AN INTRINSIC APPROACH TO POLYNOMIAL INVARIANTS FOR LINKS IN 3-MANIFOLDS

EFSTRATIA KALFAGIANNI

Abstract. We study framed links in irreducible $\mathbb{Z}$-homology 3-spheres and we provide sufficient conditions under which an invariant defined on *singular links* with one transverse double point gives rise to an invariant of framed links. Our results also hold for framed links in *atoroidal*, $\mathbb{Q}$-homology 3-spheres. As an application we obtain a 2-variable formal power series invariant for framed links that satisfies the axioms of the Kauffman polynomial. For links in S^3 our results provide a new construction of the classical Kauffman polynomial. In general, the coefficients of the power series are *finite type* link invariants and can be thought of as realizations of the Witten $SO(n)$-perturbative invariants for links in 3-manifolds.

1. Introduction

The Kauffman polynomial is a 2-variable Laurent polynomial invariant for links in S^3 [13] that has interesting applications and connections with contact geometry. More specifically, the degree in one of the variables of the Kauffman polynomial provides an upper bound for the Thurston-Bennequin norm of Legendrian links [7], [20]. The inequality is known to be sharp for several classes of links (e.g. alternating links) and the proof of this sharpness has led to deeper connections between knot polynomials and contact geometry [18].

The original constructions of the Kauffman polynomial rely heavily on features of link projections in S^3 which makes them hard to generalize in other 3-manifolds [13]. However, the polynomial can be re-casted as a sequence of infinite power series the coefficients of each of which are Vassiliev invariants (a.k.a. *finite type* invariants) [21]. The theory of finite type invariants generalizes to links in 3-manifolds using algebraic topology [22] or intrinsically 3-dimensional topology techniques [11]. As an application of the approach of [11] in [12] we obtained a HOMFLY power series link invariant for links in a large class of $\mathbb{Q}$-homology 3-spheres. In this paper we study *framed links* in irreducible $\mathbb{Z}$-homology 3-spheres and in *atoroidal* $\mathbb{Q}$-homology 3-spheres. We give conditions under which an invariant that is defined on *singular links* with one transverse double point gives rise to an invariant of framed links (Theorem 2.1). This allows us to construct a formal

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power series framed link invariant obeying the Kauffman polynomial skein relations. The coefficients of this series are *finite type* framed link invariants and can be thought of as realizations of the Witten $SO(n)$-perturbative invariants [23] . In principle, these coefficients should be recoverable from the Le-Murakami-Ohtsuki universal perturbative invariant [16] by considering *weight systems* corresponding to certain representations of the Lie algebras $so(n)$. Our approach here is quite different from this line and exhibits the interplay between skein framed link theory and the topology of 3-manifolds.

For an orientable 3-manifold M let $\hat{\pi}$ denote the set of *non-trivial* conjugacy classes of $\pi_1(M)$. An m-component *link* is a collection of m unordered oriented circles, tamely and disjointly embedded in M. Hence a link is homotopically equivalent to an unordered m-tuple of elements in $\hat{\pi} \cup \{1\}$. In every homotopy class of links, we will fix, once and for all, a link CL and call it a *trivial link*. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^* and U^1 will be abbreviated to U later on. We will denote by $\mathcal{CL}^*$ the set of all *trivial links* with none of their components homotopically trivial. Every link L is homotopic to a certain $CL^* \coprod U^k$ for some $CL^* \in \mathbb{C}\mathcal{L}^*$, possibly empty. Link theory in the 3-manifold M seeks to understand how two links can differ up to a (tame) isotopy if they are homotopic. In this paper we need to consider framed links: To define a notion of link framing we will consider our links as cores of embedded annuli in M. That is a framed link will be a piecewise linear embedding $L : P \times I \longrightarrow M$, where P is a disjoint union of oriented S^1's. The framing on a component $L_i \subset L$ is the linking number of a boundary component of the annulus with the core L_i. Thus the framing of an m-component link can be given as an m-tuple of integers $\mathbf{f} := (\mathbf{f_1}, \dots, \mathbf{f_m})$. Two framed links are isotopic if the corresponding thickened links are isotopic.

Let L_+, L_- , L_o and L_∞ denote four framed links that are identical everywhere except in a 3-ball B in M. There, under a suitable projection of the parts in B, L_+, L_- and L_o intersect at a positive crossing, a negative crossing, and an orientation consistent smoothing of a crossing, respectively. The link L_∞ intersects B at the smoothing that is not consistent with orientation of L_+, L_-. See Figure 1. Also for every framed link , we will denote by L_r, L_l links that are identical to L everywhere except in a 3-ball where they differ by full twists as shown in Figure 2.

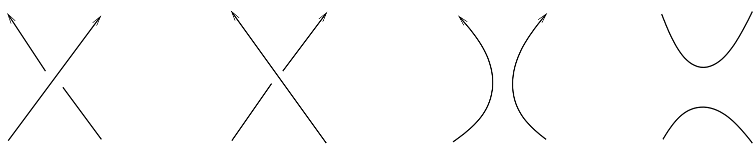

FIGURE 1. The parts L_+, L_- and L_o and L_∞ in B.

Let $\mathcal{L}$ be the set of isotopy classes of framed links in M and let $\hat{\Lambda} := \mathbb{C}[[x,\ y]]$ denote the ring of formal power series in variables x, y over $\mathbb{C}$. Now we consider the power series $a := e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} \dots$ and $z := e^x - e^{-x} = 2x + \frac{x^3}{3} + \frac{x^5}{60} \dots$.

Theorem 1.1. *Let M be an oriented, $\mathbb{Q}$-homology sphere with $\pi_2(M) = 0$ and such that if $H_1(M) \neq 0$, M is* atoroidal. *Suppose we are given values $R_M(U)$ and $R_M(CL^*)$ for every $CL^* \in \mathcal{CL}^*$, that are independent of the orientation of the link. Then, there is a unique map $R_M : \mathcal{L} \to \hat{\Lambda}$ such that.*
(i) R_M satisfies the Kauffman skein relation

$$R_M(L_+) - R_M(L_-) = z\big[R_M(L_o) - R_M(L_\infty)\big],$$

for every skein quartuple of links L_+, L_- and L_o and L_∞.
(ii) $R_M(L_r) = aR_M(L)$ and $R_M(L_l) = a^{-1}R_M(L)$
(iii) For every $L \in \mathcal{L}$, $R_M(L)$ is independent of the orientation of L.

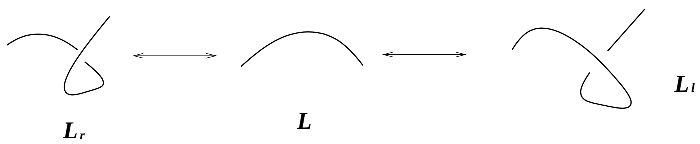

FIGURE 2. L_r and L_l are obtained by a full twist from L.

Let $\Lambda := \mathbb{C}[a^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in a and z. As we will discuss in Section 4, for links in S^3, if we choose the value $R_{S^3}(U)$ to be in Λ then $R_{S^3}(L) \in \Lambda$, for every $L \in \mathcal{L}$. Thus Theorem 1.1 provides another construction of the classical Kauffman polynomial [13]. This leads to the following question:

Question 1.2. For which 3-manifolds M can we choose the trivial links $CL^* \in \mathcal{CL}^*$ so that if the values $R_M(U)$ and $R_M(CL^*)$ lie in Λ, then $R_M(L) \in \Lambda$ for every $L \in \mathcal{L}$?

In [12] we constructed a formal power series invariant that satisfies the HOMFLY skein change formula for (unframed) links in large classes of $\mathbb{Q}$-homology 3-spheres. In [5], Cornwell showed that in Lens spaces the analogue of Question 1.2 for this HOMFLY series has a positive answer. Thus he obtained a 2-variable HOMFLY polynomial for links in Lens spaces. Then, generalizing the celebrated Franks-William-Morton inequality, he shows that the degree of one variable of this polynomial provides upper bounds to the Thurston-Bennequin invariant of Legendrian links in Lens spaces [6]. It is plausible that the techniques of [6] can be applied to the study of the invariants of the current paper and could lead to further connections with contact geometry.

For null-homologous links in 3-manifolds there is a natural framing coming from self-linking number. When working in $\mathbb{Z}$-homology 3-spheres the notion of framing given above agrees with this natural framing. There are several notions of self-linking number that have been considered for non null-homologous links: For example Chernov [4] works with an integer self-linking number while Baker and Etnyre [1] discuss a rational valued self-linking number for links in $\mathbb{Q}$-homology 3-spheres. Here for $\mathbb{Q}$-homology 3-spheres we work with an integer framing that is previously considered, for example, in [16]. and is closer to the spirit to that of [4].

