A NOTE ON QUANTUM 3-MANIFOLD INVARIANTS AND HYPERBOLIC VOLUME

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Abstract. For a closed, oriented 3-manifold M and an integer $r > 0$, let $\tau_r(M)$ denote the $SU(2)$ Reshetikhin-Turaev-Witten invariant of M, at level r. We show that for every $n > 0$, and for $r_1, \ldots, r_n > 0$ sufficiently large integers, there exist infinitely many non-homeomorphic hyperbolic 3-manifolds M, all of which have different hyperbolic volume, and such that $\tau_{r_i}(M) = 1$, for $i = 1, \ldots, n$.

Key words: Brunnian link, colored Jones polynomial, Reshetikhin-Turaev-Witten invariants, hyperbolic volume.

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1. introduction

Throughout the paper, M will denote a closed orientable 3-manifold. The Reshetikhin-Turaev-Witten $SU(2)$-invariants of M are complex valued numbers ([RT], [KiM]) parametrized by positive integers (levels). Given a positive integer $r \in \mathbb{N}$, let $\tau_r(M)$ denote the $SU(2)$ invariant of M at level r. In the case that M is hyperbolic, let $\mathrm{vol}(M)$ denote the hyperbolic volume of M. It has been speculated that $\mathrm{vol}(M)$ is determined by the entire collection of the invariants $T_M := \{\tau_r(M) |\ r = 1, 2, ..\}$ (see [Mu]). However, at the moment, it is not clear what the precise statement of a conjecture in this direction should be. In this paper we are concerned with the question of the extent to which the finite sub sequences of T_M determine $\mathrm{vol}(M)$. The main result is the following theorem that shows that for most finite sub sequences the answer to this question is an emphatic *no*.

Theorem 1.1. *Fix $n > 0$. There is a constant $C_n > 0$ such that for every n-tuple of integers $r_1, \ldots, r_n > C_n$, there exist and infinite sequence of hyperbolic 3-manifolds $\{M_k\}_{k \in \mathbb{N}}$ such that*
(a) For $i = 1, \ldots, n$, and every $k \in \mathbb{N}$ we have $\tau_{r_i}(M_k) = 1$; and
(b) $\ldots > \mathrm{vol}(M_{k+1}) > \mathrm{vol}(M_k) > \ldots \mathrm{vol}(M_0) > \frac{n}{2}$.

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Given a framed link L in S^3, there is a sequence of Laurent polynomials $\{J_N(L,t)\}_{N\in\mathbb{N}}$; the colored Jones polynomials ([KiM]). For the trivial knot U, if equipped with the 0-framing, we have

$$J_N(U,t) = [N] := \frac{t^N - t^{-N}}{t - t^{-1}}.$$

To prove Theorem 1.1 we need the following:

Theorem 1.2. *Given integers $n > 0$ and $r_1, \ldots, r_n > 2$, there is a knot $K \subset S^3$ such that for any common framing on K and U we have*

$$J_N(K, e_{r_i}) = J_N(U, e_{r_i}) \quad \text{for all} \quad N \in \mathbb{N}.$$

Here, for $i = 1, \ldots, n$, $e_{r_i} := e^{\frac{2\pi\sqrt{-1}}{r_i}}$ is a primitive r_i-th root of unity. For fixed n, if $r_1, \ldots, r_n$ are sufficiently large, then K can be chosen hyperbolic with $\mathrm{vol}(S^3 \smallsetminus K) > n$. Furthermore, if M is a hyperbolic 3-manifold obtained by $\frac{p}{q}$-surgery on K, for some $|q| > 12$, then we have

$$\mathrm{vol}(M) \;\geq\; (1 - \frac{127}{q^2})^{\frac{3}{2}}\; n. \tag{1}$$

The proof of the first part of Theorem 1.1 uses a result of Lackenby ([L]) and a construction of [K]. For the remaining claims, we need Thurston's hyperbolic Dehn surgery theorem ([Th]) and a result proved jointly with Futer and Purcell ([FKP]).

Corollary 1.3. *Given an integer $n > 0$, there is a sequence of hyperbolic 3-manifolds $\{M_i\}_{i\in\mathbb{N}}$ and an increasing sequence of positive integers $\{m_i\}_{i\in\mathbb{N}}$ such that*

$$\mathrm{vol}(M_i) > \frac{n}{2} \quad \text{and} \quad \tau_{m_i}(M_i) = 1,$$

for all $i \in \mathbb{N}$.

