CROSSING CHANGES OF FIBERED KNOTS

EFSTRATIA KALFAGIANNI

Abstract. We prove that a crossing circle L of a fibered knot K bounds a disc in the complement of K, if and only if there is a crossing change supported on L that doesn't change the isotopy class of K. We also sow that if a knot K is n-adjacent to a fibered knot K', for some $n > 1$, then either the genus of K is larger that of K' or K is isotopic to K'. This statement leads to criteria for detecting non-fibered knots and it has some applications in the theory of finite type 3-manifold invariants.

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1. Introduction

An open question in classical knot theory is the question of when a crossing change in a knot changes the isotopy class of the knot. The purpose of this paper is to answer this question for fibered knots.

A crossing disc for a knot $K \subset S^3$ is an embedded disc $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic intersection number. A crossing change on K can be achieved by twisting D or equivalently by performing appropriate Dehn surgery of S^3 along the crossing circle ∂D. The crossing is called nugatory if and only if ∂D bounds an embedded disc in the complement of K. This disc and D form a 2-sphere that decomposes K into a connected sum, where some of the summands may be trivial. Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. Problem 1.58 of Kirby's Problem List ([GT]), asks whether the converse is true: That is, if a

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crossing change in a knot K yields a knot isotopic to K is the crossing nugatory?

In the case that K is the trivial knot an affirmative answer follows from a deep result of D. Gabai ([Ga]) that describes the behavior of the Thurston norm under Dehn filling (see [ScT]). In [To], I. Torisu obtained an affirmative answer for 2-bridge knots. He also observed that in general the question can be reduced to the analogous question for prime knots. The argument of [To] first uses the so called "Montesinos trick" to realize the crossing change on a knot by Dehn surgery in the 2-fold cover of S^3 branching over the knot. The 2-fold covers of 2-bridge knots are lens spaces and the result follows by applying the cyclic surgery theorem of Culler, Gordon, Luecke and Shalen ([CGLS]). In this paper we will show the following.

Theorem 1.1. *Let K be a fibered knot. A crossing change in K yields a knot isotopic to K if and only if the crossing is nugatory.*

The main ingredients of the proof of Theorem 1.1 are the aforementioned result of Gabai on the Thurston norm and Dehn surgery, and a result of D. Kotschick about the "commutator length" of certain products of Dehn twists in the mapping class group of surfaces. To give a brief outline of the proof, let K be a fibered knot such that a crossing change on K gives a knot K' that is isotopic to K. The complement of K can be fibered over S^1 with fiber, say S; a minimum genus Seifert surface of K. Roughly speaking, Gabai's result implies that the crossing change from K to K' can be achieved along an arc that is properly embedded and essential on S. Equivalently, the crossing change can be achieved by twisting K along a meridian disc D of a handlebody neighborhood, say $N = N(S)$, of the fiber. Using properties of curves on handlebodies and properties of Heegaard splittings we are able to show that, in many cases, ∂D must bound an embedded disc whose interior is disjoint from K. In the remaining cases, using the uniqueness properties of knot complement fibrations, the problem reduces to the question of whether a power of a Denh twist on the surface ∂N along the curve ∂D, can be written as a single commutator in the mapping class group of the surface. In the recent years, this question has arisen in the study of Lefschetz fibrations and the theory of symplectic 4-manifolds and has been studied in this context ([EKo], [BrKo], [Ko], [KrO]). In particular, a result of Kotschick ([Ko]) implies that a product of Dehn twists of the same sign, along a collection of disjoint, homotopically essential curves on an orientable surface cannot be written as a single commutator in the mapping class group of the surface. Using this result, we show that the assumption that K is isotopic to K' implies that ∂D must be contractible on ∂N. This, in turn, easily implies that the crossing corresponding to D is nugatory.