The paper is organized as follows: In Section 2 we formulate the problem of integrating singular link invariants to invariants of framed links. Then we state an "integrability theorem" and prove it for *atoroidal* $\mathbb{Q}$-homology spheres. In Section 3 we treat manifolds containing essential tori and in Section 4 we construct the Kauffman power series invariant. Finally, in Section 5 we give some concluding remarks and questions.

Acknowledgement. I thank Chris Cornwell for his interest in this work and for several stimulating questions about link theory in 3-manifolds that motivated me to go back and work on this project.

2. Link and singular link Invariants

2.1. **Link Framings.** Let P be a disjoint union of m oriented circles. A *framed link* in an oriented 3-manifold M is a piecewise linear embedding $L : P \longrightarrow M$ such that each component, $L_i \subset L(P)$ is endowed with the homotopy class of non-singular normal vector field. If M is a $\mathbb{Z}$-homology sphere (i.e. $H_1(M) = 0$) there is a natural choice of framing for links: For every component L_i of an oriented link $L : P \longrightarrow M$, let $N(L_i)$ denote a regular neighborhood of L_i. There is unique simple closed curve $L'_i \subset \partial(N(L_i)$ that is homologically trivial in $M \smallsetminus N(L_i)$; this curve bounds an embedded, bi-collared surface S_i in $M \smallsetminus N(L_i)$ (a Seifert surface). The homotopy class of any non-singular normal vector field along L_i is determined by the algebraic intersection number of L_i with the surface S_i. Thus the framing of an m-component link is determined by an m-tuple of integers $\mathbf{f} := (\mathbf{f_1}, \ldots, \mathbf{f_m})$. Two m-component links L and L' (of the same framing) are equivalent if there is an isotopy $h_t : M \longrightarrow M$, $t \in [0, 1]$ such that $h_0 = id$, $L' = h_1(L)$ and for every $t \in [0, 1]$, $h_t(L)$ has the same framing as this of L and L'.

An m-component *singular link* of order n is a piecewise-linear immersion $L : P \longrightarrow M$ whose only singularities are exactly n transverse double points. Given a double point of a singular link of order n we can obtain two singular links L_+ and L_- of order $n-1$ by resolving the double point in two ways (see below for more details.) For a singular link $L_\times$ with one double point the resolutions L_+, L_- represent framed links. We note that if the double point $\times$ occurs between two different components of P, then the framings of L_+, L_- are the same. For a singular link $L_\times$ with one double point the resolutions L_+, L_- represent framed links. We note that if the double point $\times$ occurs between two different components of P, then the framings of L_+, L_- are the same. We define the framing of $L_\times$ to be

$$\mathbf{f}(L_\times) := (\mathbf{f_1}, \dots, 0, \dots, 0, \dots, \mathbf{f_m}), \tag{0}$$

where the entry 0 occurs at the indices corresponding to the components on L involved at the double point of $L_\times$. If the double point $\times$ occurs on a single component, say P_i, then we define the framing of $L_\times$ by

$$\mathbf{f}(L_\times) := (\mathbf{f_1}, \dots, \mathbf{f_i}(L_+) - \mathbf{f_i}(L_-), \dots, \mathbf{f_m}). \tag{1}$$

Then we continue inductively to extent the framing on singular links of any order. With this understanding we will be considering framed singular links in the rest of the paper.

Two framed singular links of the same order are equivalent if there is an ambient isotopy of M that brings one to the other within the class of framed singular links of order n. We will denote by $\mathcal{L}^{(n)}$ (resp. $\mathcal{L}$) the set of isotopy classes of all framed singular links of order n (resp. links) in M.

Now suppose that M is a $\mathbb{Q}$-homology 3-sphere (i.e. $H_1(M)$ is finite). To define a notion of link framing in M we will consider our links as embeddings of annuli. That is a framed link will be a piecewise linear embedding $L : P \times I \longrightarrow M$. The framing on a component $L_i \subset L$ is the linking number of a boundary component of the annulus with core L_i. Thus the framing of an m-component link can be given as an m-tuple of integers $\mathbf{f} := (\mathbf{f_1}, \dots, \mathbf{f_m})$. Two framed links are isotopic if the corresponding thickened links are isotopic. Again we extend the framing on singular links as above.

2.2. **Four resolutions of double points.** Let $p \in M$ be a transverse double point of a framed singular link L. Then $L^{-1}(p)$ consists of two points $p_1,\ p_2 \in P$. There are disjoint 1-simplices, σ_1 and σ_2, on P with $p_i \in \operatorname{int}(\sigma_i)$, $i = 1,\ 2$ such that for a small ball neighborhood B of p

$$L \cap B = L(\sigma_1) \cup L(\sigma_2)$$

Moreover, there is a proper 2-disc D in B such that $L(\sigma_1)$, $L(\sigma_2) \subset D$ intersect transversally at p. One assumes that isotopies of singular links carry the ball disc pair (B, D) through for all the double points of L [17]. We resolve a transverse double point of a singular link of order n in different ways: $L(\sigma_1) \cup L(\sigma_2)$ intersects ∂D at four points and since σ_i inherits an orientation from that of P we can talk about the initial point and terminal point of σ_i and $L(\sigma_i)$. Now choose arcs a_1, a_2, b_1, b_2 with disjoint interiors such that

(1) a_1 and a_2 go from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_1)$ and lie in distinct components of $\partial B \setminus \partial D$; and

(2) b_1 and b_2 lie on ∂D with b_1 going from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_2)$ and b_2 from the initial point of $L(\sigma_2)$ to the terminal point of $L(\sigma_1)$. The complement of $b_1 \cup b_2$ in ∂D consists of two arcs, say c_1, c_2. The orientation of M and that of $L(\sigma_2)$ define an orientation

of $a_1 \cup a_2$; suppose this induced orientation agrees with the one of a_1 and is opposite to that of a_2. Define

$$L_+ = \overline{K \setminus K(\sigma_2)} \cup a_1$$

$$L_- = \overline{K \setminus K(\sigma_2)} \cup a_2$$

$$L_o = \overline{K \setminus (K(\sigma_2) \cup K(\sigma_1))} \cup (b_1 \cup b_2)$$

$$L_\infty = \overline{K \setminus (K(\sigma_2) \cup K(\sigma_1))} \cup (c_1 \cup c_2)$$

Now L_+, L_- and L_o are oriented framed singular links of order $n-1$. The link $L_\infty = \overline{K \setminus (K(\sigma_2) \cup K(\sigma_1))} \cup (c_1 \cup c_2)$, only makes sense as an unoriented link. We will denote by $\mathcal{L}^{(n)}$ (resp. $\mathcal{L}$) the set of isotopy classes of framed, oriented, singular links of order n (resp. links) in M.

2.3. **Integration of singular link invariants.** Given an abelian group $\mathbb{A}$, and framed link invariant $F : \mathcal{L} \longrightarrow \mathbb{A}$, can extend it to an invariant of framed singular links as follows: Let $L_\times \in \mathcal{L}^{(1)}$ be a framed singular link where $\times$ stands for the only double point. Then the two resolutions of L_+, $L_- \in \mathcal{L}$ represent links. We define

$$f(L_\times) = F(L_+) - F(L_-). \tag{2}$$

Continuing inductively we can extend the invariant on singular links in $\mathcal{L}^{(n)}$, for all $n \in \mathbf{N}$. We are interested in reversing this process; the reverse process is usually referred to as *integration* of the singular link invariant to an invariant of links (cf [17], [12], [11]). In this section we deal with the following question: Suppose that we are given an invariant of framed singular links $f : \mathcal{L}^{(1)} \longrightarrow \mathbb{A}$. Under what conditions is there a link invariant $F : \mathcal{L} \longrightarrow \mathbb{A}$ so that (2) holds for all singular links $L_\times \in \mathcal{L}^{(1)}$? We will address this question for links in $\mathbb{Q}$-homology 3-spheres with trivial π_2.

Let N be an orientable, compact 3-manifolds with or without boundary. A map $\Phi : S^1 \times S^1 \longrightarrow N$ is called *essential* if it induces an injection on π_1 and it cannot be homotoped to a map $\Phi' : S^1 \times S^1 \longrightarrow \partial N$. Otherwise, Φ is called *inessential*. The manifold N is called *atoroidal* if there are no *essential* maps $S^1 \times S^1 \longrightarrow N$.

Theorem 2.1. *Suppose that M is a $\mathbb{Q}$-homology sphere with $\pi_2(M) = 0$ and such that that if $H_1(M) \neq 0$, then M is atoroidal. Let $f : \mathcal{L}^{(1)} \longrightarrow \mathbb{A}$ be an invariant of framed singular links with one double point. Suppose that $\mathbb{A}$ is torsion free and that the invariant f has the following properties.*

$$f(L_{\times r}) = f(L_{r \times}). \tag{3}$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}), \tag{4}$$

for every $L_{\times\times} \in \mathcal{L}^{(2)}$. Then there exists a framed link invariant F such that f is derived from F via equation (2).