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2. The Proofs

2.1. **Some properties of the colored Jones polynomials.** A crossing disc of a knot J is an embedded disc $D \subset S^3$ that intersects J only in its interior exactly twice geometrically and with zero algebraic intersection number. The curve ∂D is a crossing circle for J. A knot K is said to be obtained from J by a generalized crossing of order $r \in \mathbb{Z}$ iff K is the result of J under surgery of S^3 along ∂D with surgery slope $\frac{1}{r}$.

Definition 2.1. (Definition 1.1, [L]) *Let $r \in \mathbb{N}$ and let J and K be two 0-framed knots in S^3. We say K and J are congruent modulo $(r,\ 2)$, iff K is obtained from J by a collection of generalized crossing changes of order r supported on disjoint crossing discs. In this case we will write $J \equiv K(\text{mod}(r,\ 2))$.*

For $j \in N$, let K^j denote the j-th parallel cable of K formed with 0-framing and let $J(K^j, t)$ denote the Jones polynomial of K^j. We need the following result of Lackenby.

Lemma 2.2. (Corollary 2.8, [L]) *Let $r > 2$ be an integer and let $e_r := e^{\frac{2\pi\sqrt{-1}}{r}}$ denote a primitive r-th root of unity. Suppose that J and K are 0-framed knots in S^3. If $J \equiv K(\text{mod}(r,\ 2))$, then,*

$$J(K^j, e_r) = J(J^j, e_r), \quad \text{for all} \quad j \in \mathbb{N}.$$

We recall that for a 0-framed link L the value $J_N(L, e_r)$ is a linear combination of $J(L^j, e_r)$, with the coefficients of the combination being constants independent of L (Theorem 4.15, [KiM]). Using this fact and Lemma 2.2 we have:

Corollary 2.3. *Let $r > 2$ be an integer and let $e_r := e^{\frac{2\pi\sqrt{-1}}{r}}$ denote a primitive r-th root of unity. Suppose that J and K are 0-framed knots in S^3. If $J \equiv K(\text{mod}(r,\ 2))$, then, for every integer $q > 0$, we have $J_N(K^q, e_r) = J_N(J^q, e_r)$, for all $N \in \mathbb{N}$. In particular, we have $J_N(K, e_r) = J_N(J, e_r)$, for all $N \in \mathbb{N}$.*

2.2. **A construction of hyperbolic Brunnian links.** To prove Theorem 1.2 we will need the following lemma which summarizes results proved in [K] and uses results proved jointly with Askitas in [AK]. Below we will sketch the proof referring the reader to the original references for details.

Lemma 2.4. *For every $n > 0$, there is an $(n+1)$-component link $L_n := U \cup K_1 \cup \ldots K_n$ with the following properties:*

(a) L_n is Brunnian; that is every proper sublink of L_n is a trivial link.

(b) For $i = 1 \ldots, n$, K_i bounds a crossing disc $D_i \subset S^3$ of U.

(c) L_n is hyperbolic; that is the interior of the 3-manifold $\overline{M_n := S^3 \setminus \eta(L_n)}$ admits a complete hyperbolic metric of finite volume. Here, $\eta(L_n)$ denotes a tubular neighborhood of L_n.

(d) For $n > 1$, any collection of generalized crossing changes along any collection of discs formed by a proper subset of $\{D_1, \ldots, D_n\}$, leaves U unknotted.

(e) Every knot obtained by a collection of n generalized crossing changes of order $r_1, \ldots, r_n > 0$ along $D_1, \ldots D_n$, respectively, is non-trivial.

Proof: For $n = 1$, we take $L_1 := U \cup K_1$ to be the Whitehead link and for $n = 2$, we take $L_2 := U \cup K_1 \cup K_2$ to be the 3-component Borromean link (see Figure 1). It is well known that they are both hyperbolic ([Th]). For $n = 2$, condition (d) is clearly satisfied and for $n = 1$ and $n = 2$, condition (e) is true since the resulting knot will be a non-trivial twist knot.

