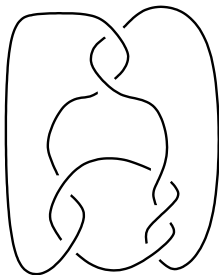


# Polyhedral decompositions, essential surfaces and colored Jones knot polynomials

joint with D. Futer and J. Purcell

November 4, 2011

# Given: Diagram of a knot or link



= 4-valent graph with over/under crossing info at each vertex.

## Quantum Topology

- Knot invariants esp. colored Jones polynomials

## Geometric topology

- Incompressible surfaces in knot complements
- Geometric structures and data esp. hyperbolic geometry and volume

Long term goal: Develop a setting to study both sides and relate them.

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...
  - Ribbon graphs to Jones polynomials...

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...
  - Ribbon graphs to Jones polynomials...
- Applications:
  - Give diagrammatic conditions for state surface incompressibility.
  - Understand JSJ-decompositions of surface complements... emphasis on “Guts” and volume estimates...

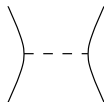
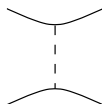
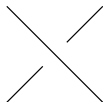
- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...
  - Ribbon graphs to Jones polynomials...
- Applications:
  - Give diagrammatic conditions for state surface incompressibility.
  - Understand JSJ-decompositions of surface complements... emphasis on “Guts” and volume estimates...
- Colored Jones polynomial (CPJ) relations:
  - Boundary slopes relate to degrees of CJP.

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...
  - Ribbon graphs to Jones polynomials...
- Applications:
  - Give diagrammatic conditions for state surface incompressibility.
  - Understand JSJ-decompositions of surface complements... emphasis on “Guts” and volume estimates...
- Colored Jones polynomial (CPJ) relations:
  - Boundary slopes relate to degrees of CJP.
  - Coefficients measure how far certain surfaces are from being fibers

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...
  - Ribbon graphs to Jones polynomials...
- Applications:
  - Give diagrammatic conditions for state surface incompressibility.
  - Understand JSJ-decompositions of surface complements... emphasis on “Guts” and volume estimates...
- Colored Jones polynomial (CPJ) relations:
  - Boundary slopes relate to degrees of CJP.
  - Coefficients measure how far certain surfaces are from being fibers
  - Guts  $\rightarrow$  relate CJP and volume of hyperbolic knots.

# Graphs

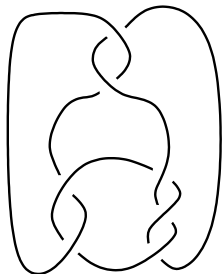
Two choices for each crossing,  $A$  or  $B$  resolution.



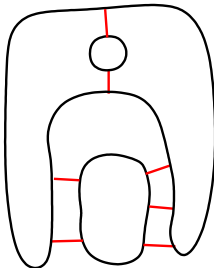
- Choice of  $A$  or  $B$  resolutions for all crossings: *state*  $\sigma$ .
- Result: Planar link without crossings. Components: *state circles*.
- Form a **graph** by adding edges at resolved crossings. Call this graph  $H_\sigma$ .  
( Note:  $n$  crossings  $\rightarrow 2^n$  state graphs)

# Example states

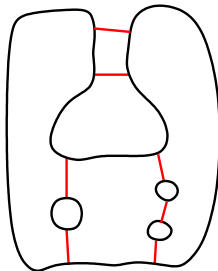
Link diagram



All A state



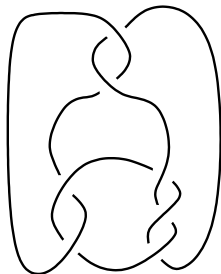
All B state



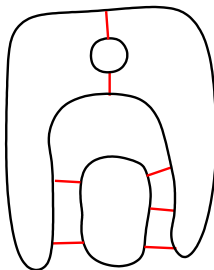
Above:  $H_A$  and  $H_B$ .

# Example states

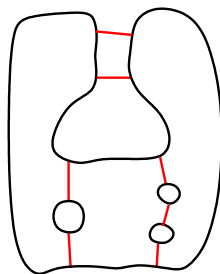
Link diagram



All A state



All B state



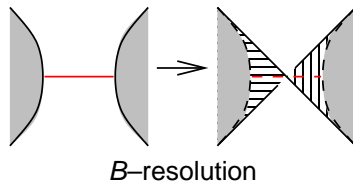
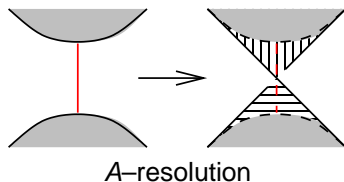
Above:  $H_A$  and  $H_B$ .

- The colored Jones polynomials of the knot can be calculated from  $H_A$  or  $H_B$ : *spanning graph expansion* arising from the Bollobas-Riordan *ribbon graph* polynomial (Dasbach-F-K-Lin-Stoltzfus , 2006).

# State surface

Using graph  $H_\sigma$  and link diagram, form the *state surface*  $S_\sigma$ .

- Each state circle bounds a disk in  $S_\sigma$  (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



# Example state surfaces

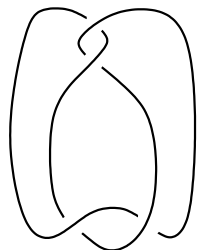
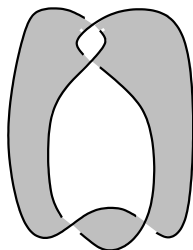
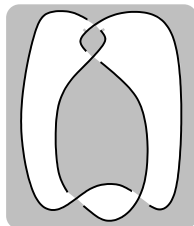


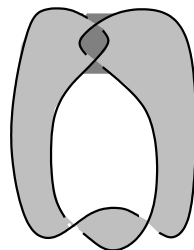
Fig-8 knot



$S_A$



$S_B$



Seifert surface

- For alternating knots:  $S_A$  and  $S_B$  are checkerboard surfaces.