Theorem 1.1 says that an essential crossing change always changes the isotopy class of a fibered knot. It is natural to ask whether the crossing change produces a simpler or more complicated knot with respect to some knot complexity. A complexity function whose interplay with crossing changes has been studied using the theory of taut foliations and sutured 3-manifolds is the knot genus. For example [ScT] studies the interplay of link genus and Conway skein moves. Simple examples show that a single crossing change may decrease or increase the genus of a knot even if one stays within the class of fibered knots. However by replacing a crossing change by the more refined notion of knot adjacency studied in [KL], we are able to derive stronger conclusions. To state a result in this direction, we recall that K is called 2-adjacent to K' if K admits a projection that contains two crossings such that changing any of them or both of them simultaneously, transforms K to K'.

Theorem 1.2. *Let $g(K)$ and $g(K')$ denote the genera of K and K', respectively. Suppose that K' is a fibered knot and that K is 2-adjacent to K'. Then, either $g(K) > g(K')$ or K is isotopic to K'.*

It is well known that the Alexander polynomial can be used to detect non-fibered knots. Theorem 1.2, and its generalization given in Section 7, can be used to obtain criteria for detecting non-fibered knots when the Alexander polynomial provides inconclusive evidence. This direction is explored in [KL1] where we also provide applications in the theory of finite type invariants of 3-manifolds.

We organize the paper paper as follows: In Section 2 we summarize the mapping class group results that we need for the proof of Theorem

1.1. In Section 3 we discuss basic facts about curves and discs in handlebodies and prove two technical lemmas that we use in subsequent sections. In Section 4 we summarize some well known uniqueness properties of knot complement fibrations. Then, in Section 5 we develop a setting relating fibrations of knot complements and Heegaard splittings of S^3, from the point of view needed in the rest of the paper. In Section 6, we study nugatory crossings of fibered knots and we prove Theorem 1.1; in fact we prove a more general result. In Section 7 we study adjacency to fibered knots and prove Theorem 1.2. In Section 8 we discuss some examples and possible generalizations of the results here.

Throughout the paper we work in the PL or the smooth category.

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2. Commutator lengths and Dehn twists

2.1. **Commutators in the mapping class group.** Let Σ_k denote a closed oriented surface of genus k and let Γ_k denote the mapping class group of Σ_k. That is Γ_k is the group of isotopy classes of orientation preserving homeomorphisms $\Sigma_k \longrightarrow \Sigma_k$. Let $\Gamma'_k := [\Gamma_k,\ \Gamma_k]$ denote the commutator subgroup of Γ_k. An element $f \in \Gamma'_k$ is written as a product of commutators. The commutator length of f, denoted by $c(f)$, is the minimum number of factors needed to express f as a product of commutators. In the recent years, the growth of the commutator length of Dehn twists has been studied using methods from the theory of symplectic four-manifolds ([EKo]), [BrKo], [Ko], [KrO]). Here we will need the following result of D. Kotschick.

Theorem 2.1. [Theorem 7, [Ko]] *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geq 2$. Suppose that $f \in \Gamma_k$ is a product of right-handed Dehn twists along homotopically essential disjoint simple closed curves $a_1, \ldots, a_m \subset \Sigma_k$. If $f^q \in \Gamma'_k$, for some $q > 0$, then we have*

$$c(f^q) \geq 1 + \frac{qm}{18k-6}.$$

Remark 2.2. For a single Dehn twist ($m = 1$) along a separating curve Theorem 2.1 was proved in [EKo]. The assumption that the curve on which the Dehn twist takes place be separating was removed in Theorem 12 of [BrKo].

Remark 2.3. It is known that the abelianization of Γ_k is finite ([Mac], [Mu]). In particular, for $k > 2$, Γ_k is know to be a prefect group ([Po]); that is, we have $\Gamma_k = \Gamma'_k$. Hence, every element $f \in \Gamma_k$ can be written as a product of commutators and $c(f)$ is well defined. Thus, for $k > 2$, Theorem 2.1 applies to every power of a product of right-handed Dehn twists along essential disjoint simple closed curves. The abelianization of Γ_2 is known to be isomorphic to the cyclic group of order 10 and it is generated by any Dehn twist along a non-separating simple curve on Σ_2. In this case, Theorem 2.1 applies to powers divisible by 10.