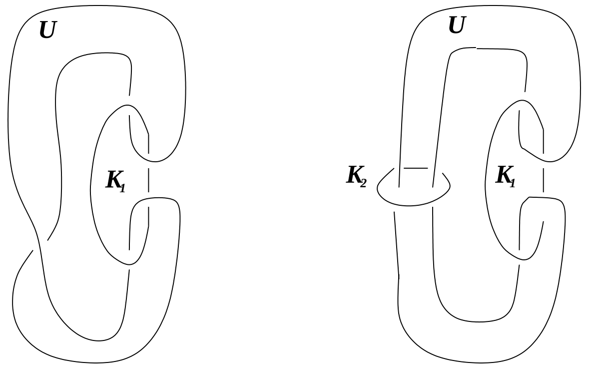

FIGURE 1. The link L_n for $n = 1$ and $n = 2$.

For $n > 2$ the construction of a link L_n as claimed above, is given in the proof of Theorem 3.1 of [K]. We now outline the construction using the terminology of [AK] and [K]. Recall that a knot $\hat{K}$ is n-adjacent to the unknot, for some $n \in \mathbf{N}$, if $\hat{K}$ admits an embedding containing n generalized crossings such that changing any $0 < m \leq n$ of them yields an embedding of the unknot. A collection of crossing circles corresponding to these crossings is called an n-trivializer. Theorem 3.5 of [K] proves the following: For every $n > 2$, there exists a knot $\hat{K}$ that is n-adjacent to the unknot, and it admits an n-trivializer $K_1 \cup \ldots \cup K_n$ such that the link $L_n^* = \hat{K} \cup K_1 \cup \ldots \cup K_n$ is hyperbolic. Let $D_1, \ldots, D_n$ denote crossing discs corresponding to $K_1, \ldots, K_n$, respectively. Now let L_n denote the $(n+1)$-component link obtained from L_n^* by performing the n generalized crossing changes (along $D_1, \ldots, D_n$) that exhibit $\hat{K}$ as n-adjacent to the unknot, simultaneously. We can write $L_n := U \cup K_1 \cup \ldots \cup K_n$, where U is the unknot resulting from $\hat{K}$ after these crossing changes. Since the links L_n^* and L_n differ by twists along a collection of discs, they have homeomorphic complements.

We conclude that L_n is hyperbolic. Thus L_n satisfies conditions (b) and (c) of the statement of the lemma. Next, let us focus on a proper sub collection of crossing discs; say, without loss of generality, $D_1, \ldots, D_{n-1}$. Since $K_1 \cup \ldots K_n$ is an n-trivializer for $\hat{K}$, by the definition of adjacency to the unknot, $K_1 \cup \ldots \cup K_{n-1}$ is an $(n-1)$-trivializer By Theorem 2.2 of [AK], and its proof, U bounds an embedded disc, say Δ, in the complement of $K_1 \cup \ldots \cup K_{n-1}$ such that $D_i \cap \Delta$ is a single arc properly embedded on Δ. It follows that $U \cup K_1 \cup \ldots \cup K_{n-1}$ is the trivial link and that every collection of generalized crossing changes supported along $D_1, \ldots, D_{n-1}$ leaves U unknotted. This proves (b) and (d). To see (e) suppose, on the contrary, that there is a collection of n generalized crossing changes of order $r_1, \ldots, r_n > 0$ along $D_1, \ldots, D_n$, respectively, that leaves U unknotted. Let U' denote the result of U after performing the crossings changes along $D_1, \ldots, D_{n-1}$ only, leaving D_n intact. By (b) and (d), $U' \cup K_1 \cup \ldots \cup K_{n-1}$ is the trivial link. Then, a crossing change of order $r_n > 0$ along D_n leaves U' unknotted. By the argument in the proof of Theorem 2.2 in [AK], we conclude that L_n is the trivial link. This is a contradiction since L_n is hyperbolic. □

2.3. **Proof of Theorem 1.2.** Fix $n \in \mathbb{N}$ and let L_n be a link as in Lemma 2.4. We will consider the component U of L_n as a 0-framed unknot in S^3. Given an n-tuple of integers $\mathbf{r} := (r_1, \ldots, r_n)$, with $r_i > 2$, let $M_n(\mathbf{r})$ denote the 3-manifold obtained from M_n as follows: For $1 \leq i \leq n$, perform Dehn filling with slope $\dfrac{1}{r_i}$ along $\partial\eta(K_i)$. Let $K := U(\mathbf{r})$ denote the image of U in $M_n(\mathbf{r})$. Clearly, $M_n(\mathbf{r}) := \overline{S^3 \smallsetminus \eta(U(\mathbf{r}))}$. Note, that since the linking number of K_i and U is zero the framing on K induced by that of U is zero. Thus K is a 0-framed knot in S^3 that is obtained from U by a generalized crossing of order $r_1, \ldots, r_n$ along $D_1, \ldots D_n$, respectively. By Lemma 2.4 (e), K is non-trivial.