Notation. The four singular links in $\mathcal{L}^{(1)}$ appearing in (4) are obtained by resolving the double points of $L_{\times\times}$ one at a time. The two singular links in (3) only differ locally as shown in Figure 3; in particular they are inadmissible. Note that (3) and (4) imply that $f(L_{\times l}) = f(L_{l\times})$, where $L_{\times l}$ and $L_{l\times}$ are also shown in Figure 3.

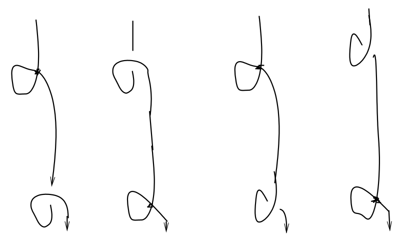

FIGURE 3. From left to right: $L_{\times r}$, $L_{r\times}$, $L_{\times l}$, $L_{l\times}$

Definition 2.2. A singular link L is called *inadmissible* if there is a 2-disc $D \subset M$ such that $L \cap D = \partial D$ and exactly one double point of L lies on ∂D. Otherwise the singular link is called *admissible.* A crossing change on a link that produces an inadmissible singular link as intermediate step will be called an *inadmissible crossing change.* Notice that the links of Figure 3 are inadmissible.

Theorem 2.1 is similar to Theorem 3.16 of [11] and Theorem 3.1.2 of [12]; the difference is that the later results are about invariants of ordinary (unframed) link isotopy. For arguments and technical aspects that are similar to these in [11], [12] we will often refer the reader in these manuscripts for details and terminology.

2.4. **Loop space and framing control.** Let $L \in \mathcal{L}$ be a framed link in M. We also use L to denote a representative $L : P \longrightarrow M$, of L. Let $\mathcal{M}^L(P, M)$ denote the space of maps $P \longrightarrow M$ homotopic to L equipped with the compact-open topology. For every framed link $L' \in \mathcal{M}^L(P, M)$, we choose a homotopy $\phi_t : P\times[0, 1] \longrightarrow M$ such that $\phi_0 = L'$ and $\phi_1 = L$. After a small perturbation, we can assume that for only finitely many points $0 < t_1 < t_2 < \cdots < t_n < 1$, ϕ_t is not an embedding. Moreover, we can assume that ϕ_{t_i}, for $i = 1, 2, \ldots, n$ are singular framed links of order 1 (i.e. $\phi_{t_i} \in \mathcal{L}^{(1)}$). For different $t's$ in an interval of $[0,\ 1]\smallsetminus\{t_1,\ t_2,\ \ldots, t_n\}$, the corresponding framed links are equivalent. When t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another. Since we are interested in framed link equivalence invariants, for closed homotopies from a framed link to itself the "total framing change" should be zero. Next we introduce a measure of this framing change.

Definition 2.3. Let $\Phi : P \times [0,\ 1] \longrightarrow M$ be a generic homotopy between m-component links L, L' as above. Let $0 < t_1 < \cdots < t_n < 1$, denote the values of t for which $\phi_{t_i} \in \mathcal{L}^{(1)}$. Define

$$\Delta \mathbf{f}_{\Phi} := (\Delta \mathbf{f_1}, \dots, \Delta \mathbf{f_m}),$$

where, for $i = 1, \dots, m$,

$$\Delta \mathbf{f_i} := \sum_{j=1}^{n} \delta^i_j \epsilon_j \mathbf{f_i}(\phi_{t_j}), \tag{5}$$

and the framings $\mathbf{f_i}(\phi_{t_j})$ are given by the formulae (0)-(1). Here δ^i_j is 1 if the double point of ϕ_{t_j} lies on the i-th component and 0 otherwise, and the sign $\epsilon_j = \pm 1$ is determined as follows: If $\phi_{t_j+\delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of ϕ_{t_j} then $\epsilon_j = 1$. Otherwise $\epsilon_j = -1$. Thus, $\Delta \mathbf{f}_{\Phi}$ records the total framing change along Φ. We will say that a closed homotopy Φ is *framing preserving* iff we have $\Delta \mathbf{f}_{\Phi} = \mathbf{0} := (0, \dots, 0)$.

2.5. **Beginning the proof of Theorem 2.1.** Let $L' \in \mathcal{M}^L(P, M)$ be a framed link. Choose a generic homotopy $\phi_t : P \times [0,1] \longrightarrow M$ such that $\phi_0 = L$ and $\phi_1 = L$. Let $0 < t_1 < t_2 < \cdots < t_n < 1$ denote the points where ϕ_t is not an embedding. Recall that $\phi_{t_i} \in \mathcal{L}^{(1)}$ such that for different $t's$ in an interval of $[0,\ 1] \diagdown \{t_1,\ t_2,\ \dots, t_n\}$, the corresponding framed links are equivalent. When t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another. We define

$$F(L') = F(L) + \sum_{i=1}^{n} \epsilon_i f(\phi_{t_i}) \tag{6}$$

Here $\epsilon_i = \pm 1$ is determined as follows: If $\phi_{t_i+\delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of ϕ_{t_i} then $\epsilon_i = 1$. Otherwise $\epsilon_i = -1$.

To prove that F is well defined we have to show that modulo "the integration constant" $F(L)$, the definition of $F(L')$ is independent of the choice of the homotopy. For this we consider a *framing preserving* closed homotopy $\Phi : P \times S^1 \longrightarrow M$ from L to itself. After a small perturbation, we can assume that there are only finitely many points $x_1, x_2, \dots, x_n \in S^1$, ordered cyclicly according to the orientation of S^1, so that $\phi_{x_i} \in \mathcal{L}^1$ and ϕ_x is equivalent to ϕ_y for all $x_i < x, y < x_{i+1}$. To prove that F is well defined we need to show that

$$X_{\Phi} := \sum_{i=1}^{n} \epsilon_i f(\phi_{t_i}) = 0 \tag{7}$$

where $\epsilon_i = \pm 1$ is determined by the same rule as above.

The proof of (7), which occupies the remaining of Section 2 and Section 3, will be divided into several steps. In this section we will give the proof of (7) for closed homotopies in atoroidal 3-manifolds and in the next section we deal with essential tori.

To continue, we first need to introduce some notation. Suppose that P has m components; that is

$$P = \coprod_{i=1}^{m} P_i$$

where each P_i is an oriented circle. Let $L : P \longrightarrow M$ be a link. Pick a base point $p_i \in P_i$ and let a_i denote the homotopy class of $L(P_i)$ in $\pi_1(M, L(p_i))$. Finally, we denote by $Z(a_i)$ the centralizer of a_i in $\pi_1(M, L(p_i))$. We begin with the following well known lemma (see, for example, the proof of Proposition 4.3 of [17]).

Lemma 2.4. *Suppose that M is an orientable 3-manifold with $\pi_2(M) = 0$ and let the notation be as above. Then*

$$\pi_1(\mathcal{M}^L(P, M),\ L) \cong \oplus_{i=1}^{m} Z(a_i).$$

2.6. **Integrating around inessential tori.** Here we show how to derive (7) in the case where the closed homotopy Φ represents a collection of inessential tori in M. Since $\partial M = \emptyset$ this means that the induced map $(\Phi_i)_* : \pi_1(P_i \times S^1) \longrightarrow \pi_1(M)$ has non-trivial kernel.

Lemma 2.5. *Suppose that a framing preserving closed homotopy Φ can be extended to a map $\hat{\Phi} : P \times D^2 \longrightarrow M$, where D^2 is a 2-disc with $\partial D^2 = \{*\} \times S^1$. Then, $X_\Phi = 0$.*

Proof. We perturb $\hat{\Phi}$, relatively ∂D^2, to be in almost general position in the sense of Proposition 1.1 of [11]. Then the set

$$S_{\hat{\Phi}} := \{x \in D^2 \mid \hat{\phi}_x := \hat{\Phi}(P \times \{x\}) \text{ is not an embedding}\},$$

is an 1-dimensional embedded sub complex of D^2 with properties (1)-(5) given in Proposition 1.1 of [11]. The vertices of $S_{\hat{\Phi}}$ in the interior of D^2 are of valence one or of valence four (see Figure 4).

Each edge of $S_{\hat{\Phi}}$, corresponds to a singular link of order 1. So by using the given invariant f we can assign an element of $\mathbb{A}$ to every edge of $S_{\hat{\Phi}}$ and we can reduce (7) to local conditions; one around each interior vertex of $S_{\hat{\Phi}}$. Before we explain this better, we observe that the condition $f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times})$ implies that the value X_Φ does not depend on the order in which the crossing changes around $\Phi = \hat{\Phi}|P \times \partial D^2$ are made. Thus, without loss of generality, we may assume that the valence one vertices of $S_{\hat{\Phi}}$ correspond to *inadmissible crossing changes* on ∂D^2: More specifically, with the notation as above, for $i = 1, \ldots, s$, $\phi_{x_i} \in \mathcal{L}^1$ is an *inadmissible* singular link and for $i = s, \ldots, n$, the singular link $\phi_{x_i} \in \mathcal{L}^1$ is *admissible*. In particular, there are s edges of $S_{\hat{\Phi}}$ emanating from $x_1, \ldots, x_s$ respectively and ending at an interior vertex of valence one and these are the only valence one vertices of $S_{\hat{\Phi}}$.