For a simple closed curve $a \subset \Sigma_k$ let T_a denote the right hand Dehn twist about a; then the left hand Dehn twist about a is T_a^{-1} (compare, Figure 2). We will need the following corollary of Theorem 2.1.

Corollary 2.4. *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geq 2$. Let $a \subset \Sigma_k$ be a simple closed curve. Suppose that there exist $g, h \in \Gamma_k$ such that*

$$T_a^q = [g, h] = ghg^{-1}h^{-1},$$

for some $q \neq 0$. Then c is homotopically trivial on Σ_k.

Proof: First suppose that $q > 0$. By assumption, we have $T_a^q \in \Gamma'_k$. If a is homotopically essential on Σ_k then Theorem 2.1 applies to give that

$$c(T_a^q) \geq 1 + \frac{q}{18k-6} > 1.$$

On the other hand, since T_a^q is by assumption a simple commutator we have $c(T_a^q) \leq 1$ which is a contradiction. Thus, a has to be homotopically trivial.

If $q < 0$ then just apply the argument above to T_n^{-q}. □

The proof of Theorem 2.1 given in [Ko] relies on the theory of Lefschetz fibrations, which, as the author points out, is sensitive to the chirality of Dehn twists. In fact, the argument of [Ko] breaks down if one allows f to be a product of right-handed Dehn twists and their inverses and, as the following example shows, Theorem 2.1 is not true in this case. In subsequent sections we will discuss how this situation is reflected when one tries to apply Theorem 2.1 to the study of crossing changes that do not alter the isotopy class of fibered knots (see Example 8.2).

Example 2.5. [Example 3.3, [Ko]] Suppose that $a \subset \Sigma_k$ is an essential simple closed loop on a closed oriented surface of genus at least two. Let $g : \Sigma_k \longrightarrow \Sigma_k$ be an orientation preserving homeomorphism such that $a \cap g(a) = \emptyset$. We will also use g to denote the mapping class of g. Set $b := g(a)$ and set $f := T_a T_b^{-1}$. In the mapping class Γ_k we have $g T_a g^{-1} = T_{g(a)}$ or equivalently $T_a = g^{-1} T_b g$. Since a, b are disjoint we also have $T_a T_b^{-1} = T_b^{-1} T_a$. Thus

$$f^q = (T_a T_b^{-1})^q = T_a^q T_b^{-q} = (g^{-1} T_b g)^q T_b^{-q} = [g^{-1}, T_b^q],$$

for all $q > 0$. Hence we have $c(f^q) = 1$ showing that Theorem 2.1 and Corollary 2.4 are not true in this case.

2.2. **When is T_a^q trivial?** It is known that if a is homotopically essential on Σ then no non-trivial power T_a^q ($0 \neq q \in \mathbf{Z}$) is isotopic to the identity on Σ_k. This statement is well known to researchers working on mapping class groups: It is for example asserted in [BiLM] when the authors state that the kernel of the reduction homomorphism corresponding to an essential simple closed curve a is the free abelian group generated by Dehn twists along a. Below we show how to derive the statement using properties of intersection numbers steaming from Thurston's study of surface homeomorphisms ([FLP]).

