Guts of surfaces and colored Jones knot polynomials

David Futer, Efstratia Kalfagianni, and Jessica S. Purcell

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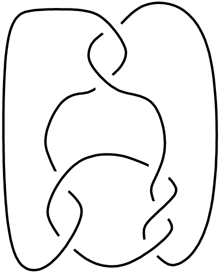
Given: Diagram of a knot or link



= 4–valent graph with over/under crossing info at each vertex.

Quantum Topology

- Knot invariants esp. colored Jones polynomials

Geometric topology

- Incompressible surfaces in knot complements
- Geometric structures and data esp. hyperbolic geometry and volume

Long term goal: Develop a setting to study both sides and relate them.

Outline

- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...

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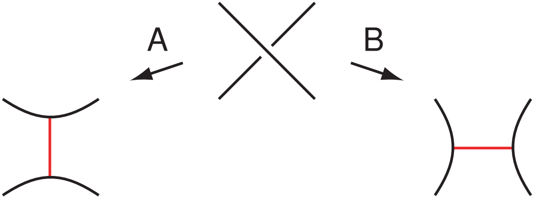
  - Coefficients measure "size" of part of JSJ-decomposition
  - Guts $\rightarrow$ relate CJP and volume of hyperbolic knots.

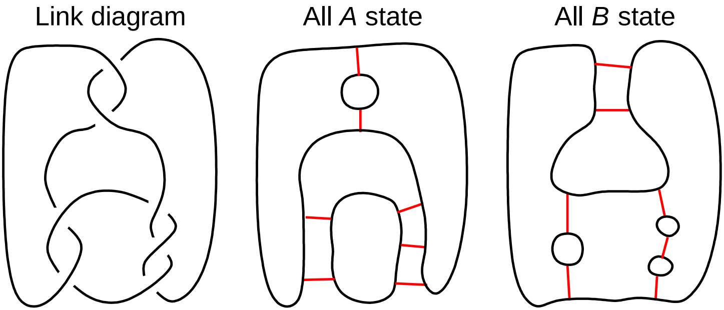
Graphs

Two choices for each crossing, *A* or *B* resolution.



- Choice of *A* or *B* resolutions for all crossings: *state* σ.
- Result: Planar link without crossings. Components: *state circles*.
- Form a graph by adding edges at resolved crossings. Call this graph H_σ. (Note: n crossings $\rightarrow 2^n$ state graphs)

Example states



Above: H_A and H_B.

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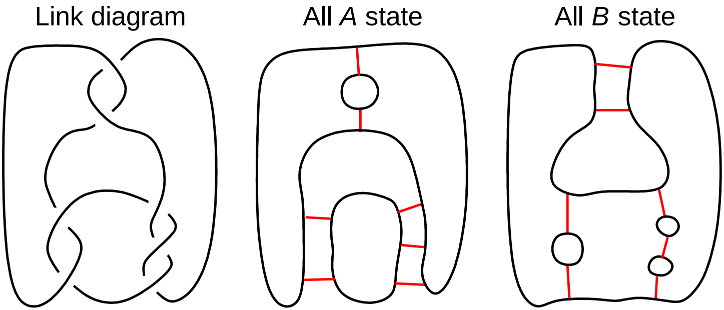


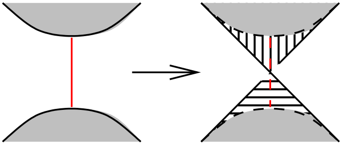
Above: H_A and H_B.

- The colored Jones polynomials of the knot can be calculated from H_A or H_B: *spanning graph expansion* arising from the Bollobas-Riordan *ribbon graph* polynomial (Dasbach-F-K-Lin-Stoltzfus , 2006).

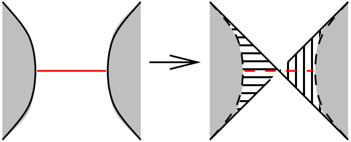
State surface

Using graph H_σ and link diagram, form the *state surface* S_σ.

- Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



A–resolution



B–resolution

Example state surfaces

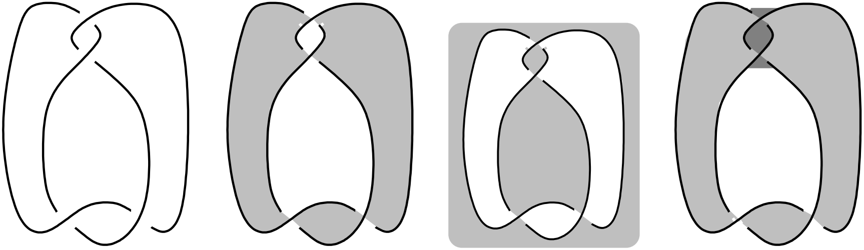


Fig-8 knot S_A S_B Seifert surface

- For alternating knots: S_A and S_B are checkerboard surfaces.

Example state surfaces

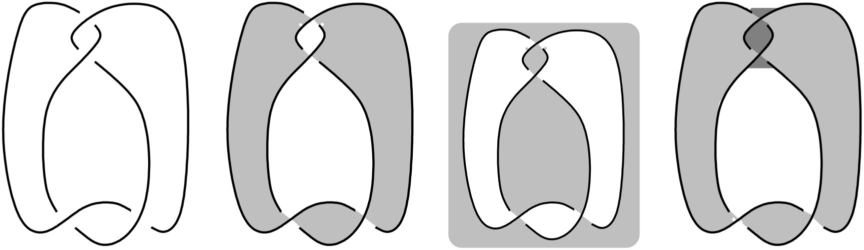


Fig-8 knot

S_A

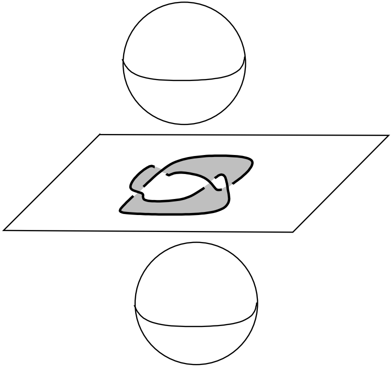
S_B

Seifert surface

- For alternating knots: S_A and S_B are checkerboard surfaces.
- For alternating knots S_A and S_B are *essential*: incompressible, ∂-incompressible (Menasco-Thistlethwaite, Lackenby)

Polyhedral decomposition prototype

Menasco (1984): Expand balloons above and below 2-sphere of alternating projection, obtain polyhedral decomposition of link complement (two 3-cells).



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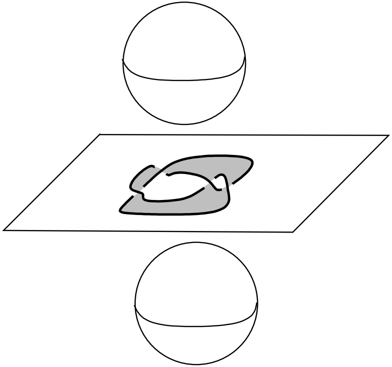
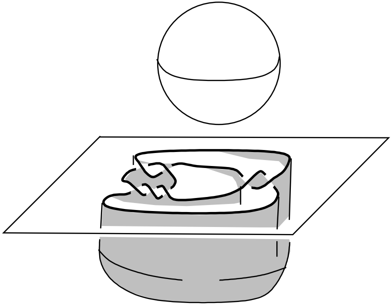


For alternating knots this gives polyhedral decomposition of checkerboard surface complement.—- This is the picture we seek to generalize to all knots.

Polyhedral decomposition of surface complement

General case:

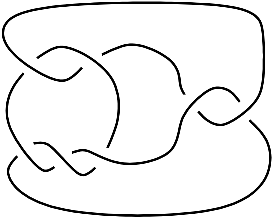
S_A (or S_B) hangs below plane of projection. Need more balloons.

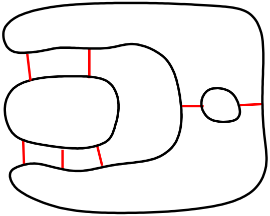


Polyhedral decomposition of complement of S_A

3–cells:

- One "upper" 3–cell, on top of plane of projection.
- One "lower" 3–cell for each nontrivial component of complement of state circles in A–resolution.