# Example state surfaces

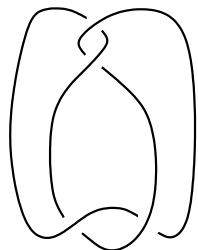
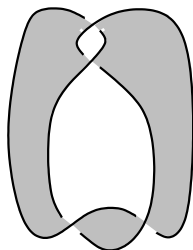
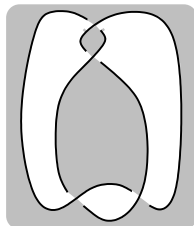


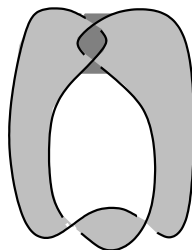
Fig-8 knot



$S_A$



$S_B$

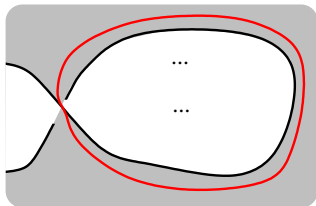


Seifert surface

- For alternating knots:  $S_A$  and  $S_B$  are checkerboard surfaces.
- For alternating knots  $S_A$  and  $S_B$  are *essential*: incompressible,  $\partial$ -incompressible (Menasco-Thistlethwaite, Lackenby)

# When are state surfaces incompressible?

Not always: If  $H_A$  has edge with both endpoints on a single state circle, then we form boundary compression disk:



That's the only thing that can go wrong.

## Theorem

(Ozawa, Futer-K-Purcell) *The following are equivalent:*

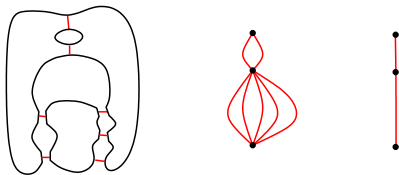
- $H_A$  has no edge with both endpoints on a single state circle
- $S_A$  is incompressible and boundary incompressible.

Ozawa proof is different; uses Murasugi sum arguments.— We see more about  $S_A$ .

For example: When is  $S_A$  a fiber in the complement of  $K = \partial S_A$ ?

Recall the graph  $H_A$ .

- Collapse each state circle of  $H_A$  to a vertex to obtain the state graph  $G_A$ .
- Remove redundant edges. Result is a graph  $G'_A$ .



## Theorem

(FKP) The complement  $S^3 \setminus K$  fibers over  $S^1$  with fiber  $S_A$  if and only if the reduced graph  $G'_A$  is a tree.

**Exercise:** Derive Stallings's classical result: Positive closed braids are fibered.

# Geometry and topology of the state surface complement

Take JSJ-decomposition of  $S^3 \setminus S_A$ . There are no essential tori. Cut along essential annuli into components:

- 1 Solid tori,
- 2  $I$ -bundles over surfaces,
- 3 *Simple* pieces (admitting complete hyperbolic metric).

The union of all the hyperbolic pieces is the *guts*:  $\text{Guts}(S^3 \setminus S_A)$ .

- $\text{Guts}(S^3 \setminus S_A) = \emptyset \Leftrightarrow S^3 \setminus S_A$  is a union of  $I$ -bundles and solid tori (i.e. *a book of  $I$ -bundles*).
- $\chi(\text{Guts}(S^3 \setminus S_A))$  measures how far  $S_A$  is from being “fiber-like” (*a fibroid*).

Notation: Set  $\chi_-(Y) = \max\{-\chi(Y_i), 0\}$ , for  $Y$  =connected cell complex. For non-connected  $Y$  sum  $\chi_-$ 's over connected components.

## Theorem

Let  $D(K)$  be a diagram such that  $H_A$  has no edge with both endpoints on a single state circle, and let  $S_A$  be the essential spanning surface determined by this diagram. Then

$$\chi_-(Guts(S^3 \setminus S_A)) = \chi_-(G'_A) - \|E_c\|,$$

where  $\|E_c\| \geq 0$  is a diagrammatic quantity.

In several instances  $\|E_c\| = 0$ . Examples:

- 1 Alternating links,
- 2 Montesinos Links,
- 3 Closures of positive braids where each exponent is at least 3.

## Guts relates to volume:

### Theorem (Agol–Storm–W. Thurston 2005)

For  $K$  hyperbolic

$$\text{Vol}(S^3 \setminus K) \geq v_8 \chi_-(\text{Guts}(S^3 \setminus S_A)),$$

here  $v_8 \approx 3.66\dots$  is the volume of a regular ideal octahedron.

### Corollary

Let  $D = D(K)$  be a prime  $A$ -adequate diagram of a hyperbolic link  $K$ . Then

$$\text{Vol}(S^3 \setminus K) \geq v_8 (\chi_-(\mathbb{G}'_A) - \|E_c\|).$$

Application example: Volume and twist number

### Theorem (Lackenby, 2005)

Let  $D$  be a reduced alternating diagram of a hyperbolic link  $K$ . Then

$$\frac{v_8}{2} (t(D) - 1) \leq \text{Vol}(S^3 \setminus K) < 10v_3 (t(D) - 1),$$

where  $v_3 = 1.0149\dots$  is the volume of a regular ideal tetrahedron.