Corollary 2.6. *Suppose that $T_a^q = 1$ in the mapping class group $\Gamma := \Gamma(\Sigma_k)$, $k > 0$. Then, either a is homotopically trivial on Σ or $q = 0$.*

Proof: Suppose that the curve a is not homotopically trivial on Σ and that $T_a^q = 1$ in the mapping class group $\Gamma(\Sigma)$. We will argue that $q = 0$. First suppose that a is a non-separating loop on Σ. Then, we can find an embedded loop b that intersects a exactly once. Orient a, b so that the algebraic intersection (see Figure 1) of a, b is 1; that is $< a, b >= 1$. In $H_1(\Sigma)$ we have

$$T_a^q(b) = b + q < a, b > a = b + qa.$$

Thus we have $< T_a^q(b), b >=< b, b > +q < a, b >$ which, since $T_a^q(b) = b$, gives $q = 0$ as desired. If a is separating, we appeal to the geometric intersection number. For b a simple closed loop on Σ let $i(a, b)$ denote the intersection number; the minimum number of intersections in the isotopy classes of a and b. Since we assumed that a is homotopically essential on Σ, we can find b so that $i(a, b) \neq 0$. Proposition 1, Exposé 4 [FLP] implies the following:

$$i(T_a^q(b),\ b) = |q|\ (i(a,\ b))^2.$$

Since $T_a^q = 1$, we have $0 = i(b,\ b) = i(T_a^q(b),\ b)$. Thus we obtain $|q|\ (i(a,\ b))^2 = 0$; which implies that $q = 0$. $\square$

Suppose that Σ_k is the boundary of a handlebody N_k of genus k and let Δ_k denote the subgroup of Γ_k consisting of those classes that can be represented by homeomorphisms $\Sigma_k \longrightarrow \Sigma_k$ that extend on N_k. Clearly, if the loop a bounds an embedded disc $D \subset N_k$ then $T_a^q \in \Delta_k$ since T_a^q extends to a Dehn twist of N_k along D. The following result shows that the converse is also true:

Theorem 2.7. ([O], [Mc]) *Let a be a simple loop on $\Sigma_k = \partial N_k$ and let T_a denote the right hand side Dehn twist along a. Suppose that there is and integer $q \neq 0$, such that T_a^q is isotopic on Σ_k to a homeomorphism $h : \Sigma_k \longrightarrow \Sigma_k$ that extends to N_k; that is $T_a^q \in \Delta_k$. Then, a bounds a disc in N_k.*

2.3. **Commutators in the extended mapping class group.** Corollary 2.4 is not true for elements in the extended mapping class group of Σ_k; that is if one allows orientation reversing homeomorphisms of Σ_k. Below we give a simple example; John McCarthy has constructions of more interesting examples ([M]) showing that a Dehn twist in the extended mapping class group can be written as a single commutator of two pseudo-Anosov homeomorphisms.

Example 2.8. Let T_c denote the right-handed Dehn twist along a simple closed curve $c \subset \Sigma_k$. Let $g : \Sigma_k \longrightarrow \Sigma_k$ be an orientation-reversing homeomorphism of Σ_k such that $g(c) = c$. Since g is orientation-reversing we have $gT_cg^{-1} = T^{-1}_{g(c)}$. Since $g(c) = c$, it follows that $gT_cg^{-1} = T_c^{-1}$. Hence, $g^{-1}T_c^{-1}g = T_c$ and thus

$$[T_c, g^{-1}] = T_cg^{-1}T_c^{-1}g = T_c^2.$$

Notation. To simplify our notation, throughout the paper, we will use $\Sigma := \Sigma_k$ to denote an oriented surface of any genus $k \geq 0$ and $\Gamma := \Gamma_k$ to denote the mapping class group of Σ.

3. Handlebodies and disc-busting curves

In this section we recall some terminology and basic properties about handlebodies and prove some auxiliary results needed in the remaining of the paper. As in Section 2, Σ will denote a closed oriented surface.