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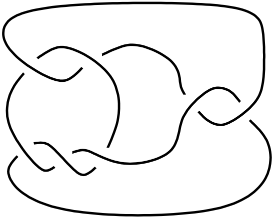

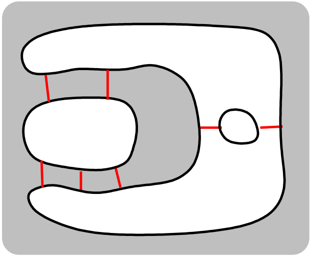

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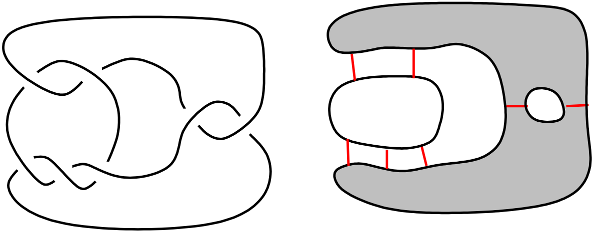
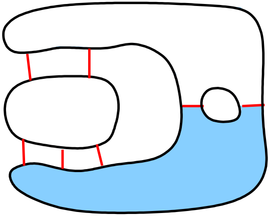


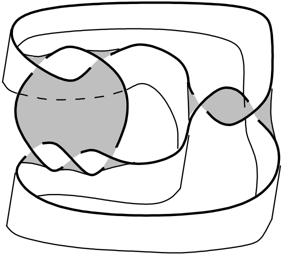
Two nontrivial components

Polyhedral decomposition of complement of S_A, continued

“Faces”:

- Portions of 3–cell meeting S_A. Shade these.
- Disks lying slightly below plane of projection, with boundary on S_A.
  - One disk for each region of graph H_A.





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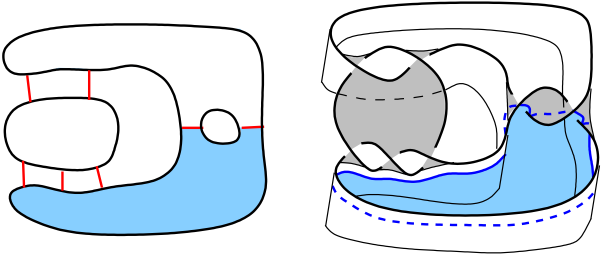
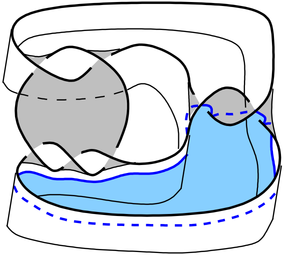


Polyhedral decomposition of complement of S_A, continued

Ideal edges:

- Run from undercrossing to undercrossing, adjacent to region of H_A.

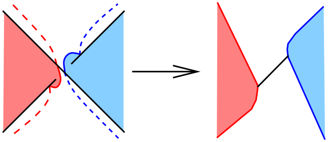


Ideal vertices:

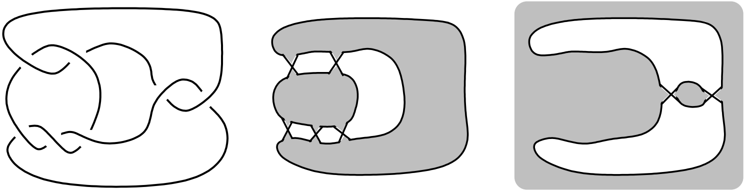
- On the link. Portions of the link visible from inside the 3–cell.

Combinatorics of lower polyhedra:

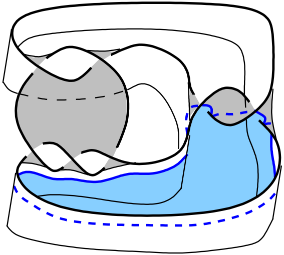
Ideal edges lie below plane of projection, so cut off view of link from below *except* at an undercrossing.



Result: Polyhedron is identical to checkerboard polyhedron of alternating sublink.