For every interior vertex of $S_{\hat{\Phi}}$ we draw a small circle C around it, so that the number of points in $C \cap S_{\hat{\Phi}}$ is equal to the valence of the vertex. See

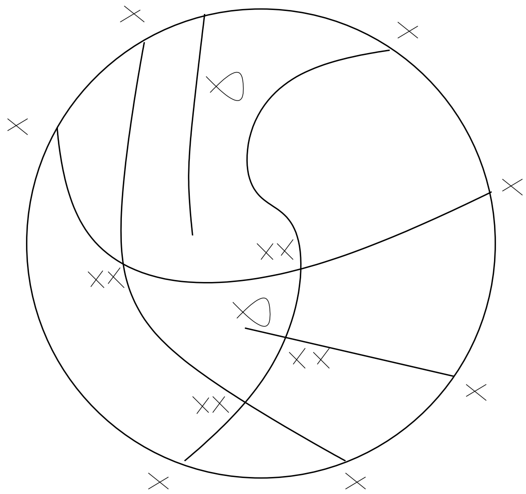

FIGURE 4. The set of singularities $S_{\hat{\Phi}}$ with the types of double points they represent.

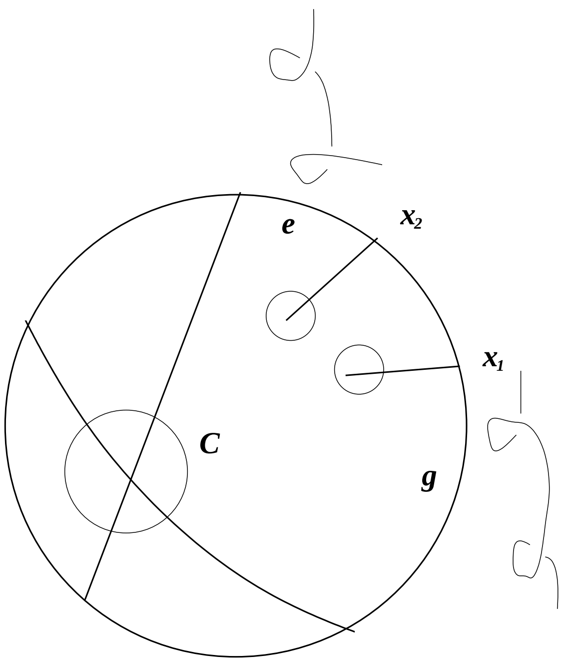

FIGURE 5. The singular links ϕ_{x_1}, ϕ_{x_2} form a pair of type $L_{\times r}, L_{r\times}$ (or $L_{\times l}, L_{l\times}$) shown in Figure 3.

Figure 5. Let $C_1, \ldots, C_s$ denote the circles encircling the valence one vertices of $S_{\hat{\Phi}}$ and let Γ denote the disjoint union of the circles surrounding the vertices of valence four. First we look at a vertex of valence four: Then the four points in $C \cap S_{\hat{\Phi}}$ correspond exactly to the four singular links appearing

in equation (4). Since (4) implies $f(L_{\times +}) - f(L_{\times -}) - f(L_{+\times}) + f(L_{-\times}) = 0$, we have

$$\sum_{x \in \Gamma \cap S_{\hat{\Phi}}} \epsilon_x f(\hat{\phi}_x) = 0 \tag{8}$$

where $\hat{\phi}_x := \hat{\Phi}(P \times \{x\})$. Now observe that

$$\sum_{i=s+1}^{n} \epsilon_i f(\phi_{x_i}) = \sum_{x \in \Gamma \cap S_{\hat{\Phi}}} \epsilon_x f(\hat{\phi}_x).$$

Hence the last equation and (8) imply that

$$X_\Phi = \sum_{i=1}^{s} \epsilon_i f(\phi_{x_i}). \tag{9}$$

Since Φ is *framing preserving*, we have $\Delta \mathbf{f}_\Phi = \mathbf{0} := (0, \ldots, 0)$.

Claim: We have $\Delta \mathbf{f}_C = (0, \ldots, 0)$, for every loop $C \in \Gamma$.

Proof of Claim: By Definition 2.3, the i-th coordinate of $\Delta \mathbf{f}_C$ can have non-zero contributions only during crossing changes involving the i-th component. Let $L_{\times\times} \in \mathcal{L}^{(2)}$ and suppose that one of the double points occurs between the components L_k, L_l and the second double point occurs between L_r and L_w. If $i \neq k, l, r, w$ then $\Delta \mathbf{f_i} = 0$, by definition. Next observe that, by the definition of framing of singular links given in (0)-(1) , if in a singular link $L_\times \in \mathcal{L}^{(1)}$ the double point occurs between the components $L_a, L_b \subset L_\times$, then $\mathbf{f_i}(L_\times) = 2\delta_a^b$, for every $i \in \{a, b\}$. This implies that $\mathbf{f_i}(L_{\times +}) - \mathbf{f_i}(L_{\times -}) = 0$ for each $i \in \{k, l\}$. Similarly, $\mathbf{f_i}(L_{+\times}) - \mathbf{f_i}(L_{-\times}) = 0$ for each $i \in \{r, w\}$. Based on these observations and Definition 2.3 an easy case-by-case direct examination shows that $\Delta \mathbf{f_i} = 0$, for all $i \in \{k, l, r, w\}$: For example suppose that both endpoints of $L_{\times\times}$ lie on the component $L_k := L_{\times\times}(P_k)$. Then, by formula (5), we have $\Delta \mathbf{f_i} = 0$ for $i \neq k$ and $\Delta \mathbf{f_k} = \mathbf{f_k}(L_{\times +}) - \mathbf{f_k}(L_{\times -}) - \mathbf{f_i}(L_{+\times}) + \mathbf{f_i}(L_{-\times})$ which is equal to zero since $\mathbf{f_k}(L_{\times +}) - \mathbf{f_i}(L_k \times -) = \mathbf{f_i}(L_{+\times}) - \mathbf{f_k}(L_{-\times}) = 2$.

To continue we note that the claim above implies that

$$\Delta \mathbf{f}_\Gamma := \sum_{C \in \Gamma} \Delta \mathbf{f}_C = (0, \ldots, 0).$$

Since we have

$$\Delta \mathbf{f}_\Phi = \sum_{i=1}^{s} \epsilon_i \mathbf{f}(\phi_{x_i}) + \Delta \mathbf{f}_\Gamma = (0, \ldots, 0),$$

we conclude that $\sum_{i=1}^{s} \epsilon_i \mathbf{f}(\phi_{x_i}) = (0, \ldots, 0)$. This in turn implies that the full twists on L resulting from the inadmissible crossing changes are divided into pairs of twists of opposite sign and they can be eliminated by framed link isotopy (see Figure 4). Now local condition (3) implies that the right hand side of (9) must be identically zero. Thus we have verified the global condition that $X_\Phi = 0$ as desired. $\square$

Lemma 2.6. *Let M be a $\mathbb{Q}$-homology sphere with $\pi_2(M) = 0$. Suppose, moreover, that $\pi_1(M)$ is infinite and that the framed link L to begin with has no homotopically trivial components. Let $\Phi \subset \mathcal{M}^L(P, M)$ be a framing preserving homotopy such that, for every $i = 1, \dots, m$, the restriction $\Phi|P_i \times S^1 \longrightarrow M$ is inessential. Then, exists a map $\tilde{\Phi} : P \times D^2 \longrightarrow M$ such that*

$$X_{\partial\tilde{\Phi}} = aX_{\Phi}$$

for some $a \in \mathbb{Z}$. Here $\partial\tilde{\Phi} = \tilde{\Phi}|P \times \partial D^2$ and D^2 is a 2-disc.