We extended the list of manifolds for which we can compute explicitly the Euler characteristic of the guts and can be used to derive results analogous Lackenby's. Samples:

## Theorem

*Let  $D(K)$  be a diagram of a hyperbolic link  $K$ , obtained as the closure of a positive braid with at least three crossings in each twist region. Then*

$$\frac{2v_8}{3} t(D) \leq \text{Vol}(S^3 \setminus K) < 10v_3(t(D) - 1).$$

In this case  $\|E_c\| = 0$ . Similar results for: Montesinos links, Conway sums of alternating tangles...

# Colored Jones polynomials

For a knot  $K$  we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{n-1} + \cdots + \beta'_n t^{m+1} + \alpha'_n t^{k_n}.$$

- Some properties:
  - $J_{K,n}(t)$  is determined by the Jones polynomials of certain cables of  $K$ .
  - The sequence  $\{J_{K,n}(t)\}_n$  is *q-holonomic*: for every knot the CJP's satisfy linear recursion relations (Garoufalidis-Le , 2004). Then for every  $K$ 
    - 1 Degrees  $m_n, k_n$  are quadratic (quasi)-polynomials in  $n$
    - 2 Coefficients  $\alpha_n, \beta_n \dots$  satisfy recursive relations in  $n$ .

# Colored Jones polynomials

For a knot  $K$  we write its  *$n$ -colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{n-1} + \cdots + \beta'_n t^{m+1} + \alpha'_n t^{k_n}.$$

- Some properties:
  - $J_{K,n}(t)$  is determined by the Jones polynomials of certain cables of  $K$ .
  - The sequence  $\{J_{K,n}(t)\}_n$  is  *$q$ -holonomic*: for every knot the CJP's satisfy linear recursion relations (Garoufalidis-Le , 2004). Then for every  $K$ 
    - 1 Degrees  $m_n, k_n$  are quadratic (quasi)-polynomials in  $n$
    - 2 Coefficients  $\alpha_n, \beta_n \dots$  satisfy recursive relations in  $n$ .
- Properties manifest themselves in strong forms for knots with *state graphs* that have **no edge with both endpoints on a single state circle!**

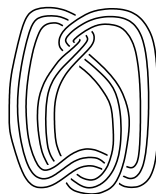
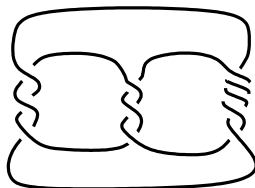
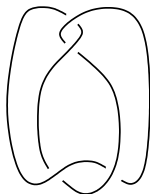
# Adequate links

Lickorish–Thistlethwaite 1987: Introduced  $A$ -adequate links ( $B$ -adequate links) in the context of Jones polynomials.

## Definition

A link is  $A$ -adequate if has a diagram with its graph  $H_A$  has no edge with both endpoints on the same state circle.

$A$  or  $B$ -adequate: all alternating knots, Montesinos knots, positive braids, negative braids, “most” arborescent knots, blackboard cables of adequate knots, “most” knots on tables up to 15 crossings.



# Jones polynomials and adequate links

Properties of interest:

- 1 The Jones polynomial detects the unknot within the class of  $A$ -adequate knots.
- 2 coefficients  $|\alpha'_n| = 1$  are independent of  $n$ :  $\alpha'_K := |\alpha_n|$ .
- 3  $\min \deg J_{K,n}(t)$  quadratic polynomial in  $n$ ; can be calculated explicitly.
- 4 (Dasbach-Lin) Coefficients  $|\beta'_n|$  are independent of  $n$ :  
 $\beta'_K := |\beta_n| = 1 - \chi(\mathbb{G}'_A)$ .

# Jones polynomials and adequate links

Properties of interest:

- 1 The Jones polynomial detects the unknot within the class of  $A$ -adequate knots.
- 2 coefficients  $|\alpha'_n| = 1$  are independent of  $n$ :  $\alpha'_K := |\alpha_n|$ .
- 3  $\min \deg J_{K,n}(t)$  quadratic polynomial in  $n$ ; can be calculated explicitly.
- 4 (Dasbach-Lin) Coefficients  $|\beta'_n|$  are independent of  $n$ :  
 $\beta'_K := |\beta_n| = 1 - \chi(\mathbb{G}'_A)$ .

We have analogous properties for  $B$ -adequate.

# Restate Theorems proved earlier:

- 1 Diagram is  $A$ -adequate  $\Leftrightarrow S_A$  incompressible and boundary incompressible.
- 2  $S_A$ =state surface corresponding to  $A$ -adequate diagram of  $K$ . The complement  $S^3 \setminus K$  fibers over  $S^1$  with fiber  $S_A \Leftrightarrow \beta'_K = 0$
- 3  $\beta'_K = 1 \Leftrightarrow S_A$  is a fibroid (**but not a fiber!**) with  $\chi(S^3 \setminus S_A) = \chi(G_A) - \chi(G'_A)$ .
- 4 In general,  $\beta'_K$  measures distance of  $S_A$  from being fiber.

$$\beta'_K - 1 = \chi_-(\text{Guts}(S^3 \setminus S_A)) + \|E_c\|.$$

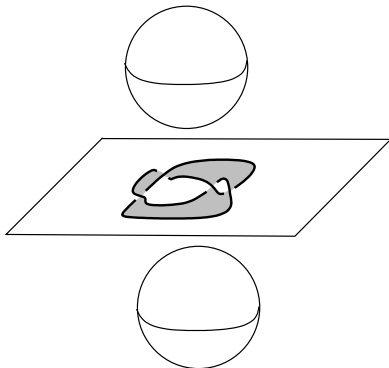
- 5 Volume estimates of hyperbolic knots in terms of coefficients of CJP.