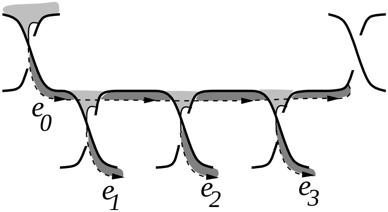
Combinatorics of upper polyhedron:

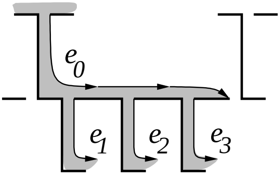


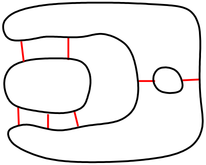
- **“ Faces”:** Shaded “faces” contain innermost disks, White faces correspond to regions of H_A.
- **Ideal edges** start and end at undercrossings, stay adjacent to single region of H_A.
- **Ideal vertices** are connected components of overcrossings = diagram components in usual diagram of link (with breaks at undercrossings).

Combinatorics of upper polyhedron, continued

- Sketch ideal edges onto usual projection of link diagram, or onto H_A.
- Edges bound *white disks*, *shaded "faces"*.
- Shaded faces: *Innermost disks*, along with *tentacles* adjacent to ideal edges.







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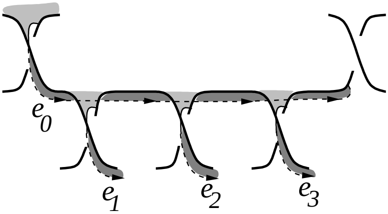

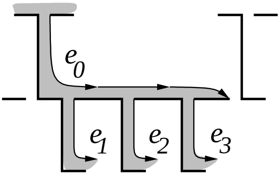

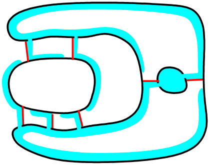

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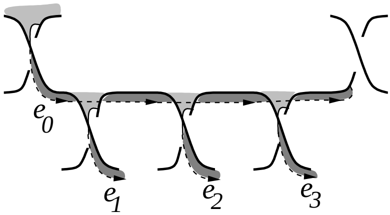
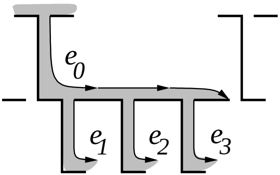
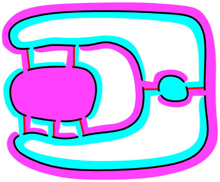


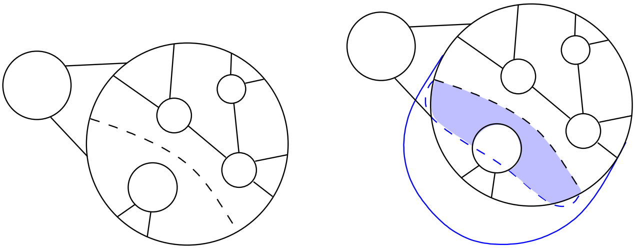




One additional issue

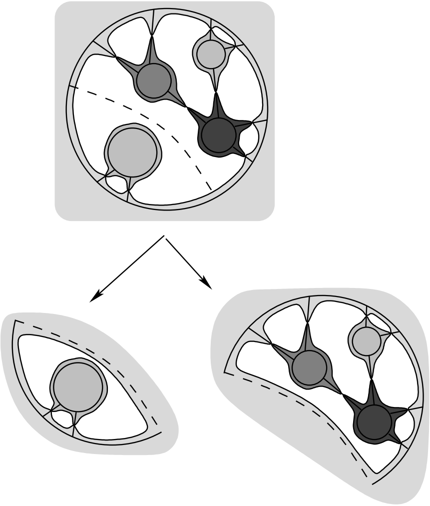
Lower polyhedra may not give *prime* alternating links.

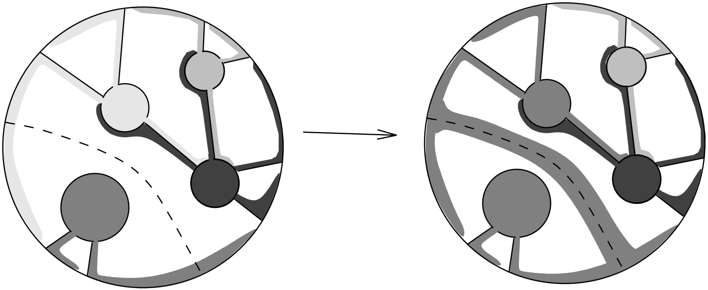
Example:



- See bigon in polyhedral decomposition.
- Fix: Modify polyhedra — surger along bigon.
  - Splits 3–cell into two.
  - Splits white disk into two.
  - In upper polyhedron: Connects two shaded “faces” along arc.

