Dehn twists

In this section will prove Theorem 1.1. In fact, we will work in a more general context as we will consider generalized crossing changes.

5.1. Nugatory crossing changes in fibered knots. Let K be a knot in S^3 and let $q \in \mathbb{Z}$. A generalized crossing of order q on K is a set C of $|q|$ twist crossings on two strings that inherit opposite orientations from any orientation of K. If K' is obtained from K by changing all the crossings in C simultaneously, we will say that K' is obtained from K by a generalized crossing change of order q (see Figure 1). Notice that if $|q| = 1$, K and K' differ by

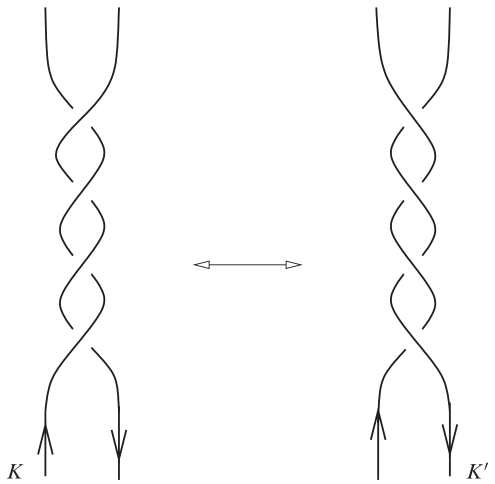

Figure 1. The knots K and K' differ by a generalized crossing change of order $q = -4$.

an ordinary crossing change, while if $q = 0$, we have $K = K'$. A crossing disc for K is an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ twice with zero algebraic intersection number. Performing $\frac{1}{-q}$-surgery on $L := \partial D$, for $q \in \mathbb{Z}$, changes K to another knot $K' \subset S^3$. Clearly, K' is obtained from K by a generalized crossing change of order q. The boundary $L := \partial D$ is called a crossing circle supporting the generalized crossing change.

Definition 5.1. A generalized crossing supported on a crossing circle L of a knot K is called nugatory if and only if $L := \partial D$ bounds an embedded disc in the complement of K. This disc and D form an embedded 2-sphere that decomposes K into a connected sum where some of the summands may be trivial.

Clearly, changing a nugatory crossing does not change the isotopy class of a knot. It is an open question whether, in general, the converse is true ([10], Problem 1.58). The answer is known to be *yes* in the case when K is the unknot [13] and when K is a 2-bridge knot [15]. To these we add the following theorem.

Theorem 5.2. *Let K be a fibered knot and let K' be a knot obtained from K by a generalized crossing change. If K' is isotopic to K, then a crossing circle L supporting this crossing change bounds an embedded disc in the complement of K.*

5.2. Preliminaries. Let C be a generalized crossing of order $q \neq 0$ of a fibered knot K. Let K' denote the knot obtained from K by changing C and let D be a crossing disc for C with $L := \partial D$.

Lemma 5.3. *Suppose that $M_L := \overline{S^3 \setminus (\eta(K) \cup \eta(L))}$ is reducible. Then L bounds a disc in the complement of K. Thus, in particular, the crossing change from K to K' is nugatory.*

Proof. Let Δ be an essential 2-sphere in M_L; $\eta(K)$ and $\eta(L)$ must lie in different components of $M_L \setminus \Delta$. Isotope Δ so that its intersection with D is minimal in M_L. Then $\Delta \cap D$ is a collection of simple closed curves, each parallel to ∂D on D. Thus $K \cup L \subset S^3$ is a split link. Since L is unknotted, it bounds a disc in the complement of K. □

In view of Lemma 5.3, we may assume that M_L is irreducible. Since the linking number of L and K is zero, K is homologically trivial in the complement of L. It is known that this implies that K bounds a Seifert surface in the complement of L. Let S be a Seifert surface that is of minimum genus among all such Seifert surfaces. Since S is incompressible, after an isotopy we can arrange so that the closed components of $S \cap D$ are homotopically essential in $D \setminus K$. But then each such component is parallel to ∂D on D and by further modification we can arrange so that $S \cap D$ is an arc that is properly embedded on S. The surface S gives rise to Seifert surfaces S and S' of K and K', respectively.