Proof. For $i = 1, \dots, m$, let $T_i := P_i \times S^1$, $\Phi_i := \Phi|T_i$, $l_i := P_i \times \{*\}$, and $m_i := \{*\} \times S^1$. Since Φ_i is inessential we can find a closed curve $\lambda_i \subset T_i$ such that the image $\Phi_i(\lambda_i)$ is homotopically trivial in M. Since we assumed that $\pi_1(M)$ is infinite and $\pi_2(M) = 0$, it follows that $\pi_1(M)$ is torsion free [8]. Thus we can take λ_i to be a simple closed curve, for all $i = 1, \dots, m$. In $\pi_1(T_i)$ we have $[\lambda_i] = a_i[m_i]+b_i[l_i]$, for some co-prime integers a_i, b_i. If $a_i = 0$ for some $i = 1, \dots, m$, then $\Phi_i(l_i)$ is homotopically trivial in M. Thus, one of the components of the link $\Phi(P \times \{*\})$ is homotopically trivial. Since we assumed otherwise we have that $a_i \neq 0$, for $i = 1, \dots, m$. Let $q_i : \tilde{T}_i \longrightarrow T_i$ be the covering of T_i corresponding to the subgroup $a_i\mathbb{Z}\oplus\mathbb{Z}$ of $\pi_1(T_i) = \mathbb{Z}\oplus\mathbb{Z}$. Let $\tilde{l}_i$, $\tilde{\lambda}_i$, $\tilde{m}_i$ denote the lifting of l_i, λ_i, and m_i, respectively. We have

$$[\tilde{\lambda}_i] = [\tilde{m}_i] + b_i[\tilde{l}_i]. \tag{10}$$

Note that the curve $\Phi_i(q_i(\tilde{\lambda}_i))$ is homotopically trivial in M. Thus we can extend the map $\Phi_i \circ q_i$ on a 2-disc D_i with $\partial D_i = \tilde{\lambda}_i$. Since $\pi_2(M) = 0$, $\Phi_i \circ q_i : \tilde{T}_i \longrightarrow M$ extends to a map $\tilde{\Phi}_i : S^1 \times D_i \longrightarrow M$, where $\tilde{l}_i \times \tilde{\lambda}_i = \tilde{T}_i$. We will set

$$\tilde{\Phi} := \coprod_{i=1}^{m} \tilde{\Phi}_i$$

Claim: We have that

$$X_{\partial\tilde{\Phi}} = aX_{\Phi}$$

where $|a| = max\{|a_1|, \dots, |a_m|\}$.

Proof of Claim: We identify the curves $\tilde{\lambda}_i$ by a common parameterization, and call the result $\tilde{\lambda}$. The parameterization should be such that corresponding points on the $\tilde{\lambda}_i$s map, under the q_i's, to the same point on the parameter space of Φ. This induces a common parameterization of the curves $\tilde{m}_i$. Identify them and call the result $\tilde{m}$. Now, $\tilde{\Phi}$ induces a map $\tilde{l} \times \tilde{m} \longrightarrow M$, where

$$\tilde{l} = \coprod_{i=1}^{m} \tilde{l}_i$$

We continue to denote this map by $\tilde{\Phi}$. Clearly, we have

$$\tilde{\Phi}(\tilde{l} \times \{x\}) = \coprod_{i=1}^{m} \Phi_i(P_i \times \{q_i(x)\})$$

for every x on $\tilde{m}$. Notice that each point on the parameter space of Φ, for which $\Phi(P)$ is not an embedding, corresponds to $|a|$ points $x \in \tilde{m}$ for which $\tilde{\Phi}(\tilde{l} \times \{x\})$ is not an embedding. We observe that, because of (10), the quantity $X_{\partial\tilde{\Phi}}$ doesn't change if we replace the parameter space $\tilde{m}$, by $\tilde{\lambda}$. Now, the claim follows easily.

□

2.7. **Theorem 2.1 for atoroidal manifolds.** Before we can proceed with the proof of the theorem we need two additional lemmas.

Lemma 2.7. *Consider $\Phi, \Phi' : S^1 \longrightarrow \mathcal{M}^L(P, M)$ two* framing preserving *closed homotopies from L to itself. Suppose that Φ, Φ' are freely homotopic as loops in $\mathcal{M}^L(P, M)$. Then $X_\Phi = X_{\Phi'}$. Furthermore, there is a group homomorphism $\psi : \pi_1(\mathcal{M}^L(P, M),\ L) \longrightarrow \mathbb{A}$ defined by $\psi([\Phi]) = X_\Phi$.*

Proof. As in the proof Lemma 3.3.2 of [12] we show that there is a map $\hat{\Phi} : D^2 \longrightarrow \mathcal{M}^L(P, M)$ such that if we let $\Psi := \hat{\Phi}|D^2$, then $\Psi : S^1 \longrightarrow \mathcal{M}^L(P, M)$ is a *framing preserving* closed homotopy from L to itself with

$$X_\Psi = X_\Phi - X_{\Phi'}.$$

Then Lemma 2.5 implies $X_\Psi = 0$ and thus $X_\Phi = X_{\Phi'}$. Next observe that $\pi_1(\mathcal{M}^L(P, M),\ L)$ can be represented by *framing preserving* a closed homotopy from L to itself. For, suppose that Φ is a closed homotopy from L to itself as an unframed link: That is the links $L := \phi_0(P)$ and $L_1 = \phi_1(P)$ have different framings. We can adjust $\phi_1(P)$ by a sequence of inadmissible crossing changes so that it achieves the same framing as L. This changes Φ to a framing preserving closed homotopy that represents the same element of $\pi_1(\mathcal{M}^L(P, M),\ L)$. Now define $\psi : \pi_1(\mathcal{M}^L(P, M),\ L) \longrightarrow \mathbb{A}$ by $\psi(\alpha) = X_\Phi$, where Φ is a framing preserving homotopy representing the class $\alpha \in \pi_1(\mathcal{M}^L(P, M),\ L)$. It follows easily that ψ is a group homomorphism. □

The next lemma is proved as Lemma 3.2.5 in [12] where one replaces Lemma 3.3.2 of that proof by Lemma 2.7 above. We point out that the proof of this lemma uses the hypothesis that the group $\mathbb{A}$ in which the invariants take values is torsion free.

Lemma 2.8. *Suppose that M is a $\mathbb{Q}$-homology 3-sphere, with $\pi_2(M) = 0$. Let $L : P \longrightarrow M$ be a framed link and let $\Phi : P \times S^1 \longrightarrow M$ be a* framing preserving *closed homotopy from L to itself. Moreover, assume that for some $1 \leq i \leq m$, we have $a_i = 1$. Let $P' := P \smallsetminus P_i$ and let Φ' denote the restriction of Φ on P'. If $X_{\Phi'} = 0$, then $X_\Phi = 0$.*

We are now ready to give the proof of Theorem 2.1 in the case where M is an atoroidal $\mathbb{Q}$-homology 3-sphere.

Theorem 2.9. *Suppose that M is a $\mathbb{Q}$-homology 3-sphere, with $\pi_2(M) = 0$. Then the conclusion of Theorem 2.1 is true for M.*

Proof. As before, P is a disjoint union of oriented circles and

$$\Phi : P \times S^1 \longrightarrow M$$

is a closed framing preserving homotopy from some link $L : P \longrightarrow M$ to itself. Let $f : \mathcal{L}^1 \longrightarrow \mathbb{A}$ be a singular link invariant satisfying (3) and (4) of the statement of Theorem 2.1. We have to show that

$$X_\Phi = 0$$

where X_Φ is the signed sum of values of f around Φ defined in (7).

First suppose that $\pi_1(M)$ is finite. By Lemma 2.4, the fundamental group $\pi_1(\mathcal{M}^L(P, M),\ L)$ is finite and, by Lemma 2.7, there is a group homomorphism $\psi : \pi_1(\mathcal{M}^L(P, M),\ L) \longrightarrow \mathbb{A}$, with $\psi([\Phi]) = X_\Phi$. Since $\mathbb{A}$ is torsion free ψ is the trivial homomorphism and thus $X_\Phi = 0$.

Now suppose that $\pi_1(M)$ is infinite. If the link L to begin with contains no homotopically trivial components, then since we assumed that M is atoroidal, Lemma 3.4 applies to conclude that $X_{\partial\tilde{\Phi}} = aX_\Phi$, where $\tilde{\Phi} : P \times D^2 \longrightarrow M$. By Lemma 2.5, $X_{\partial\tilde{\Phi}} = aX_\Phi = 0$ and thus, since $\mathbb{A}$ is torsion free, $X_\Phi = 0$.

Next suppose that all the components of L are homotopically trivial; that is $a_i = 1$, for $i = 1, \ldots, m$. Then, by Lemma 2.4,

$$\pi_1(\mathcal{M}^L(P, M),\ L) \cong \oplus_i^m \pi_1(M, L(p_i).$$

Since $H_1(M)$ is finite, the above equality implies that the abelianization of $\pi_1(\mathcal{M}^L(P, M),\ L)$ is a finite group. By Lemma 2.7 we have a group homomorphism $\psi : \pi_1(\mathcal{M}^L(P, M),\ L) \longrightarrow \mathbb{A}$, with $\psi([\Phi]) = X_\Phi$. Since $\mathbb{A}$ is abelian, ψ factors through the abelianization of $\pi_1(\mathcal{M}^L(P, M),\ L)$; a finite group. But since $\mathbb{A}$ is torsion free, ψ is the trivial homomorphism. This $X_\Phi = 0$ for all framing preserving loops Φ in $\mathcal{M}^L(P, M)$.