Example: Lower polyhedron splits in two





Generic form of Upper polyhedron

Summary: Ideal polyhedral decomposition

Result:

The above procedure gives an ideal polyhedral decomposition of $S^3 \backslash\backslash S_A$ if "faces" are simply connected. This happens when H_A has no edge with both endpoints on a single state circle!

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Use these polyhedra and normal surface theory to study the topology of $S^3 \backslash\backslash S_A$.

Smallest complexity normal surfaces: Normal bigons

A *normal bigon* is a disk in normal form whose boundary consists of two arcs:

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Proposition (FKP)

Under the above polyhedral decomposition, if the graph H_A has no edge with both endpoints on a single state circle, then there are no normal bigons in the polyhedra.

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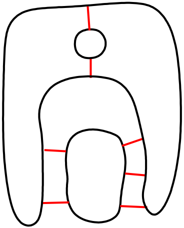

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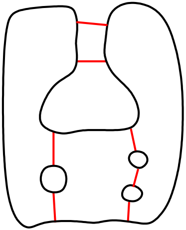


Non-example

Application: Essential state surfaces

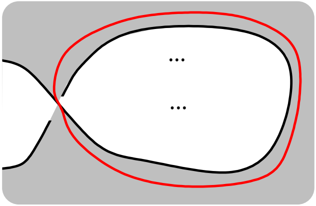
We obtain a new proof of a theorem recently proved by M. Ozawa.

Theorem

The following are equivalent:

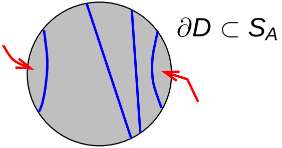
- *H_A has no edge with both endpoints on a single state circle*
- *S_A is incompressible and boundary incompressible.*

Proof. If H_A has edge with both endpoints on a single state circle, then we form boundary compression disk:



Proof continued

Suppose H_A has no such edge but is compressible or ∂-compressible: Put compressing disk, boundary compressing disk into normal form.
A compressing disk D for S_A would meet white faces of polyhedra in arcs. Outermost arc on D forms a *normal bigon*. Contradiction.



Normality implies boundary arc of boundary compressing disk E lies in a single polyhedron. Outermost intersection of E with white face cuts off a disk E' which cannot be a normal bigon, so contains boundary arc. But then $E \setminus E'$ is a disk meeting white faces, obtain normal bigon. Contradiction. □

Colored Jones polynomial prelims

For a knot K we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{n-1} + \cdots + \beta'_n t^{m+1} + \alpha'_n t^{k_n}.$$

- Some properties:
  - $J_{K,n}(t)$ is determined by the Jones polynomials of certain cables of K.
  - The sequence $\{J_{K,n}(t)\}_n$ is *q-holonomic*: for every knot the CJP's satisfy linear recursion relations (Garoufalidis-Le , 2004). Then for every K
    1. Degrees m_n, k_n are quadratic (quasi)-polynomials in n
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    1. Degrees m_n, k_n are quadratic (quasi)-polynomials in n
    2. Coefficients $\alpha_n, \beta_n \ldots$ satisfy recursive relations in n.
- Properties manifest themselves in strong forms for knots with *state graphs* that have no edge with both endpoints on a single state circle!

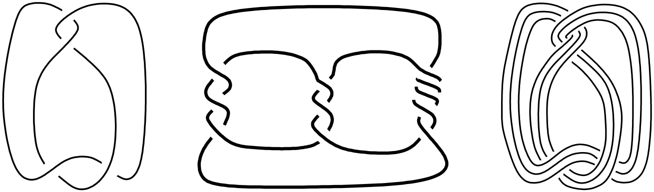
Adequate links

Lickorish–Thistlethwaite 1987: Introduced *A*–adequate links (*B*–adequate links) in the context of Jones polynomials.

Definition

A link is *A*–adequate if has a diagram with its graph H_A has no edge with both endpoints on the same state circle.

A or *B*-adequate: all alternating knots, Montesinos knots, positive braids, negative braids, arborescent knots, blackboard cables of adequate knots....