$$\lim_{n \rightarrow \infty} \frac{\min \text{degree of } J_K^n(t)}{n^2} = \text{slope of } S_A$$

- 6 Growth rate of degree of CJP =boundary slope of  $S_A$
- 7 Relations of Jones polynomial and volume: old and new

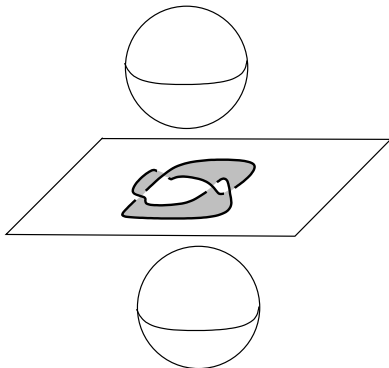
# Polyhedral decomposition prototype

Menasco (1984): Expand balloons above and below 2-sphere of alternating projection, obtain polyhedral decomposition of link complement (two 3-cells).



# Polyhedral decomposition prototype

Menasco (1984): Expand balloons above and below 2-sphere of alternating projection, obtain polyhedral decomposition of link complement (two 3-cells).

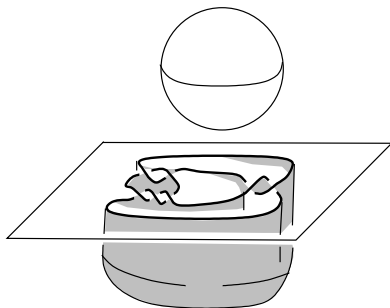


For **alternating** knots this gives polyhedral decomposition of checkerboard surface complement.— This is the picture we seek to generalize to **all** knots.

# Polyhedral decomposition of surface complement

## General case:

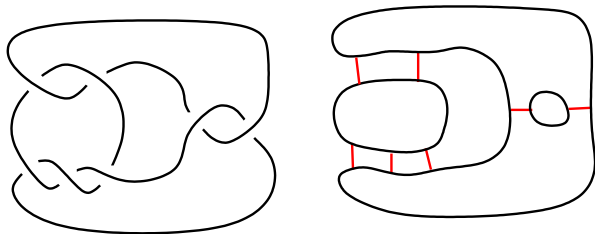
$S_A$  (or  $S_B$ ) hangs below plane of projection. Need more balloons.



# Polyhedral decomposition of complement of $S_A$

## 3-cells:

- One “upper” 3-cell, on top of plane of projection.
- One “lower” 3-cell for each nontrivial component of complement of state circles in  $A$ -resolution.

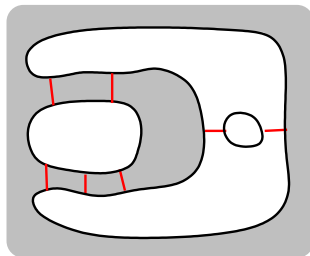
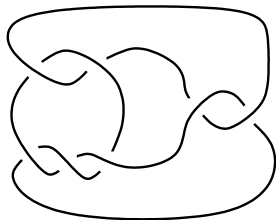


Two nontrivial components

# Polyhedral decomposition of complement of $S_A$

## 3-cells:

- One “upper” 3-cell, on top of plane of projection.
- One “lower” 3-cell for each nontrivial component of complement of state circles in  $A$ -resolution.

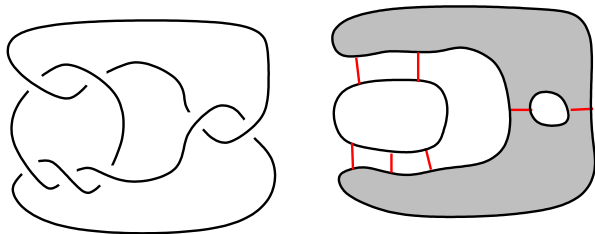


Two nontrivial components

# Polyhedral decomposition of complement of $S_A$

## 3-cells:

- One “upper” 3-cell, on top of plane of projection.
- One “lower” 3-cell for each nontrivial component of complement of state circles in  $A$ -resolution.

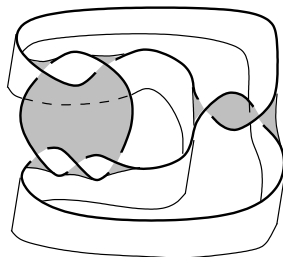
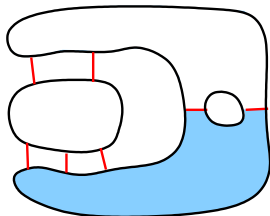


Two nontrivial components

# Polyhedral decomposition of complement of $S_A$ , continued

## “Faces”:

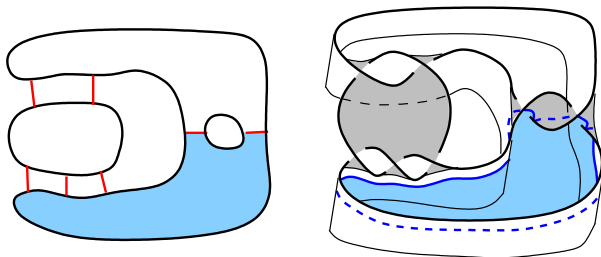
- Portions of 3–cell meeting  $S_A$ . Shade these.
- Disks lying slightly below plane of projection, with boundary on  $S_A$ .
  - One disk for each region of graph  $H_A$ .