Proposition 5.4. *Suppose that K is isotopic to K'. Then S and S' are Seifert surfaces of minimal genus for K and K', respectively.*

Proof. We can consider the surface S properly embedded in M_L so that it is disjoint from the component $\partial\eta(L)$ of ∂M. The assumptions on irreducibility of M_L and on the genus of S imply that the foliation machinery of Gabai [7] applies. In particular, S is taut in the Thurston norm. The manifolds $M := \overline{S^3 \backslash \eta(K)}$ and $M' := \overline{S^3 \backslash \eta(K')}$ are obtained by Dehn filling of M_L along $\partial(\eta(L))$. By [7], Corollary 2.4, S can fail to remain taut in the Thurston norm (i.e. genus minimizing) in at most one of M and M'. But since K is isotopic to K', M is homeomorphic to M'. Thus S remains taut in both of M and M'. This implies that S and S' are Seifert surfaces of minimal genus for K and K', respectively. □

Next we restrict ourselves to fibered knots and recall the assumptions that we have to work with from the statement of Theorem 5.2: K and K' are fibered knots that are isotopic; S and S' are minimum genus Seifert surfaces, for K and K', respectively.

5.3. An *HN*-model for M' from Dehn surgery. With the notation of Section 4, there is a fibration $M := \overline{S^3 \backslash \eta(K)} = S \times J/h$ with monodromy $h : S \to S$. With $N := S \times [0, 1]$ and $\Sigma := \partial N$ we have an HN-model (Σ, g) corresponding to the fibration of M. We can think of the Heegaard splitting of S^3 corresponding to the fibration $M = S \times J/h$ as the quotient

$$N \cup_g (-N) := N \sqcup (-N)/\{y \sim ig(y) \mid y \in \Sigma\}. \tag{8}$$

We will further assume that the crossing circle L is embedded on Σ so that D is a meridian disc of N. We will furthermore assume that the embedding of L on Σ is chosen so that, up to isotopy in M, the geometric intersection $|K \cap D|$ is minimal. Note that since we assumed that $M_L := \overline{S^3 \backslash (\eta(K) \cup \eta(L))}$ is irreducible, this minimum intersection must be non-zero. Let $\tau : N \to N$ denote the right-handed Dehn twist of N along the meridional disc D and let $T_L := \tau \,|\, \Sigma$, where $L = \partial D$. We have $\tau^{-q}(S) = S'$ and $\tau^{-q}(K) = K'$. Recall that $M := S \times J/h$ and that $M' := \overline{S^3 \backslash \eta(K')}$ is obtained from M_L by Dehn filling along $\partial \eta(L)$ with slope $\dfrac{1}{-q}$. Next we use that information to construct an HN-model for M'. The proof of Lemma 5.5 follows a known process of passing between gluing maps of Heegaard splittings and Dehn surgeries of 3-manifolds (compare [1], pp. 86–87).

Lemma 5.5. *With the above notation, (Σ, gT_L^{-q}) is an HN-model for $M' = \overline{S^3 \backslash \eta(K')}$.*

Proof. By assumption, (Σ, g) is an HN-model corresponding to the fibration $M = S \times J/h$. Let A denote an annulus on Σ that supports T_L and let $B := g(A)$. We will think of this HN-splitting of M as the quotient

$$H \cup_g (-N) := H \sqcup (-N)/\{y \sim ig(y) \mid y \in \Sigma\}, \tag{9}$$

where $H \subset N$. We consider the complement $M_L := \overline{S^3 \backslash (\eta(K) \cup \eta(L))}$ as the pre-quotient space

$$H \cup_{g^1} (-N), \quad \text{where } g^1 := g \,|\, (\Sigma \backslash A) : \Sigma \backslash A \to \Sigma \backslash B. \tag{10}$$

Thus we can think of the torus $\mathcal{T} := A \cup B$ as the boundary torus of a tubular neighborhood of L. Let α be an arc that is properly embedded and essential on A such that it intersects L exactly once and let $\beta := g(\alpha)$. Now $\mu := \alpha \cup \beta$ is the meridian of $\mathcal{T}$ and $\lambda := L$ is the longitude which we will orient so that their algebraic intersection number on $\mathcal{T}$, denoted by $\langle \lambda, \mu \rangle$, is one. Since T_L is supported in A, it can be considered as a Dehn twist on $\mathcal{T}$. We have

$$T_L^q(\mu) = \mu - q\lambda = T_L^q(\alpha) \cup \beta.$$

(Recall that, in general, if a, b are simple closed curves on $\mathcal{T}$, we have $T_a(b) = b + \langle a, b\rangle a$. Since $\langle \lambda, \mu \rangle = 1$, we have $T_L^{-1}(\mu) = \mu + \lambda$, which explains the change of sign between the power T_L^q and the coefficient of λ in $T_L^q(\mu)$ in the equations above.)