Jones polynomials and adequate links

- Properties:
  - (L-T) The Jones polynomial detects the unknot within the class of A-adequate knots.
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- Similar properties for B-adequate.
- Restate Theorem proved earlier:
  - Diagram is A–adequate $\Leftrightarrow$ S_A incompressible and boundary incompressible.
- Theorem has several applications:
  - Volume estimates of hyperbolic knots
  - Relations of topology complement of S_A to colored Jones polynomials
  - Relations of Jones polynomial and volume: old and new

Application of incompressibility of S_A

1. S_A gives Stavros Garoufalidis' recent conjecture on Jones slope for A–adequate knots. (S_B for B–adequate.)

Theorem (FKP)

For an A–adequate diagram,

$$\textit{slope of all–A state surface} = \lim_{n\to\infty} \frac{-4}{n^2} k_n,$$

$k_n := \textit{min}\ \deg J_{K,n}(t)$.

(The conjecture is that every limit of a subsequence of the sequence of the right is a boundary slope of an incompressible, boundary incompressible surface.)

2. Applications to geometry and volume — guts.

Application: Guts

Take JSJ-decomposition of $S^3 \backslash\backslash S_A$. There are no essential tori. Cut along essential annuli into components:

1. Solid tori,
2. I–bundles over surfaces,
3. *Simple* pieces (admitting complete hyperbolic metric).

The union of all the hyperbolic pieces is the *guts*: $\text{Guts}\,(S_A)$.

- $\text{Guts}\,(S_A) = \emptyset \Leftrightarrow S^3 \backslash\backslash S_A$ is a union of I-bundles and solid tori (i.e. *a book of I-bundles*).
- $\chi(\text{Guts}\,(S_A))$ measures how far S_A is from being “fiber-like” (*a fibroid*).

Guts relates to volume:

Theorem (Agol–Storm–Thurston 2005)

For K hyperbolic

$$Vol\,(S^3 \smallsetminus K) \geq v_8 \, |\chi(Guts\,(S_A))|,$$

here $v_8 \approx 3.66...$ is the volume of a regular ideal octahedron.

Guts relates to Jones polynomials

Recall the tail coeffs of CJP for *A*-adequate: $|\alpha'| = 1$, $|\beta'|$.

Theorem (FKP)

K prime link with A-adequate diagram without nugatory crossings. Then,

$$|\chi(Guts\,(S_A))| \leq |\beta'|.$$

Similarly for B-adequate diagram without nugatory crossings,

$$|\chi(Guts\,(S_B))| \leq |\beta|.$$

Corollary

K as above. If $|\beta'| = 0$ *then* S_A *is "fiber-like" (a fibroid).*

- In Theorem above we have equalities for:
  - Alternating links (Lackenby, Dasbach-Lin, 2005)
  - Montesinos links (FKP)
  - When state graph H_A has no 2-edge loops...
- Combining with Agol–Storm–Thurston, gives relations of CPJ and volume

A closer look at the topology of $S^3 \backslash\backslash S_A$

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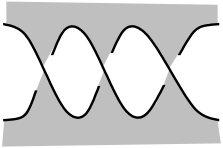
- Relate these I–bundles to combinatorial properties of state graph H_A; they relate to *2-edge loops* of H_A.
- What is the difference $O(S_A) := |\beta'| - |\chi(\text{Guts}\,(S_A))|$? What obstructs to equality in general?
- $O(S_A)$ depends on the A-adequate diagram used: For several families (e.g. Montesinos links) we can find diagrams so that $O(S_A) = 0$. We do not know if this is true in general (work in progress).

Example: I–bundles

A *twist region* is a non-empty string of bigons arranged end to end.