# Polyhedral decomposition of complement of $S_A$ , continued

## “Faces”:

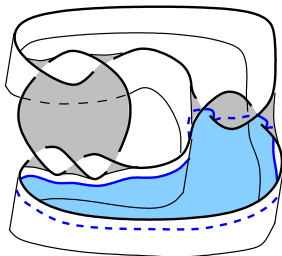
- Portions of 3–cell meeting  $S_A$ . Shade these.
- Disks lying slightly below plane of projection, with boundary on  $S_A$ .
  - One disk for each region of graph  $H_A$ .



# Polyhedral decomposition of complement of $S_A$ , continued

## Ideal edges:

- Run from undercrossing to undercrossing, adjacent to region of  $H_A$ .

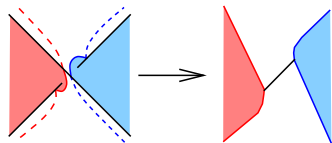


## Ideal vertices:

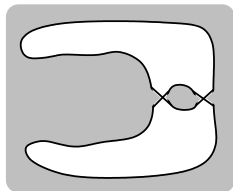
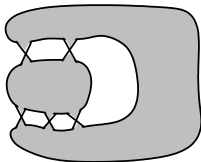
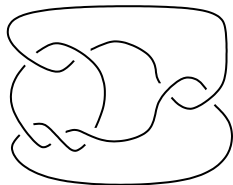
- On the link. Portions of the link visible from inside the 3-cell.

# Combinatorics of lower polyhedra:

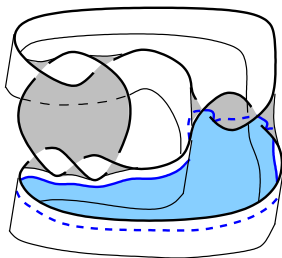
Ideal edges lie below plane of projection, so cut off view of link from below *except* at an undercrossing.



Result: Polyhedron is identical to checkerboard polyhedron of alternating sublink.



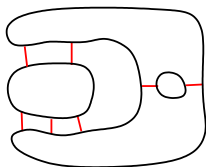
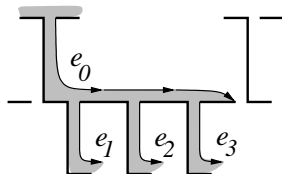
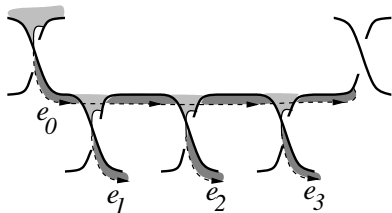
# Combinatorics of upper polyhedron:



- “**Faces**”: **Shaded “faces”** contain innermost disks, **White faces** correspond to regions of  $H_A$ .
- **Ideal edges** start and end at undercrossings, stay adjacent to single region of  $H_A$ .
- **Ideal vertices** are connected components of overcrossings = **diagram components** in usual diagram of link (with breaks at undercrossings).

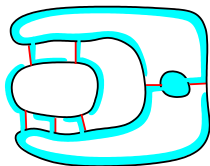
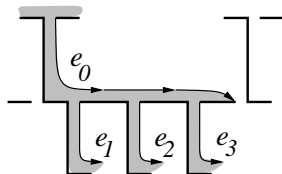
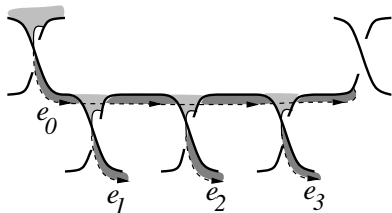
# Combinatorics of upper polyhedron, continued

- Sketch ideal edges onto usual projection of link diagram, or onto  $H_A$ .
- Edges bound *white disks*, *shaded "faces"*.
- Shaded faces: *Innermost disks*, along with *tentacles* adjacent to ideal edges.



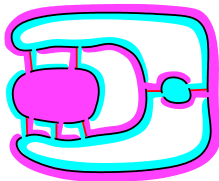
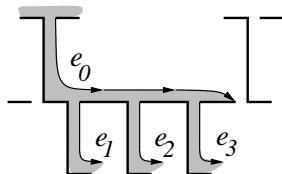
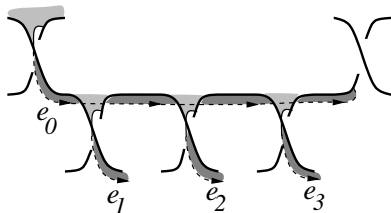
# Combinatorics of upper polyhedron, continued

- Sketch ideal edges onto usual projection of link diagram, or onto  $H_A$ .
- Edges bound *white disks*, *shaded "faces"*.
- Shaded "faces": *Innermost disks*, along with *tentacles* adjacent to ideal edges.



# Combinatorics of upper polyhedron, continued

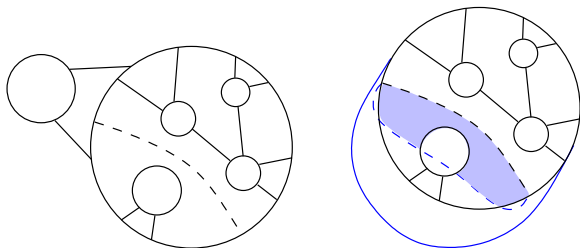
- Sketch ideal edges onto usual projection of link diagram, or onto  $H_A$ .
- Edges bound *white disks*, *shaded faces*.
- Shaded faces: *Innermost disks*, along with *tentacles* adjacent to ideal edges.