Now we have

$$\alpha' \cup \beta = \mu - q\lambda.$$

Let $M_L(q)$ denote the 3-manifold obtained from M_L by $\dfrac{1}{-q}$ Dehn filling on $\mathcal{T}$. From the discussion above, in order to obtain $M_L(q)$ one needs to attach a solid torus to $\mathcal{T}$ in such a way so that the meridian is attached along the curve μ. It follows that $H \cup_{gT_L^{-q}} (-N)$ is an HN-splitting for $M_L(q)$. But since by assumption we have $M_L(q) = \overline{S^3 \backslash \eta(K')} = M'$, it follows that (Σ, gT_L^{-q}) is an HN-model for M'. □

5.4. Understanding the HN-model (Σ, gT_L^{-q}). In view of the conventions adapted earlier, N is thought of as a product $S \times I$ and K is embedded on $\Sigma := \partial N$. The Dehn twist $\tau^{-q} : N \to N$ changes K to K' and the product structure of N to $S' \times I$. By our assumptions, each of K, K' split Σ into two bounded surfaces that are incompressible in N. Let A be an annulus on Σ supporting the restriction $T_L := \tau \mid \Sigma$ so that the core of A is L and the intersection $A \cap K$ consists of two properly embedded, disjoint arcs, say α_1, α_2, each of which intersects L exactly once. We set $B := g(A)$, $\gamma := g(K)$, $\gamma' := g(K')$ and $z := g(L)$. By construction, we have $g \mid K = \mathrm{id}$. Thus, $g^{-1}(K) = K$, $B \cap \gamma = \alpha_1 \cup \alpha_2$. We have

$$\gamma' := g(K') = g\big(T_L^{-q}(K)\big) = g\big(T_L^{-q}\big(g^{-1}(K)\big)\big) = gT_L^{-q}g^{-1}(K) = T_{g(L)}^{-q}(K),$$

where the last equation follows from the fact that in the mapping class group we have $gT_Lg^{-1} = T_{g(L)}$. Thus γ' is the result of $\gamma := g(K) = K$ under a non-trivial power of a Dehn twist along $z := g(L)$ supported on B. We will think of the HN-splitting of $M' = \overline{S^3 \backslash \eta(K')}$ corresponding to the model (Σ, gT_L^{-q}) as the quotient

$$M' = H \cup (-N)/\{y \sim igT_L^{-q}(y) \mid y \in \Sigma\}, \tag{11}$$

and we will identify the corresponding Heegaard splitting of S^3 with the quotient

$$N \sqcup (-N)/\{y \sim igT_L^{-q}(y) \mid y \in \Sigma\}. \tag{12}$$

Lemma 5.6. *Push $g(L)$ slightly in the interior of N and let $\eta\big(g(L)\big)$ denote a tubular neighborhood of it in there. If $\overline{N \backslash \eta\big(g(L)\big)}$ is reducible, then $M_L := \overline{S^3 \backslash \eta(K) \cup \eta(L)}$ is reducible.*

Proof. Since $\overline{N \backslash \eta\big(g(L)\big)}$ is reducible, $g(L)$ must lie in a 3-ball in N. It follows that $K \cup L \subset S^3$ is a split link, thus M_L is reducible. □

In view of Lemma 5.6 and our earlier assumption that M_L is irreducible, we may assume that $\overline{N \backslash \eta\big(g(L)\big)}$ is irreducible. For $i = 0, 1$, let $S_i := S \times \{i\}$. The boundary ∂N is the union $S_0 \cup E \cup S_1$, where $E = \partial S \times (0,1)$. Let Σ_0, Σ_1 denote the image of S_0, S_1, respectively, under the Dehn twist $T_{g(L)}^{-q}$. Then, for $i = 0,1$, $\partial\Sigma_i = \gamma' \times \{i\}$.

Lemma 5.7. *The surfaces* Σ_0, Σ_1 *are incompressible in* N.