3.1. **Preliminaries.** Here, we follow the exposition in [RR]: A collection of mutually disjoint simple closed curves $\mathbf{x} := \{x_1, \ldots, x_n\} \subset \Sigma$ is called a *complete system* iff Σ cut along these curves is a connected planar surface. A *super-complete system* is a collection of mutually disjoint simple curves $\mathbf{x} \subset \Sigma$ that contains a complete system. In the case of a super-complete system of curves $\mathbf{x}$ we will always assume that they are mutually non-isotopic on Σ. Given a super-complete system $\mathbf{x} \subset \Sigma$ let $N(\mathbf{x})$ denote the handlebody, with boundary Σ, obtained by attaching a collection of discs, say $\mathbf{D}$, along $\mathbf{x}$ and then attaching 3-balls. The collection $\mathbf{D}$ is called a (super)-complete system of meridian discs for $N(\mathbf{x})$.

Remark 3.1. Let $\mathbf{x} := \{x_1, \ldots, x_n\}$ and $\mathbf{y} := \{y_1, \ldots, y_n\}$ be super-complete systems of curves on Σ. We will say that $\mathbf{x}$, $\mathbf{y}$ are equivalent (or they define the same handlebody structure) iff $N(\mathbf{x})$, $N(\mathbf{y})$ are homeomorphic by a homeomorphism that is the identity on Σ.

Given a super-complete system of curves $\mathbf{x} := \{x_1, \ldots, x_n\}$ and an additional simple closed curve y on Σ then, after orienting Σ and all the curves one can assign a sign to each transverse intersection point of $y \cap \mathbf{x}$ using the convention of Figure 1.

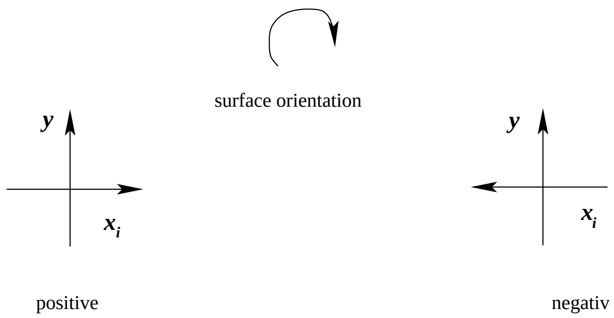


FIGURE 1. Sign conventions for forming $w(\mathbf{x}, y)$.

Then one can read off a cyclic word $w(\mathbf{x}, y)$ in the symbols $x_1, \ldots, x_n$ as follows: Transverse y in the given orientation, and for each transverse intersection point with a x_i read x_i or ${x_i}^{-1}$ according to whether the intersection point is positive or negative according to the aforementioned fixed sign convention. We have the following:

Lemma 3.2. (Lemma 2, [RR]) *The curve y bounds a disc in the handlebody $N(\mathbf{x})$ if and only if the word $w(\mathbf{x}, y)$ reduces to the empty word by cancellation; we will write $w(\mathbf{x}, y) = 1$.*

We will say that an isotopy class of simple loops $J \subset \Sigma$ is disc-busting for a handlebody N, with $\Sigma = \partial N$, if for every meridian disc, say D, of N and every loop γ representing J, the boundary ∂D intersects γ. This means that the geometric intersection number $|\gamma \cap \partial D|$ is at least two. Throughout the paper, we will call every representative $\gamma \in J$ of a disc busting class, a disc-busting curve.

Lemma 3.3. *Suppose that S a Seifert surface of a knot $K \subset S^3$ such that S is incompressible in $S^3 \setminus K$. Then, $K := \partial S \times \{\frac{1}{2}\}$ is a disc-busting curve of the handlebody $N := S \times I$, ($I := [0,1]$). Thus, in particular, if S is a minimum genus Seifert surface of K then K is a disc-busting curve of N.*

Proof: It follows immediately from the observation that a simple closed curve $K \subset \Sigma := \partial N$ is disc-busting if and only if the surface $\Sigma \setminus K$ is incompressible in N. □

Next we state a lemma that is a special case of a well known result about incompressible surfaces in I-bundles which we will need in subsequent sections. The proof is given in [W].