To handle the general case let $i(L)$ denote the number of components of L that are homotopically trivial. The proof is by induction on $i(L)$. In the light of our discussion above the conclusion is true if $i(L) = 0$ or $i(L) = m$. Thus we may assume that $i(L) \neq 0, m$. Let $L_i \subset L$ be a component that is homotopically trivial and let $L' := L \setminus L_i$. Also let Φ be a framing preserving closed homotopy from L to itself and let Φ' denote the restriction of Φ on P', where $P' := P \setminus P_i$. Since $i(L') < i(L)$, by induction, $X_{\Phi'} = 0$. Then, by Lemma 2.8, $X_\Phi = 0$. □

3. Integration of invariants in toroidal 3-manifolds

To study the question of integrability of singular link invariants in *toroidal* 3-manifolds we need several results from the theory of *characteristic sub manifolds* of Jaco-Shalen [9] and Johannson [10]. The statements of the results from these theories, in the form needed in our setting, are summarized in Section 2 of [11] and in Section 2 of [12]. It will be convenient for us to recall the statements we need below from therein, instead from the original references. In particular we will need the *Enclosing Theorem* and the *Torus Theorem* both stated on pp. 679 of [11]. The later, in the form needed for our purposes, is due to Scott, Casson-Jungreis and Gabai. The proof of Theorem 2.1 will be completed once we show the following.

Theorem 3.1. *Suppose that M is a $\mathbb{Z}$-homology sphere with $\pi_2(M) = 0$. Let $f : \mathcal{L}^{(1)} \longrightarrow \mathbb{A}$ be an invariant of singular links with one double point. Suppose that $\mathbb{A}$ is torsion free and that the invariant f satisfies (3) and (4) of Theorem 2.1 Then there exists a framed link invariant F such that f is derived from F via equation (2).*

Remark 3.2. The restriction to $\mathbb{Z}$-homology spheres is necessary in Theorem 3.1. As explained in Remark 3.13 of [11] and the discussion at end of Section 3 in [12], in general, "local conditions" are not sufficient for the integration of singular link invariants. When the characteristic sub manifold contains Seifert fibered components over non-orientable surfaces one needs to impose extra "non-local" conditions. Specific constructions demonstrating these phenomena are given by Kirk and Livingston in [15]. One can still obtain results analogous to Theorem 2.1 in these cases (cf Theorem 3.4.1 of [12]) ; for reasons of simplicity and clarity we've chosen not to discuss these cases here.

For the proof of Theorem 3.1 we will need the following.

Lemma 3.3. *Let M be a $\mathbb{Z}$-homology 3-sphere with $\pi_2(M) = 0$. Suppose that $\Phi : T = S^1 \times S^1 \longrightarrow M$ is an essential map. Then, there exists a map $\Phi_1 : T \longrightarrow M$ which is homotopic to Φ and such that one of the following holds:*
(i) $\Phi_1(T)$ lies on an essential embedded torus in M.
(ii) There exists an oriented surface F with $\partial F \neq \emptyset$, such Φ_1 can be extended to a trivial fiber bundle $\hat{\Phi} : S^1 \times F \longrightarrow M$ so that the image $\hat{\Phi}(\partial F \setminus T)$ is contained on a collection of embedded tori in M.

Proof. By the Torus Theorem, and the discussion at the end of Section 2 of [12], either M is Haken or it is a Seifert fibered 3-manifolds that fibers over S^2 with three or less exceptional fibers.

First, suppose M is Haken. Then by the Enclosing Theorem, there is a Seifert fibered sub manifold $S \subset M$ and a homotopy $\Phi'_t : T \longrightarrow M$ such that $\Phi'_0 = \Phi$ and $\Phi'_1(T) \subset S$. If $\Phi'_1(T)$ can be further homotoped in S so that it lies on a component of ∂S then we have the conclusion of (i) above. Otherwise, by the classification of essential tori in Haken Seifert fibered spaces (Proposition 2.11 of [11]) we can homotope Φ'_1 in S to a map $\Phi_1 : T \longrightarrow S$ which is *vertical* with respect to the fibration.

Next suppose that M is a Seifert fibered space. By Proposition 2.2.5 of [12] , Φ is homotopic to a map $\Phi_1 : T \longrightarrow M$ which is vertical with respect to the fibration of M.

Thus, in both cases, either (i) holds or we have a Seifert fibered manifold $S \subseteq M$ such that Φ is homotopic to a map $\Phi_1 : T \longrightarrow M$ that is vertical with respect to the fibration of S. This means that Φ_1 is a composition $\Psi \circ q$, where q is a covering map from the torus T to each self and $\Psi : T \longrightarrow S$ is an immersion without triple points. Then, there exists a decomposition $T = S^1 \times S^1$ such that

a) $\Psi(S^1 \times \{*\})$ maps onto a regular fiber h, of S

b) We have $p(\Psi(\{*\} \times S^1)) = p(T)$ on the orbit surface B of S.

Let H (respectively, Q) denote the curve $S^1 \times \{*\}$ (respectively, $\{*\} \times S^1$ on T. Now $\alpha := p(T)$ is an immersed closed curve on B with singularities finitely many transverse double points. A neighborhood N:=$N(\alpha) \subset B$, of α on B is an oriented planar surface. Choose N small enough so that $Y := p^{-1}(N)$ contains no exceptional fibers of S. Now $p : Y \longrightarrow N$ is an S_1-bundle and since $H^2(N) = 0$ this fiber bundle is trivial. Choose a trivialization $Y \cong S^1 \times N$ so that N is embedded as a cross section. Pick a base point $p \in N$ and arcs from p to the components of ∂N whose homotopy classes freely generate $\pi_1(N)$; we pick one arc for each such component. Assume that these arcs intersect α only at its double points; let $x_1, \ldots, x_s$ denote the resulting generators of $\pi_1(N)$. Write α as a word in these generators. Say

$$[\alpha] = x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}.$$

We extend $\Phi_1 | \{*\} \times S^1$ to $\hat{\Phi} : (F,\ \partial F) \longrightarrow (N,\ \partial N)$ where F is a planar surface such that: (i) $\pi_1(F)$ has a free generator for every copy of x_{i_j} in the word above; (ii) the induced map $\hat{\Phi}_* : \pi_1(F) \longrightarrow \pi_1(N)$ is onto; (iii) $\hat{\Phi}_*([Q]) = x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$. Now consider the pull-back $\hat{\Phi}^*(Y)$ of Y via $\hat{\Phi}$; by assumption the pull-back of the cross-section α is a cross section of the $\hat{\Phi}^*(Y)$. Extending this cross section over F we have the desired conclusion. $\square$

We now recall that the proof of Theorem 3.1 is reduced to showing (7) for every framing preserving loop $\Phi \subset \mathcal{M}^L(P, M)$ (where, the loop is viewed as a closed homotopy from the framed link L to itself.) The next Lemma is proved by the argument in the proofs of Lemma 3.3.3 and Theorem 3.12 of [12] where one replaces Lemma 3.3.2 of [12] by Lemma 3.3; we won't repeat the details here. We note that the hypothesis that $H_1(M) = 0$ implies that the orbit space of the characteristic sub manifold of M is oriented. Thus Lemma 3.3.1 of [12] applies. See remarks at the end of Section 3 in [12] or Remark 3.13 of [11] for discussion on these hypotheses. We should also remark that after Φ has been homotoped to a map $\Phi_1 : P \times S^1 \longrightarrow M$ that is vertical with respect to the fibration of s Seifert fibered sub manifold of M in Lemma 3.3 we may need to adjust by a number of inadmissible crossing changes so that Φ_1 is still framing preserving.

Lemma 3.4. *Suppose M is a $\mathbb{Z}$-homology sphere with $\pi_2(M) = 0$ and let $\Phi \subset \mathcal{M}^L(P, M)$ be a framing preserving loop. Suppose that L doesn't contain homotopically trivial components. Suppose moreover that if, for some $i = 1, \ldots, m$, $\Phi | P_i \times S^1 \longrightarrow M$ induces an injection on π_1 then it* *cannot be homotoped on an embedded torus in* M*. Then, there exists a map* $\tilde{\Psi} : P \times D^2 \longrightarrow M$ *such that* $X_{\partial\tilde{\Psi}} = aX_\Phi$*, for some* $a \in \mathbb{Z}$*. Here,* $\partial\tilde{\Psi} = \tilde{\Psi}|P \times \partial D^2$ *and* D^2 *is a 2-disc.*

3.1. **The completion of the proof of Theorem 2.1.** As before, P is a disjoint union of oriented circles and

$$\Phi : P \times S^1 \longrightarrow M$$

is a *framing preserving* closed homotopy from some link $L : P \longrightarrow M$ to itself. We have an invariant $f : \mathcal{L}^1 \longrightarrow \mathbb{A}$ as in the statement of Theorem 3.1. We have to show that

$$X_\Phi = 0$$

where X_Φ is the signed sum of values of f around Φ defined in (7).