Proof. Suppose on the contrary that one of Σ_0, Σ_1, say Σ_0, compresses in N. Consider N as a product $S \times I$ with $g(L)$ a knot in N. By assumption, Σ_0 compresses in N. Performing the Dehn twist $T_{g(L)}^{-q}$ is equivalent to doing surgery along $g(L)$. Since $q \neq 0$, this surgery is non-trivial (Proposition 2.3). Now Σ_0 is the result of S_0 under this surgery. Thus there is a non-trivial surgery in $S \times I$ such that the surface S_0 compresses in the manifold obtained after surgery. By [14], Theorem 1, there is a simple closed homotopically essential curve $L' \subset \Sigma_0$ such that $g(L)$ and L' cobound an embedded annulus in $N = S \times I$. Furthermore, this annulus determines the slope of the surgery. This implies that $g(L)$ bounds a disc in N. But then any Dehn twist on ∂N along $g(L)$ extends to a Dehn twist on N, a homeomorphism of N. Since S_0, S_1 are incompressible, their images under any homeomorphism of N are also incompressible in N. This contradicts the assumption that Σ_0 compresses. □

Lemma 5.8. *With the notation and the setting as above, there exists a fibration of* M' *with fiber* S' *and corresponding* HN*-model* (Σ, g_1), *and an orientation preserving homeomorphism* $f : \Sigma \to \Sigma$ *such that in* $\Gamma(\Sigma)$ *we have*

$$g'' := gT_L^{-q} = fg_1f^{-1}. \tag{13}$$

Proof. We recall that the Heegaard splitting in (12) is the result of the splitting of (8) after the Dehn twist τ^{-q} on N. This twist changes the product structure of N from $S \times I$ to $S' \times I$. For $i = 0,1$, let $S_i := S \times \{i\}$. The boundary ∂N is the union $S_0 \cup E \cup S_1$, where $E = \partial S \times (0,1)$. We have

$$g''(S_i) = g\big(T_L^{-q}(S_i)\big) = g\big(T_L^{-q}\big(g^{-1}(S_i)\big)\big) = gT_L^{-q}g^{-1}(S_i) = T_{g(L)}^{-q}(S_i) = \Sigma_i.$$

By Lemma 5.7, Σ_i is incompressible in N. Now we pass to the corresponding HN-splitting of (11) and we use Q' to denote the corresponding HN-surface. Since the HN-surface of $H \cup_g (-N)$ is disjoint from a level surface of the fibration $S \times J/h$, Q' is disjoint from a neighborhood of a copy $S' \subset \mathrm{int}(H)$. By Proposition 3.1, M' is fibered with fiber S'. Let (Σ, g_1) denote the HN-model of this fibration and let Q denote the corresponding HN-surface. On the one hand M' cut along S' is a product $S' \times J$. On the other hand $M' = \overline{S^3 \backslash \eta(K)}$ is homeomorphic to

$$H \sqcup (-N)/\{y \sim ig''(y) \mid y \in \Sigma\}. \tag{14}$$

For $i = 0,1$, the surface $S_i \cup_{g''} \Sigma_i \subset Q'$ gives a properly embedded incompressible surface in M'. These two surfaces can be isotopied in M', relatively $\partial M'$, so that each becomes parallel to the fiber S' ([16], Proposition 3.1). In fact, the isotopy brings each of the surfaces onto a level surface of the fibration (Proposition 3.1). This implies that there is an orientation preserving homeomorphism $F : M' \to M'$ with $F \mid \partial M' = \mathrm{id}$ such that $F(Q) = Q'$ and $F(S' \times x) = S' \times x$, for all $x \in J$. Now applying Lemma 4.4 to the models (Σ, g_1) and (Σ, g''), we get the desired conclusion. □

5.5. Proof of Theorem 5.2. Let K, K' be fibered isotopic knots such that K' is obtained from K by a generalized crossing change, of order $q \neq 0$, supported on a crossing circle L. Let D be a crossing disc with $L := \partial D$. We will consider the Heegaard splittings of (8) and (12) so that the crossing circle L is embedded on Σ and D is a meridian disc of N. Recall that the crossing change from K to K' is now achieved by the Dehn twist τ^{-q} of N along D.

We will assume that L is homotopically essential on Σ since otherwise the crossing change from K to K' is obviously nugatory.