Proposition 3.4. (Proposition 3.1, [W]) *Let S be a compact, connected, oriented surface with one boundary component. Let $N := S \times I$ be an I-bundle over S with bundle projection $p : N \longrightarrow S$. Let F be a collection of properly embedded, incompressible surfaces in N. Suppose that ∂F can be isotoped on ∂N so that it lies on $S \times \{1\}$. Then, after an ambient deformation that is constant on ∂N, we have that $p|F_1 : F_1 \longrightarrow S$ is a homeomorphism on each component $F_1 \subset F$. This means that every component of F is parallel to $S \times \{1\}$.*

3.2. **Disc-busting curves and Dehn twists.** In this subsection we prove a couple of auxiliary lemmas that we will need in subsequent sections to finish the proof of the main results.

Lemma 3.5. *Let S be a compact, connected, oriented surface with one boundary component. Consider the handlebody $N =: S \times I$ with bundle projection $p : N \longrightarrow S$ and the simple closed curve $K := \partial S \times \{\frac{1}{2}\}$ on ∂N. Let $N'' \subset N$ be a handlebody of the same genus as that of N and set $W := \overline{N \setminus N''}$. Suppose that there exists a simple closed curve $K'' \subset \partial N''$ that is separating on $\partial N''$ and disc-busting in N''. Suppose, moreover, that K'' and K cobound an embedded annulus $E \subset W$ such that $p|E : E \longrightarrow p(E)$ is a homeomorphism, and that $\partial N'' \setminus K''$ is incompressible in W. Then, after a deformation that is constant on ∂N, we can arrange so that the restriction of p on each component of $\partial N'' \setminus \eta(K'')$ is a homeomorphism. Here, $\eta(K'')$ is a small neighborhood of K'' on $\partial N''$.*

Proof: Let $\eta(K'')$ be a small annulus neighborhood of K'' on $\partial N''$ and let S_1'', S_2'' denote the two components of $\partial N'' \setminus \eta(K'')$. We can take $\eta(K'')$ to be the intersection of a product neighborhood $E \times [\epsilon_1, \epsilon_2]$ with $\partial N''$. Now, for $i = 1, 2$, we have a spanning annulus $E_i := E \times \{\epsilon_i\}$ in W, such that ∂E_i consists of $\partial S_i''$ and a simple closed curve that is parallel to K on ∂N. Since we assumed that $p|E$ is a homeomorphism, we may assume that $p|E_i$ is a homeomorphism. The assumptions that $\partial N'' \setminus K''$ is incompressible in W and that K'' is disc-busting in N'', guarantee that F_i is a properly embedded incompressible surface in N. The conclusion follows by applying Proposition 3.4 to the collection $F := \{F_1, F_2\}$. $\square$

In order to facilitate our exposition in the rest of the subsection we need the following definition.

Definition 3.6. *Let N be a handlebody and let γ be a simple, closed, separating curve on $\Sigma := \partial N$ that is disc-busting for N. A super-complete system of curves $\mathbf{x} := \{x_1, \ldots, x_n\} \subset \Sigma$ will be called admissible for γ, iff every $x_i \in \mathbf{x}$ intersects γ at exactly two points with zero algebraic intersection number. Note, that if N and $\gamma := K$ are as in the statement of Lemma 3.3, then there is a super-compete system of curves that is admissible for γ.*

Now we state and prove the second lemma promised earlier.

Lemma 3.7. *Let N be a handlebody and let γ be a simple, closed, separating curve on $\Sigma := \partial N$ that is disc-busting for N. Suppose that there exists a super-complete system of curves $\mathbf{x} := \{x_1, \ldots, x_n\}$ that is admissible for γ and such that $N = N(\mathbf{x})$. Let z be an additional simple closed curve on Σ that intersects γ at exactly two points with zero algebraic intersection number. Let $T := T_z^q : \Sigma \longrightarrow \Sigma$ denote a non-trivial power ($q \neq 0$) of the right handed Dehn twist along z that is supported in an annulus $B \subset \Sigma$. Then, one of the following is true:*

(a) The curve $\gamma' := T(\gamma)$ is disc-busting for N.