Claim: We have $U \equiv K(\mathrm{mod}(r_i,\ 2))$, for every $1 \leq i \leq n$.

Proof of Claim: Fix $1 \leq i \leq n$. Consider $U(i)$ the knot obtained from U as follows: For every $1 \leq j \neq i \leq n$, perform a generalized crossing change of order r_j along D_j. By Lemma 2.4 (d), $U(i)$ is a (0-framed) trivial knot. Since by construction, K is obtained from $U := U(i)$ by a generalized crossing change of order r_i along D_i, we have $U \equiv K(\mathrm{mod}(r_i,\ 2))$. This proves the claim.

Now by Corollary 2.3 we have that, if both U and K are given the 0-framing, then $J_N(K, e_{r_i}) = J_N(U, e_{r_i})$, for $i = 1, \ldots, n$. Suppose now that U and K are given any framing $f \in \mathbb{Z}$. By Lemma 3.27 of [KiM], under the frame change from 0 to f both of $J_N(K,t)$ and $J_N(U,t)$ are changed by the factor $t^{f(N^2-1)}$. Thus, the equation $J_N(K,e_{r_i}) = J_N(U,e_{r_i})$ remains true for every framing. This proves the first part of the theorem.

Now we prove the remaining claims made in the statement of the theorem: By Thurston's hyperbolic Dehn filling theorem ([Th]), there is a constant $C_n := C(L_n) > 0$, such that if $r_1,\dots,r_n > C_n$ then $M_n(\mathbf{r})$ admits a complete hyperbolic structure of finite volume. Thus $K := U(\mathbf{r})$ is a hyperbolic knot. By the proof of Thurston's theorem, the hyperbolic metric on $M_n(\mathbf{r})$ can be chosen so that it is arbitrarily close to the metric of M_n, provided that the numbers $r_i >> 0$ are all sufficiently large. Thus by choosing the r_i's large we may ensure that the volume of $M_n(\mathbf{r})$ is arbitrarily close to that of M_n. Since ∂M_n has $n+1$ components, the interior of M_n has $n+1$ cusps. By [Ad], we have $\mathrm{vol}(M_n) \geq (n+1)v_3$, where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron. Thus for $r_1,\dots,r_n >> C_n$ we have

$$\mathrm{vol}(S^3 \smallsetminus K) = \mathrm{vol}(M_n) > nv_3 > n. \tag{2}$$

Now we turn our attention to closed 3-manifolds obtained by surgery of S^3 along a knot K as above: Suppose that M is a hyperbolic 3-manifold obtained by $\dfrac{p}{q}$-surgery on K. In Theorem 3.4 of [FKP] it is shown that if $|q| > 12$, then $\mathrm{vol}(M) \geq (1 - \frac{127}{q^2})^{\frac{3}{2}} \mathrm{vol}(S^3 \smallsetminus K)$. Combining this with (2) above we immediately obtain (1). This finishes the proof of the theorem. $\square$

2.4. **Proof of Theorem 1.1.** Fix $n > 0$ and $r_1,\dots,r_n > 2$ and let K be a knot as in Theorem 1.2. For a positive integer q, let $M := M_q(K)$ denote the 3-manifold obtained by surgery of S^3 along K with surgery slope $\frac{1}{q}$. Let $L := K^q$ denote the q-th cable of K formed with the 0-framing as before. Similarly let U^q denote the q-th cable of the unknot U. By Corollary 2.3, if both L and U^q are considered with 0-framing, we have $J_N(L,e_{r_i}) = J_N(U^q,e_{r_i})$, for all $N \in \mathbb{N}$. Now by Kirby calculus M can be obtained by surgery on the link $L := K^q$ and where the framing on each of the q components is 1 (see, for example, Figure 8 of [G] for details.) By formula (1.9) on page 479 of [KiM], $\tau_{r_i}(M)$ is a linear combination of the values $\{J_N(L,e_{r_i}) \mid N < r_i\}$ with the linear coefficients depending only on r_i and the linking matrix of L. From this and our earlier observations, the value of $\tau_{r_i}(M)$ remains the same if we replace L with U^q and keep the same framings. But then the 3-manifold obtained by surgery on this later framed link, which is the same as this obtained by $\frac{1}{q}$-surgery on the unknot U, is clearly S^3. Since, by [KiM], $\tau_r(S^3) = 1$, for every $r > 0$, we have

$$\tau_{r_i}(M) = \tau_{r_i}(S^3) = 1, \quad \text{for} \quad i = 1, \ldots, m.$$