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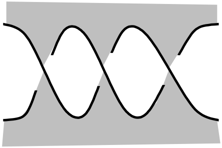

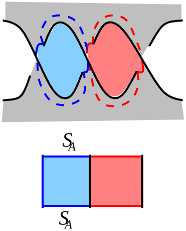

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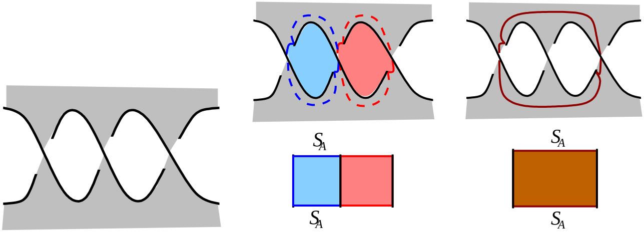

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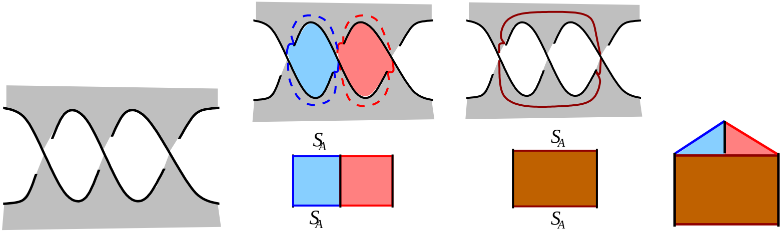

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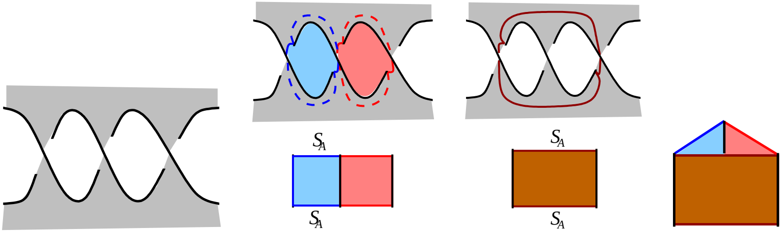

Definition

An *essential product disk (EPD)* is a normal disk with boundary consisting of two on S_A connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

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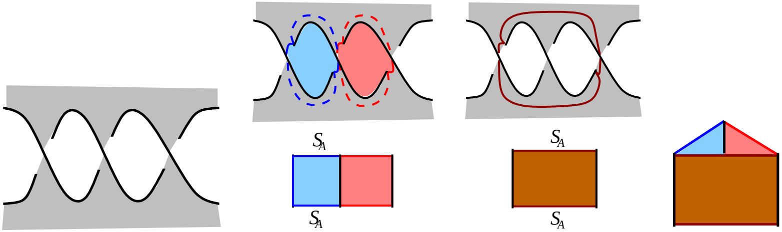

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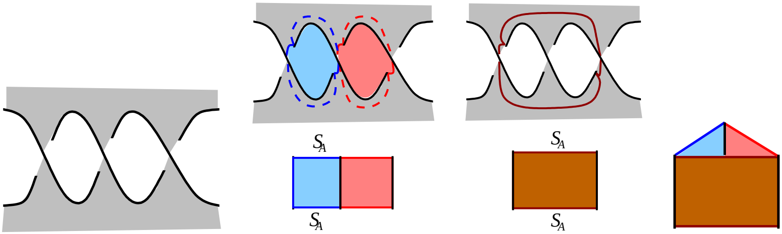

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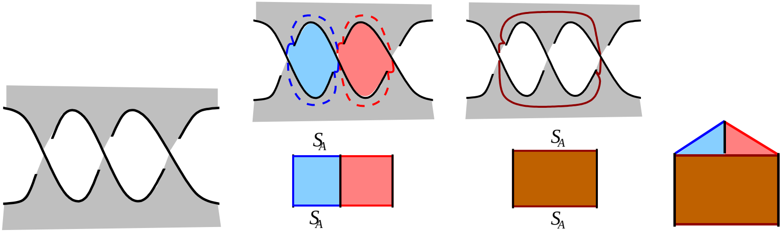

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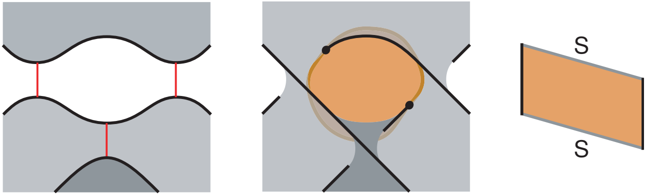
- twist regions with with more than one crossings give EPDs.
- An EPD indicates an I–bundle.
- (Lackenby) These are the only EPDs in (reduced) alternating links.