Suppose that $\Phi|P_i \times S^1 \longrightarrow M$ represents an inessential torus for all $i = 1, \ldots, m$. Then, using the argument used in the case of *atoroidal* manifolds in § 2.7 we obtain $X_\Phi = 0$.

Proof. Now suppose that some component, say $\Phi_i := \Phi|P_i \times S^1 \longrightarrow M$, can be homotoped to lie on an embedded essential torus in M. Then a theorem of Nielsen ([8], theorem 13.1) implies that after further homotopy, we may assume that Φ_i is a covering map of an embedded torus. Then, using Lemma 2.7, we conclude that Φ_i doesn't contribute to X_Φ. Thus, for our purposes, we can assume that if a component Φ_i induces injection on π_1 then it cannot be homotoped to lie on an embedded torus.

Suppose, without loss of generality, that the components $L_i \subset L$ are homotopically trivial in M for all $i \leq s$, and homotopically essential for every $i = s+1, \ldots, m$. Let Φ' denote the restriction of Φ on the disjoint union of the components $P_i \times S^1$, $i = s+1, \ldots, m$. Then by Lemma 3.4 and Lemma 2.5 we have $aX_{\Phi'} = 0$. Since we assumed that $\mathbb{A}$ is torsion free, this implies that $X_{\Phi'} = 0$. Thus, in particular, if $s = 0$, we have $X_\Phi = 0$. Now inducting on s and using Lemmas 2.4 and 2.8 we obtain the desired result. □

4. An intrinsic definition of the Kauffman power series

For links in S^3 the Kauffman polynomial is equivalent to a sequence of 1-variable Laurent polynomials $\{R_n = R_n(t)\}_{n\in\mathbf{Z}}$. They are completely determined by the following skein relations:

$$R_n(U) = 1$$

$$R_n(L_r) = t^{-(n+1)} R_n(L)$$

$$R_n(L_l) = t^{(n+1)} R_n(L)$$

$$R_n(L_+) - R_n(L_-) = (t - t^{-1})[R_n(L_o) - R_n(L_\infty)]$$

where L_+, L_-, L_o, L_∞ as in Figure 1 and L_r, L_l are as in Figure 2.

Notice that the initial value $R_n(U) = 1$ is just a normalization. Any choice of the initial value together with the rest of the relations will determine a unique R_n.

Let
$$u_n(t) = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}} + 1.$$
By the relations above one obtains
$$R_n(L \coprod U) = u_n(t)\ R_n(L)$$
where the link $L \coprod U$ is obtained from L by adding an unknotted and unlinked component U. The coefficients of the power series $R_n(x)$, obtained from $R_n(t)$ by substituting $t = e^x$, are invariants of *finite type* [**?**]. In the theorem below we reverse this procedure, and guided by the axioms above we will construct inductively power series invariants for links in 3-manifolds generalizing the $R_n(x)$'s.

Assume that M is as in Theorem 2.1. For every $n \in \mathbb{Z}$, we will construct a sequence of knot invariants
$$v_n^0,\ v_n^1, \ldots,\ v_n^m, \ldots$$
such that the formal power series
$$R_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_n^m(L) x^m$$
satisfy the axioms above, under the change of variable $t = e^x$, for every $L \in \mathcal{L}$.

We will construct our invariants inductively (induction on m) by using Theorem 2.1. More precisely, each v_n^m is going to be obtained by integrating a suitable singular link invariant determined by the v_n^j's with $j < m$.

Recall that a link invariant obtained by integrating a singular link invariant is well defined up to a collection of "integral constants" . This means that in order to define $v_n^0, v_n^1, \ldots, v_n^m, \ldots$ uniquely, we need to make a choice of "trivial links".

Let L be an n-component link and recall from §2 that $\mathcal{M}^L$ denotes the space of maps $\coprod S^1 \longrightarrow M$ which are homotopic to L. The spaces $\mathcal{M}^L$ corresponding to links with n components are in one to one correspondence with the unordered n-tuples of conjugacy classes in $\pi = \pi_1(M)$. In every such space we will fix, once and for all, a link CL and call it a trivial link. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^*. Let $\mathcal{CL}$ be the set of all trivial links and $\mathcal{CL}^*$ be the set of trivial links with all of their components homotopically non-trivial.

Theorem 4.1. *Assume that M is an orientable, $\mathbb{Q}$-homology 3-sphere with $\pi_2(M) = 0$ and such that if $H_1(M, \mathbb{Z}) \neq 0$, then M is* atoroidal *. Let $\mathcal{L}$ and $\mathcal{CL}^*$ be as above. There exists a unique sequence of complex valued link invariants $v_n^0, v_n^1, \ldots, v_n^m, \ldots,$ with given values on the links in $\mathcal{CL}^* \cup \{U\}$, such that, for $L \in \mathcal{L}$, we have:*
(i) The values $v_n^i(L)$ are independent of the orientation of L.
(ii) If we define a formal power series

$$R_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_n^m(L) x^m$$

then we have

$$R_n(U) = 1 \tag{11}$$

$$R_n(L_r) = t^{-(n+1)} R_n(L) \tag{12}$$

$$R_n(L_l) = t^{(n+1)} R_n(L) \tag{13}$$

$$R_n(L_+) - R_n(L_-) = (t - t^{-1})[R_n(L_o) - R_n(L_\infty)] \tag{14}$$

where $t = e^x = 1 + x + x^2 + \ldots$

Notation: To simplify our notation, and throughout this proof, we will write R_n instead of $R_{\{M,n\}}$.

Proof. By our assumption, the values $v_n^0(CL^*), v_n^1(CL^*), \ldots, v_n^m(CL^*), \ldots$ are given for every $CL^* \in \mathcal{CL}^*$. Hence, we can form the power series $R_n(CL^*)$. Also, we may form $R_n(U)$ using the given values $v_n^m(U)$'s.

Guided by (12)-(13) we define

$$R_n(CL_r^*) = t^{-(n+1)} R_n(CL^*), \quad \text{and} \quad R_n(CL_l^*) = t^{(n+1)} R_n(CL^*). \tag{15}$$

Now guided by these we can define the values of R_n on all framed links whose underlying isotopy class is CL^*. To explain this suppose that CL^* has s components. If L represents the framed link obtained from CL^*, with framing $\mathbf{f} = (\mathbf{f_1}, \ldots, \mathbf{f_s})$ we will have

$$R_n(L(\mathbf{f})) = t^{(n+1)\tau} R_n(CL^*) \quad \text{where} \quad \tau := \sum_{i=1}^{s} \mathbf{f_i}.$$

Using (14)-(15), and inducting on k, we can check that

$$R_n(CL^* \coprod U^k) = [u_n(t)]^{k-1} \; R_n(CL^k) \tag{16}$$

where

$$u_n(t) = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}} + 1.$$

Thus, R_n has been defined on all framed trivial links.

For every framing $\mathbf{f} = (\mathbf{f_1}, \ldots, \mathbf{f_s})$ we define the link invariant v_n^0 by

$$v_n^0(L(\mathbf{f})) = v_n^0(CL(\mathbf{f})),$$

where CL is the trivial link homotopic to L.

Inductively, suppose that the invariants $v_n^0, v_n^1, \ldots, v_n^{m-1}$ have been defined such that if we let

$$R_n^{(m-1)}(L) := \sum_{i=1}^{m-1} v_n^i(L)x^i,$$

then we have

$$R_n^{(m-1)}(L_r) = t^{-(n+1)} R_n^{(m-1)}(L) \bmod x^m \tag{17}$$

$$R_n^{(m-1)}(L_l) = t^{(n+1)} R_n^{(m-1)}(L) \bmod x^m \tag{18}$$

$$R_n^{(m-1)}(L \coprod U) = u_n(t)\ R_n(L) \bmod x^m \tag{19}$$

and

$$R_n^{(m-1)}(L_+) - R_n^{(m-1)}(L_-) = (t - t^{-1})[R_n^{(m-1)}(L_o) - R_n^{(m-1)}(L_\infty)] \bmod x^m.$$

Furthermore, suppose that these invariants do not depend on the orientation of the links.

The last equation leads us to define

$$R_n^{(m)}(L_\times) := (t - t^{-1})[R_n^{(m-1)}(L_o) - R_n^{(m-1)}(L_\infty)] \bmod x^{m+1} \tag{20}$$

We want to define the invariant v_m^n : Recall that it is already defined on all the framed links $CL(\mathbf{f})$. Next we examine the right hand side of (20). It is a polynomial of degree m, with the coefficient of x^m coming from $(t - t^{-1})[R_n^{(m-1)}(L_o) - R_n^{(m-1)}(L_\infty)]$. This term has no constant term and thus the coefficient of x^m is derived from the inductively well defined invariants v_n^i, $i = 1, 2, \ldots, m-1$. Thus the coefficient of x^m in (20) is a "new" singular link invariant. We are going to prove that it is derived from a knot invariant by using Theorem 2.1. For that we need to check that the local conditions given in (3) and (4) are satisfied. It is enough to check them modulo x^{m+1}. In what follows the symbol "$\equiv$"' will denote calculation modulo x^{m+1}.