# One additional issue

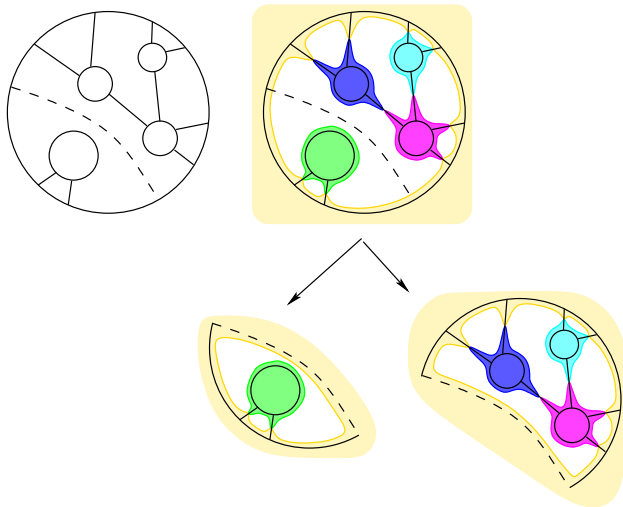
Lower polyhedra may not give *prime* alternating links.

**Example:**

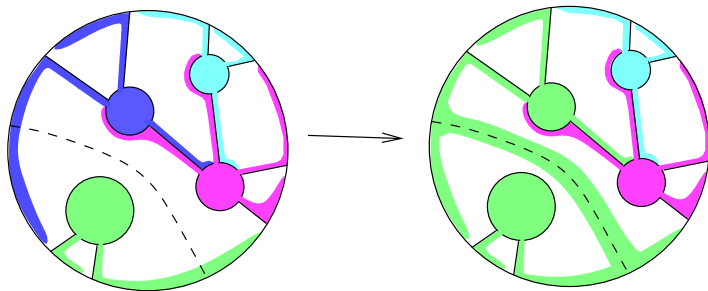


- See bigon in polyhedral decomposition.
- **Fix:** Modify polyhedra — surger along bigon.
  - Splits 3–cell into two.
  - Splits white disk into two.
  - In upper polyhedron: Connects two shaded “faces” along arc.

# Example: Lower polyhedron splits in two



# Example: Upper polyhedron



Generic form of Upper polyhedron

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

## Properties of resulting decomposition:

- All faces are simply connected.

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

## Properties of resulting decomposition:

- All faces are simply connected.
- Checkerboard colored.

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

## Properties of resulting decomposition:

- All faces are simply connected.
- Checkerboard colored.
- Ideal vertices are 4-valent.

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

## Properties of resulting decomposition:

- All faces are simply connected.
- Checkerboard colored.
- Ideal vertices are 4-valent.
- Combinatorics determined by graph  $H_A$ .

# Summary: Ideal polyhedral decomposition

## Result:

The above procedure gives an ideal polyhedral decomposition of  $S^3 \setminus S_A$  if “faces” are simply connected. This happens when  $H_A$  has no edge with both endpoints on a single state circle!

## Properties of resulting decomposition:

- All faces are simply connected.
- Checkerboard colored.
- Ideal vertices are 4-valent.
- Combinatorics determined by graph  $H_A$ .

Use these polyhedra and **normal surface theory** to study the topology of  $S^3 \setminus S_A$ .

# Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

# Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

- 1 An arc on  $S_A$  and

# Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

- 1 An arc on  $S_A$  and
- 2 An arc on a white face.

# Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

- 1 An arc on  $S_A$  and
- 2 An arc on a white face.

## Proposition (FKP)

Under the above polyhedral decomposition, if the graph  $H_A$  has no edge with both endpoints on a single state circle, then there are no normal bigons in the polyhedra.

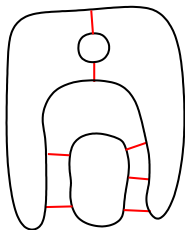
# Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

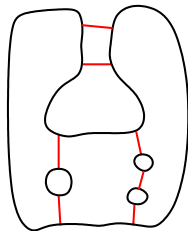
- 1 An arc on  $S_A$  and
- 2 An arc on a white face.

## Proposition (FKP)

Under the above polyhedral decomposition, if the graph  $H_A$  has no edge with both endpoints on a single state circle, then there are no normal bigons in the polyhedra.



Example

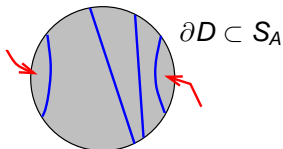


Non-example

# Application: $A$ -adequate $\Rightarrow S_A$ is essential

Suppose  $H_A$  has no such edge but is compressible or  $\partial$ -compressible: Put compressing disk, boundary compressing disk into **normal form**.

A compressing disk  $D$  for  $S_A$  would meet white faces of polyhedra in arcs. Outermost arc on  $D$  forms a **normal bigon**. Contradiction.



Normality implies boundary arc of boundary compressing disk  $E$  lies in a single polyhedron. Outermost intersection of  $E$  with white face cuts off a disk  $E'$  which cannot be a normal bigon, so contains boundary arc. But then  $E \setminus E'$  is a disk meeting white faces, obtain normal bigon. Contradiction.  $\square$

# A closer look at the topology of $S^3 \setminus S_A$

- Alexander duality:  $\chi(S^3 \setminus S_A) = \chi(S_A) = \chi(H_A)$

# A closer look at the topology of $S^3 \setminus S_A$

- Alexander duality:  $\chi(S^3 \setminus S_A) = \chi(S_A) = \chi(H_A)$
- $\chi(\text{Guts}(S_A)) + \chi(I\text{-bundle}) = \chi(S_A)$

# A closer look at the topology of $S^3 \setminus S_A$

- Alexander duality:  $\chi(S^3 \setminus S_A) = \chi(S_A) = \chi(H_A)$
- $\chi(\text{Guts}(S_A)) + \chi(I\text{-bundle}) = \chi(S_A)$
- Use polyhedral decomposition of  $S^3 \setminus S_A$  to find the characteristic  $I$ -bundles with negative  $\chi$ .