EPDs span I–bundles



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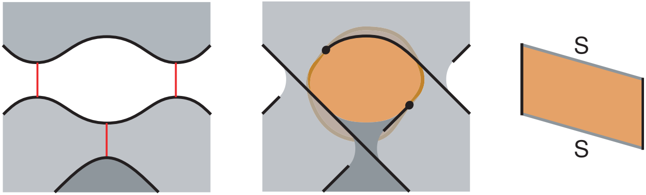

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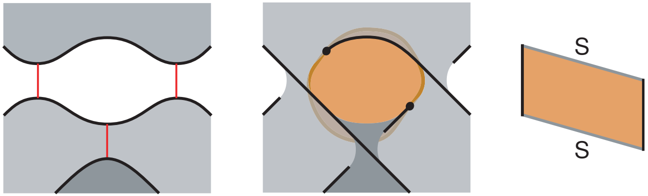
Theorem (FKP)

Let B be an I–bundle component of the JSJ decomposition of $S^3 \backslash\backslash S_A$, with $\chi(B) < 0$. Then B is spanned by EPDs, each embedded in a single polyhedron of the decomposition.



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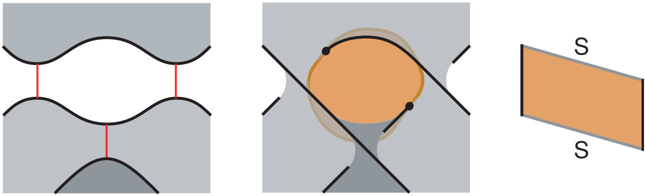

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- **Goal:** search for EPDs in polyhedra.
- Lower polyhedra: Correspond to alternating links.
  Lackenby result $\Rightarrow$ EPDs occur only at twists.

EPDs and "Upper" Polyhedron

In general, an EPD **MUST** run over a 2–edge loop in state graph H_A. The loop either:

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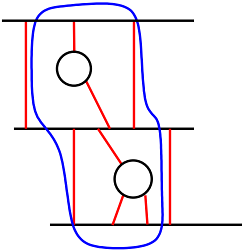
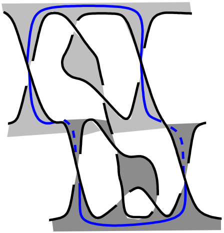
In general, an EPD **MUST** run over a 2–edge loop in state graph H_A. The loop either:

1. Corresponds to two crossings of the same twist region of a lower polyhedron, or
2. Does not.
   It may runs along a “ staircase”, with at least one stair.



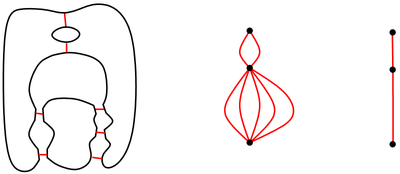


The obstruction $O(S_A)$ discussed earlier “counts” such EPD’s in “upper” polyhedron that do not *prabolically compress* (“simplify”) to EPDs in “lower” polyhedra.

Putting it all together: Reduced state graph

Recall the graph H_A.

- Collapse each state circle of H_A to a vertex.
- Remove redundant edges.



Theorem (FKP)

For *prime* links with A–adequate diagram whose graph H_A contains no 2–edge loops except those corresponding to twist regions,

$$\chi(\text{Guts}\,(S_A)) = 1 - \chi(G'_A).$$

In general,

$$|\chi(\text{Guts}\,(S_A))| \leq 1 - \chi(G'_A).$$

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Recall coefficient β' of Colored Jones polynomial for *A*–adequate links.

- (Stoimenow, 2005) $|\beta'| = 1 - \chi(G'_A)$.
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Volumes and EPDs

- **Old results:** *Highly twisted* adequate knots have volume bounded above and below by coefficients of colored Jones polynomial (FKP 2008).
- **New understanding:** Adding crossings to twist regions eliminates ”complicated EPDs”. Thus coefficients of CJP are close to Guts (S_A)

Possible generalization to all knots

- Recall:
  - For an n-crossing diagram we have 2^n *states* σ, graphs H_σ and state surfaces S_σ by considering resolutions of crossings.
  - We worked with all-A or all-B cases only.
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