If $M_L := \overline{S^3 \backslash \eta(K) \cup \eta(L)}$ is reducible, then we are done by Lemma 5.3. We will assume that $M_L := \overline{S^3 \backslash \eta(K) \cup \eta(L)}$ is irreducible. Then, by Lemma 5.6, $\overline{NN \backslash \eta(g(L))}$ is irreducible. Due to Proposition 5.4, S and S' are of minimum genus for K and K', respectively. By Lemma 5.8, there is an HN-model (Σ, g_1) that corresponds to a fibration $M' = S' \times J/h_1$ and $f : \Sigma \to \Sigma$ so that (13) is satisfied. Equivalently, we have $f^{-1} g T_L^{-q} f = g_1$. Since K and K' are isotopic knots, there is an orientation preserving homeomorphism, say Φ, of S^3 that brings K to K'. Now we have two equivalent fibered knot complements: $M' = S' \times J/h_1$ and $M = S \times J/h$. Via Lemma 4.3, Φ gives rise to a homeomorphism $\phi : \Sigma \to \Sigma$ such that

$$gT_L^{-q} = \phi g \phi^{-1} \quad \text{or} \quad T_L^{-q} = g^{-1} \phi g \phi^{-1}. \tag{15}$$

Now (15) realizes T_L^{-q} as a commutator of length one in Γ. By Corollary 2.2, L must be homotopically trivial on Σ, which contradicts the assumption that M_L is irreducible. □

Since Kotschick's result is not true in the case of twists with mixed signs, the argument above breaks down in an attempt to generalize the statement of Theorem 5.2 to multiple crossing changes. But as the following example shows, the result is, in fact, not true!

Example 5.9. Let K denote the figure eight knot as boundary of a genus one Seifert surface S obtained by Hopf plumbing two once twisted bands B_L and B_R. Consider D_1 and D_2, crossing discs of K, such that $D_1 \cap B_L$ (resp. $D_2 \cap B_R$) is an essential arc cutting B_L (resp. B_R) into a square. One can perform opposite sign twists of order four along D_1, D_2 to transform S to S', where in S' the Hopf band B_L becomes the Hopf band B_R and vice versa. The knot $K' := \partial S'$ is isotopic to K. Moreover, S and S' are clearly minimum genus Seifert surfaces for K and K', respectively. However, neither of $L_1 := \partial D_1$ or $L_2 := \partial D_2$ bounds discs in the complement of K.

6. Adjacency to fibered knots

We begin by recalling from [8] the following definition.

Definition 6.1. Let K, K' be knots. We will say that K is n-adjacent to K', for some $n \in \mathbb{N}$, if K admits a projection containing n generalized crossings such that changing any $0 < m \leqq n$ of them yields a projection of K'.

In [8] we showed the following: Given knots K and K', there exists a constant $c = c(K, K')$ such that if K is n-adjacent to K', then either $n \leqq c$ or K is isotopic to K'.

Here, using Theorem 5.2, we will show that if K' is assumed to be fibered, then we can have a much stronger result.

Theorem 6.2. *Suppose that K' is a fibered knot and that K is a knot such that K is n-adjacent to K', for some $n > 1$. Then either K is isotopic to K' or we have $g(K) > g(K')$.*

Remark 6.3. It is not hard to see that if K is n-adjacent to K', for some $n > 1$, then K is m-adjacent to K', for all $0 < m \leqq n$.

Suppose that K is n-adjacent to K' and let L be a collection of n crossing circles supporting the set of generalized crossings that exhibit K as n-adjacent to K'. Since the linking number of K and every component of L is zero, K bounds a Seifert surface S in the complement of L. Define

$$g_n^L(K) := \min\{\text{genus}(S) \mid S \text{ is a Seifert surface of } K \text{ as above}\}.$$

We recall the following.

Theorem 6.4 (Theorem 3.1, [8]). *We have*

$$g_n^L(K) = \max\{g(K), g(K')\},$$

where $g(K)$ and $g(K')$ denote the genera of K and K', respectively.

Proof of Theorem 6.2. Let K' be a fibered knot. In view of Remark 6.3, it is enough to prove that if K is a knot that is 2-adjacent to K', then either K is isotopic to K' or we have $g(K) > g(K')$. To that end, suppose that K is exhibited as 2-adjacent to K' by a two component crossing link $L := L_1 \cup L_2$. Let D_1, D_2 be crossing discs for L_1, L_2, respectively. Suppose, moreover, that $g(K) \leqq g(K')$; otherwise, there is nothing to prove. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L. As explained earlier in the paper, we can isotope S so that, for $i = 1, 2$, $S \cap \text{int}(D_i)$ is an arc, say α_i, that is properly embedded in S. For $i = 1, 2$, let K_i (resp. S_i) denote the knot (resp. the Seifert surface) obtained from K (resp. S) by changing C_i. Also let K_3 denote the knot obtained by changing C_1 and C_2 simultaneously and let S_3 denote the corresponding surface. By assumption, for $i = 1, 2, 3$, K_i is isotopic to K' and S_i is a Seifert surface for K_i. Since $g(K) \leqq g(K')$, Theorem 6.4 implies that S_i is a minimum genus surface for K_i. Observe that K_3 is obtained from K_1 by changing C_2 and that they are fibered isotopic knots. Furthermore, S_3 is obtained from S_1 by twisting along the arc $\alpha_2 \subset S$. By Theorem 5.2, L_2 bounds an embedded disc Δ_2 in the complement of K_1. Since S_3 is incompressible, after an isotopy, we can assume that $\Delta \cap S_3 = \emptyset$. Now let us consider the 2-sphere