First we check condition (4): To that end, let $L_{\times+}$ and $L_{\times-} \in \mathcal{L}^{(1)}$ be two singular links as in the left hand side of (4). From (20) we have

$$\begin{aligned}
& R_n^{(m)}(L_{\times+}) - R_n^{(m)}(L_{\times-}) \equiv \\
\equiv\ & (t - t^{-1})\big[R_n^{(m-1)}(L_{o+}) - R_n^{(m-1)}(L_{\infty+}) + o(x^m)\big] - \\
-\ & (t - t^{-1})\ \big[R_n^{(m-1)}(L_{o-}) - R_n^{(m-1)}(L_{\infty-}) + o(x^m)\big] \equiv \\
\equiv\ & (t - t^{-1})\big[R_n^{(m-1)}(L_{o+}) - R_n^{(m-1)}(L_{o-}) + o(x^m)\big] - \\
-\ & (t - t^{-1})\big[R_n^{(m-1)}(L_{\infty+}) - R_n^{(m-1)}(L_{\infty-}) + o(x^m)\big] \equiv \\
\equiv\ & (t - t^{-1})^2\big[R_n^{(m-1)}(L_{oo}) - R_n^{(m-1)}(L_{o\infty})\big] - \\
-\ & (t - t^{-1})^2\big[R_n^{(m-1)}(L_{\infty o}) - R_n^{(m-1)}(L_{\infty\infty})\big] \equiv \\
\equiv\ & (t - t^{-1})^2\big[R_n^{(m-1)}(L_{oo}) + R_n^{(m-1)}(L_{\infty\infty})\big] - \\
-\ & (t - t^{-1})^2\big[R_n^{(m-1)}(L_{\infty o}) + R_n^{(m-1)}(L_{o\infty})\big].
\end{aligned}$$

Since the result is symmetric with respect to the two double points we deduce that

$$R_n^{(m)}(L_{\times +}) - R_n^{(m)}(L_{\times -}) \equiv R_n^{(m)}(L_{+\times}) - R_n^{(m)}(L_{-\times})$$

Before we can check local condition (3), we note that if we start with an inadmissible singular link $L^1 \in \mathcal{L}^1$ and we let L be the link obtained by cutting off the double point of L^1 and the disc containing it, then we have

$$\begin{aligned} & R_n^{(m)}(L^1) \equiv \\ \equiv\ & (t-t^{-1})\big[R_n^{(m-1)}(L \coprod U) - R_n^{(m-1)}(L_\infty) + o(x^m)\big] \equiv \\ \equiv\ & (t-t^{-1})\big[(u_n(t)R_n^{(m-1)}(L) - R_n^{(m-1)}(L_\infty)\big] \equiv \\ \equiv\ & (t-t^{-1})\big[(u_n(t)R_n^{(m-1)}(L) - (R_n^{(m-1)}(L) + o(x^m))\big] \equiv \\ \equiv\ & (t-t^{-1})\big[u_n(t) - 1\big]R_n^{(m-1)}(L) \equiv \\ \equiv\ & \big[t^{(n+1)} - t^{-(n+1)}\big]R_n^{(m-1)}(L). \end{aligned}$$

Here, to obtain the third equivalence we use the fact that L and L_∞ are isotopic links and thus by the induction hypothesis we have

$$R_n^{(m-1)}(L) \equiv R_n^{(m-1)}(L_\infty) \bmod x^m.$$

Next we will check (3): Let $L_{\times r}$ and $L_{r\times} \in \mathcal{L}^{(1)}$ that differ only locally as shown in Figure 3. By (17) and (20) we have

$$\begin{aligned} & R_n^{(m)}(L_{\times r}) - R_n^{(m)}(L_{r\times}) \equiv \\ \equiv\ & (t^{(n+1)} - t^{-(n+1)})R_n^{(m)}(L_r) - (t^{(n+1)} - t^{-(n+1)})R_n^{(m)}(L_r) \equiv \\ \equiv\ & 0 \end{aligned}$$

Thus, the singular link invariant defined above is induced by a link invariant. Using the given values $\{v_n^m(CL) :\ CL \in \mathbb{C}L\}$, we can define a link invariant v_n^m, such that if we let

$$R_n^{(m)}(L) = \sum_{i=1}^{m} v_n^m(L)x^i$$

we have

$$R_n^{(m)}(L_+) - R_n^{(m)}(L_-) = R_n^{(m)}(L_\times)$$

for $L \in \mathcal{L}$ and $L_\times \in \mathcal{L}^1$.

Now its easy to check (18)-(19) mod x^{m+1}.

To finish our proof we need to show uniqueness. Inductively, we assume that $v_n^0, v_n^1, \dots, v_n^{m-1}$ are uniquely determined by their values on $\mathcal{C}L$, for every $n \in \mathbb{Z}$. Then, the conclusion for v_n^m follows from the fact that

$$v_n^m(L) = v_n^m(CL) + \sum_{i=1}^{s} \pm v_n^m(L_i)$$

where $L_1, \dots, L_s$ are singular links in $\mathcal{L}^1$, and CL is the representative of L in $\mathcal{C}L$. The non-dependence of link orientation follows similarly. □

Remark 4.2. To obtain Theorem 1.1 from Theorem 4.1 we set $z := t - t^{-1} = e^x - e^{-x}$ and $a := e^y$, where $y = (n+1)x$.

5. Concluding comments and questions

5.1. Links in S^3 are studied via projections on a sphere $S^2 \subset S^3$. Let U^m denote the m-component unlink and $U^m(\mathbf{f})$ denote then any m-component link projection $L \subset S^2$ is transformed to framed unlink by a finitely many crossing changes and *regular* isotopy moves on S^2 (i.e. isotopy using the Reidemeister moves of type II and III only). For a link projection $L \subset S^2$, we define a complexity

$$s(L) := (u(L),\ c(L)),$$

where $u(L)$ is the minimum number of crossing changes required to transform L into a framed unlink and $c(L)$ is the number of crossings of L. We order the complexities lexicographically. Let $R := R_{S^3} : \mathcal{L} \to \hat{\Lambda}$ the invariant of Theorem 1.1.

Proposition 5.1. *Define* $R(U(\mathbf{f})) = a^{-\tau}(a - a^{-1})z^{-1} + 1$, *where* $\tau := \sum_{i=1}^{m} \mathbf{f_i}$. *Then,* $R(L) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ *for every for every link. In fact,* $R(L)$ *is the two variable Kauffman polynomial.*

Proof. Note that the complexity $s(L)$ defined above has the properties that $s(L_r), s(L_i) < s(L)$ and from $s(L_-), s(L_o), s(L_\infty), s(L_+)$ always three terms are strictly less that the remaining fourth one.

Thus, the skein relations of Theorem 1.1

$$R(L_+) - R(L_-) = z\big[R(L_o) - R(L_\infty)\big],$$

$$R(L_r) = aR(L) \quad \text{and} \quad R(L_l) = a^{-1}R(L)$$

allow us to write the invariant $R(L)$ of every link L as a finite sum of the invariants of links of strictly less complexity that $s(L)$ and with coefficients in $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$. Now the desired conclusion follows by induction on $s(L)$ and the observation that $R(U(\mathbf{f})) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$. □

5.2. For an oriented 3-manifold M one can consider link projections a page of an *open book decomposition* of $M$and there are Reidemeister moves relating projections that represent isotopic links [19]. It would be interesting to study whether a complexity function, analogous to this described in §4.1 can be found to attack Question 1.2

5.3. In [5] Cornwell studies oriented links in Lens spaces through their *grid diagrams* on the standard genus one Heegaard surface of the space as studied in in [2]. He develops a skein theory that allows him to prove that the HOMFLY power series of [12] can be normalized to become Laurent polynomials. Then he uses these polynomials to obtain upper bounds for the Thurston-Bennequin norm of Legendrian links in Lens spaces [6] analogous to those obtained for links in S^3 in [20]. It is reasonable to expect that Cornwell's methods can be adapted in the non-oriented link setting to study the invariants of this paper for links in Lens spaces. These could lead to new applications to contact geometry in Lens spaces generalizing those of [18] and [7].

5.4. Recall that an invariant f, is called of finite type if there exists an integer k (the type of f) , such that the singular link invariant derived from f is zero on all singular links with more than k double points. One can see that the invariants $v_n^0,\ v_n^1, \dots\ v_n^m, \dots$ constructed above are of finite type; in fact the invariants v_n^k are of type k for all $n > 0$. It would be interesting to find a direct relation of the invariants constructed here with these coming from the $SO(n)$-perturbative Chern-Simons theory in the sense of [3], [23].

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Mathematics Department, Michigan State University, E. Lansing, MI 48824, USA

E-mail address: kalfagia@math.msu.edu