Next suppose that we have chosen the values $r_1, \ldots, r_n > 2$ large enough so that K is hyperbolic. By Thurston's hyperbolic Dehn filling theorem, if $q >> 0$ then the 3-manifold M is also hyperbolic. We may, without loss of generality, assume that $|q| > 12$. Now Theorem 1.2 implies that $\mathrm{vol}(M) \geq (1 - \frac{127}{q^2})^{\frac{3}{2}} n$, and for $q >> 12$, we can assure that

$$\mathrm{vol}(M) \ \geq \ (1 - \frac{127}{q^2})^{\frac{3}{2}} n > \frac{n}{2}.$$

Next we show that the set $A_K := \{M_q(K) \ | \ q \in \mathbb{Z}\}$, contains infinitely many non-homeomorphic 3-manifolds. By [Th], for $q >> 0$, we have $\mathrm{vol}(M_q(K)) < \mathrm{vol}(S^3 \setminus K)$ and $\lim_{q \to \infty} \mathrm{vol}(M_q(K)) = \mathrm{vol}(S^3 \setminus K)$. Thus we can find a sequence $\{M_k\}_{k \in \mathbb{N}}$ as claimed in the statement of Theorem 1.1. □

2.5. **Proof of Corollary 1.3.** We will use the notation and the setting established in the proofs Theorems 1.2 and 1.1: Fix $n > 1$ and choose $r'_1, \ldots, r'_n >> 0$ as in the proof of Theorem 1.2, so that for $K := U(\mathbf{r})$ we have $\mathrm{vol}(S^3 \setminus K) > n$, for every $\mathbf{r} := (r_1, \ldots, r_n)$ with $r_j \geq r'_j$. For $i \in \mathbb{N}$ set

$$\mathbf{r}_i := (i + r_1, \ldots, r_n) \ \text{ and } \ K_i := U(\mathbf{r}_i).$$

By the arguments in the proof of Theorem 1.1, we have

$$\mathrm{vol}(S^3 \setminus K_i) > n \ \text{ and } \ K \equiv U(\mathrm{mod}(m_i, \ 2)),$$

where $m_i := i + r_1$. Thus Corollary 2.3 and the argument in the proof of 1.1, imply that there is a 3-manifold M_i obtained by surgery along K_i such that: i) $\tau_{m_i}(M_i) = 1$; and ii) M_i is hyperbolic with $\mathrm{vol}(M_i) > \dfrac{n}{2}$. □

References

[Ad] C. C. Adams, *Volumes of N-cusped hyperbolic 3-manifolds,* J. London Math. Soc. (2) 38 (1988), no. 3, 555–565.

[AK] N. Askitas and E. Kalfagianni, *On knot adjacency,* Topology and Its Applications, Vol.126 (2002), 63-81.

[FKP] D. Futer, E. Kalfagianni and J. S. Purcell, *Dehn filling, volume, and the Jones polynomial,* Journal of Differential Geometry, in press. ArXiv:math.GT/0612138.

[G] S. Garoufalidis, On finite type invariants of 3-manifolds I, Journal of Knot Theory and Its Ramifications, 4 (1996) 441-462

[K] E. Kalfagianni, *Alexander polynomial, finite type invariants and volume of hyperbolic knots*, Algebraic and Geometric Topology, Vol. 4 (2004), 1111-1123.

[KiM] R. Kirby and P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, C)$*, Inventiones Mathematicae, vol. 105 (1991), no. 3, 473–545.

[L] M. Lackenby, *Fox's congruence classes and the quantum-SU(2) invariants of links in 3-manifolds,* Comment. Math. Helv. 71 (1996) 664-677.

[Mu] H. Murakami, *Optimistic calculations about the Witten–Reshetikhin–Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries,* in *Recent Progress towards the volume conjecture* Surikaisekikenkyusho Kokyuroku 1172, 2000, pp. 70-79.

[RT] N. Reshetikhin and V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991), no. 3, 547–597.

[Th] W. Thurston, *The Geometry and Topology of 3- manifolds*, Available at http://www.msri.org/publications/books/gt3m.

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