$$S^2 := \Delta \cup D_2.$$

By assumption, $S^2 \cap S_3$ consists of the arc $\alpha_2 \subset S_3$. Since α_1 and α_2 are disjoint, the arc α_1 is disjoint from S^2. But since K is obtained from K_1 by twisting along α_1, the circle L_2 still bounds an embedded disc in the complement of K. Hence, K is isotopic to K'. □

Remark 6.5. The trefoil knot is 2-adjacent to the unknot. Since the trefoil is a fibered knot, Theorem 6.2 implies that the unknot is not 2-adjacent to the trefoil. Thus n-adjacency is not an equivalence relation on the set of knots.

References

[1] *S. Akbulut* and *J. McCarthy*, Casson's invariant for oriented homology 3-spheres. An exposition, Math. Notes **36**, Princeton University Press, Princeton, NJ, 1990.
[2] *J. Birman*, *A. Lubotzky* and *J. McCarthy*, Abelian and solvable subgroups of the mapping class groups, Duke Math. J. **50** (1983), no. 4, 1107–1120.
[3] *V. Braungardt* and *D. Kotschick*, Clustering of critical points in Lefschetz fibrations and the symplectic Szpiro inequality, Trans. Amer. Math. Soc. **355** (2003), no. 8, 3217–3226.
[4] *G. Burde* and *H. Zieschang*, Knots, de Gruyter Stud. Math. **5**, Walter de Gruyter & Co., Berlin 1985.
[5] *H. Endo* and *D. Kotschick*, Bounded cohomology and non-uniform perfection of mapping class groups, Invent. Math. **144** (2001), no. 1, 169–175.
[6] *A. Fathi*, *F. Laudenbach* and *V. Poenaru*, Travaux de Thurston sur les surfaces, Astérisque **66–67** (1979), 1–284.
[7] *D. Gabai*, Foliations and the topology of 3-manifolds II, J. Diff. Geom. **26** (1987), 445–503.
[8] *E. Kalfagianni* and *X.-S. Lin*, Knot adjacency, genus and essential tori, Pacific J. Math. **228** (2006), no. 2, 251–276.
[9] *E. Kalfagianni* and *X.-S. Lin*, Knot adjacency and fibering, Trans. Amer. Math. Soc. **360** (2008), 3249–3261.
[10] *R. Kirby*, Problems in low-dimensional topology, in: Geometric Topology (Athens 1993), Proceedings of the 1993 Georgia International Topology Conference, Vol. 2, American Mathematical Society, Providence, RI (1997), 35–473.
[11] *M. Korkmaz* and *B. Ozbagci*, Minimal number of singular fibers in a Lefschetz fibration, Proc. Amer. Math. Soc. **129** (2001), no. 5, 1545–1549.
[12] *D. Kotschick*, Quasi-homomorphisms and stable lengths in mapping class groups, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3167–3175.
[13] *M. Scharlemann* and *A. Thompson*, Link genus and Conway moves, Comment. Math. Helv. **64** (1989), 527–535.
[14] *M. Scharlemann* and *A. Thompson*, Surgery on a knot in (Surface $\times I$), Algebr. Geom. Topol. **9** (2009), 1825–1835.
[15] *I. Torisu*, On nugatory crossings for knots, Topology Appl. **92** (1999), no. 2, 119–129.
[16] *F. Waldhausen*, On irreducible 3-manifolds which are sufficiently large, Ann. Math. **87** (1968), 56–88.

Department of Mathematics, Michigan State University, East Lansing, MI, 48824, USA
e-mail: kalfagia@math.msu.edu

Eingegangen 20. April 2010, in revidierter Fassung 17. Januar 2011