# A closer look at the topology of $S^3 \setminus S_A$

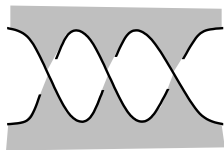
- Alexander duality:  $\chi(S^3 \setminus S_A) = \chi(S_A) = \chi(H_A)$
- $\chi(\text{Guts}(S_A)) + \chi(I\text{-bundle}) = \chi(S_A)$
- Use polyhedral decomposition of  $S^3 \setminus S_A$  to find the characteristic  $I$ -bundles with negative  $\chi$ .
- Relate these  $I$ -bundles to combinatorial properties of state graph  $H_A$ ; they relate to *2-edge loops* of  $H_A$ .

# A closer look at the topology of $S^3 \setminus S_A$

- Alexander duality:  $\chi(S^3 \setminus S_A) = \chi(S_A) = \chi(H_A)$
- $\chi(\text{Guts}(S_A)) + \chi(I\text{-bundle}) = \chi(S_A)$
- Use polyhedral decomposition of  $S^3 \setminus S_A$  to find the characteristic  $I$ -bundles with negative  $\chi$ .
- Relate these  $I$ -bundles to combinatorial properties of state graph  $H_A$ ; they relate to *2-edge loops* of  $H_A$ .
- What is the difference  $\|E_c\| = |\beta'| - |\chi(\text{Guts}(S_A))|$ ? What obstructs to equality in general?

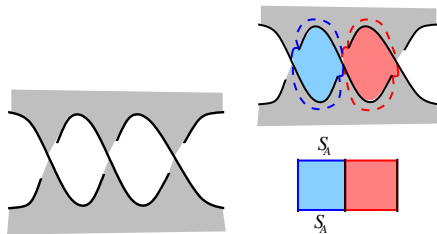
# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



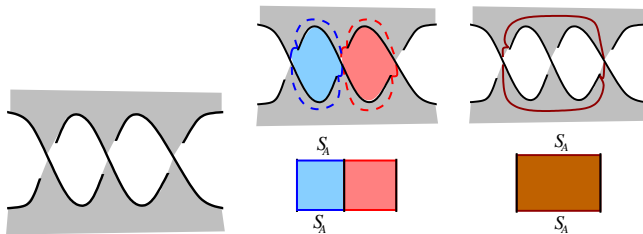
# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



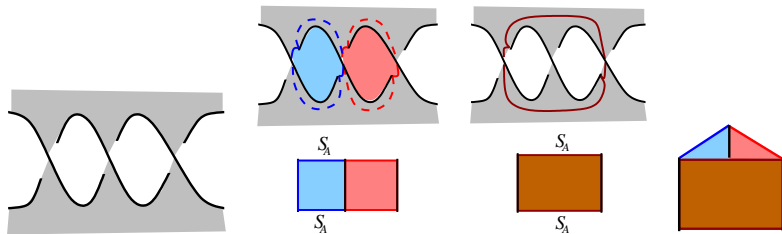
# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



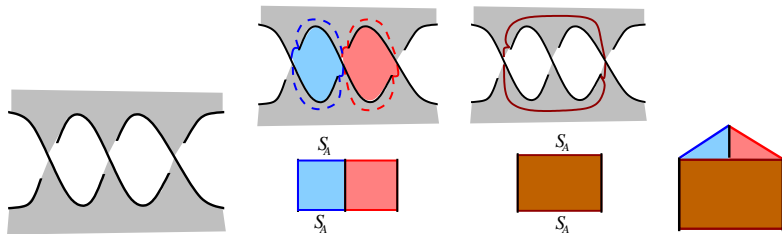
# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.

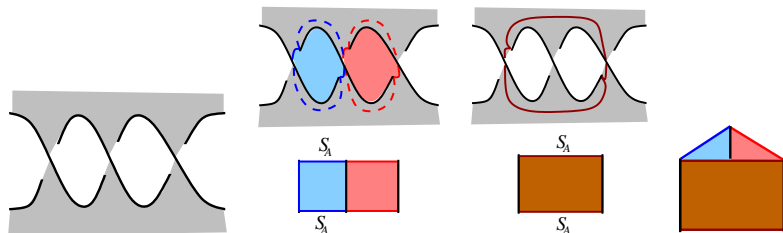


## Definition

An *essential product disk (EPD)* is a normal disk with boundary consisting of two on  $S_A$  connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



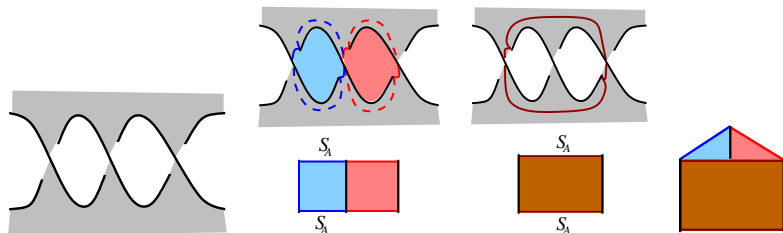
## Definition

An *essential product disk (EPD)* is a normal disk with boundary consisting of two on  $S_A$  connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

- twist regions with with more than one crossings give EPDs.

# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.



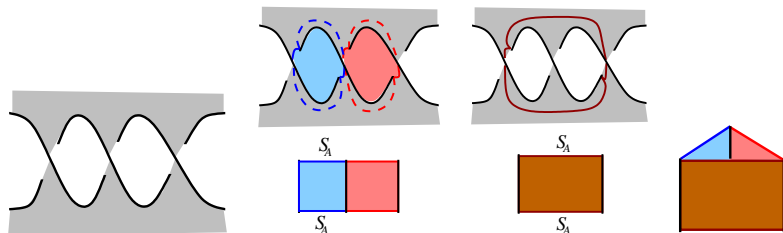
## Definition

An *essential product disk (EPD)* is a normal disk with boundary consisting of two on  $S_A$  connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

- twist regions with with more than one crossings give EPDs.
- An EPD indicates an I-bundle.

# Example: I-bundles

A *twist region* is a non-empty string of bigons arranged end to end.

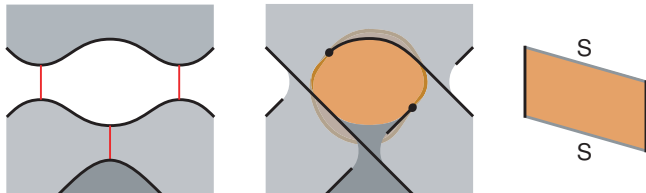


## Definition

An *essential product disk (EPD)* is a normal disk with boundary consisting of two on  $S_A$  connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

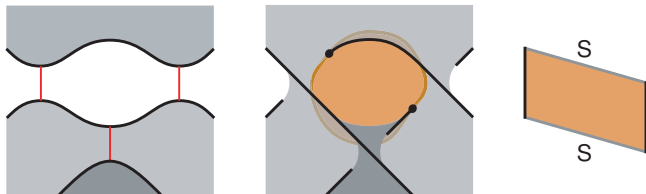
- twist regions with with more than one crossings give EPDs.
- An EPD indicates an I-bundle.
- (Lackenby) These are the only EPDs in (reduced) alternating links.

# EPDs span I-bundles



A “non-twist region” EPD

# EPDs span I-bundles

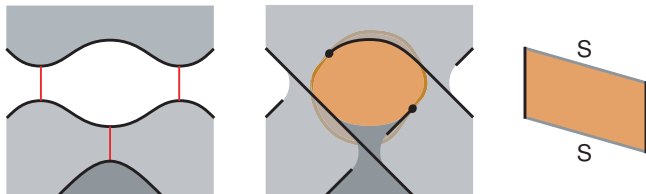


A “non-twist region” EPD

## Theorem (FKP)

*Let  $B$  be an I-bundle component of the JSJ decomposition of  $S^3 \setminus S_A$ , with  $\chi(B) < 0$ . Then  $B$  is spanned by EPDs, each embedded in a single polyhedron of the decomposition.*

# EPDs span I-bundles



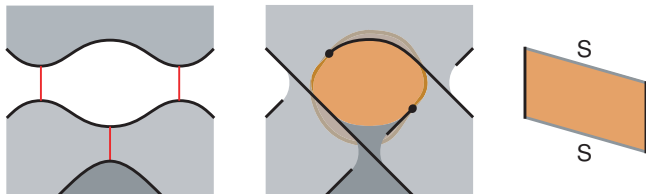
A “non-twist region” EPD

## Theorem (FKP)

*Let  $B$  be an I-bundle component of the JSJ decomposition of  $S^3 \setminus S_A$ , with  $\chi(B) < 0$ . Then  $B$  is spanned by EPDs, each embedded in a single polyhedron of the decomposition.*

- **Goal:** search for EPDs in polyhedra.

# EPDs span I-bundles



A “non-twist region” EPD

## Theorem (FKP)

Let  $B$  be an I-bundle component of the JSJ decomposition of  $S^3 \setminus S_A$ , with  $\chi(B) < 0$ . Then  $B$  is spanned by EPDs, each embedded in a single polyhedron of the decomposition.

- **Goal:** search for EPDs in polyhedra.
- Lower polyhedra: Correspond to alternating links.  
Lackenby result  $\Rightarrow$  EPDs occur only at twists.

# EPDs and “Upper” Polyhedron

In general, an EPD **MUST** run over a 2–edge loop in state graph  $H_A$ . The loop either:

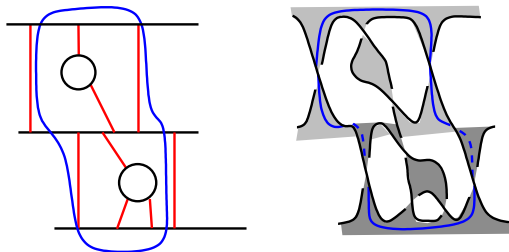
- 1 Corresponds to two crossings of the same twist region of a lower polyhedron, or

# EPDs and “Upper” Polyhedron

In general, an EPD **MUST** run over a 2–edge loop in state graph  $H_A$ . The loop either:

- 1 Corresponds to two crossings of the same twist region of a lower polyhedron, or
- 2 Does not.

**Complex EPD.** It may bound “non-trivial” parts of  $H_A$  on both sides.



The correction term  $\|E_c\|$  discussed earlier “counts” complex EPDS (In the “upper” polyhedron that do not *parabolically compress* (“simplify”) to EPDs in “lower” polyhedra.)