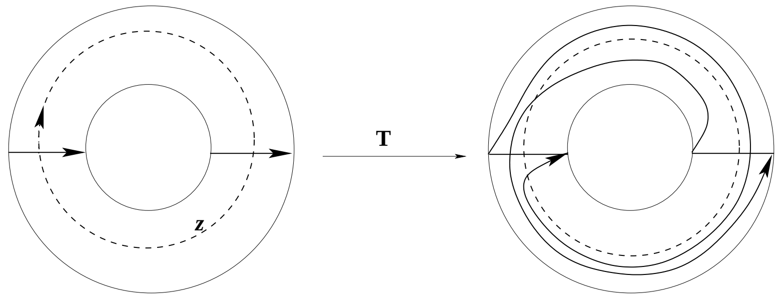
(b) We can find a super-complete system of curves $\mathbf{x}'$ that is admissible for γ and such that the following property holds: There exists $x' \in \mathbf{x}'$ such that $(x' \cap \gamma) \subset B$, $|x' \cap z| = 1$ and $|\gamma' \cap x'| = 0$. Furthermore, for every $x'' \in \mathbf{x}'$, with $x'' \neq x'$, we have $|x'' \cap B| = 0$.

Proof: We can take the annulus B to be a neighborhood of z on $\Sigma := \partial N$ so that z is the core of B. We may and will assume the following:

(i) $\bar{\gamma} := \gamma \setminus (\gamma \cap B) = \gamma' \setminus (\gamma' \cap B)$.

(ii) $\gamma \cap B$ consists of two arcs, say α_1, α_2, that are parallel in B and each of which intersects z exactly once.

(iii) $\gamma' = \bar{\gamma} \cup (\beta_1 \cup \beta_2)$, where for $i = 1, 2$, $\beta_i = T(\alpha_i)$.



(a) (b)

FIGURE 2. The annulus B with z, $B \cap \gamma$ and $B \cap \gamma'$ in the case that $q = -1$. Part (a) shows z and $\gamma \cap B = \alpha_1 \cup \alpha_2$; part (b) also incorporates the arcs $\beta_i = T(\alpha_i)$, $i = 1, 2$.

To continue, let S_1, S_2 denote the two components of $\Sigma \setminus \gamma$. Since γ is disc-busting in N each of S_1, S_2 is surface with boundary γ that is incompressible in N. Let $\mathbf{D}$ denote a super-complete collection of discs of N bounded by the curves in $\mathbf{x}$. The tactic of the proof will be to assume that $\gamma' := T(\gamma)$ is not disc-busting and show that, then, conclusion (b) must hold.

Since γ' is not disc-busting, there is a simple closed curve y that is homotopically essential on Σ, bounds a disc in N and such that $y \cap \gamma' = \emptyset$. Among all such curves we will choose one that minimizes the total geometric intersection, say $|\mathbf{x} \cap y|$, with the curves in $\mathbf{x}$. We distinguish two cases:

Case 1: Suppose that $|y \cap \mathbf{x}| = 0$. Since γ is disc-busting for N, y must intersect γ in an essential way. By (i)-(iii) above, the intersection $y \cap B$ consists of a collection of arcs each of which is parallel to $\beta_{1,2}$ in B. Then, either $\mathbf{x} \cap B = \emptyset$ or the intersection $\mathbf{x} \cap B$ consists of a collection of arcs each of which is also parallel to $\beta_{1,2}$ in B. Suppose that $\mathbf{x} \cap B = \emptyset$; thus in particular $\mathbf{x} \cap z = \emptyset$. Then, clearly, z satisfies the hypothesis of Lemma 3.2 and thus z bounds a disc in N. Then, the homeomorphism $T : \partial N \longrightarrow \partial N$ extends to a homeomorphism of N; namely a Dehn twist on N along this disc bounded by z. But then $\gamma' := T(\gamma)$ is disc busting on N; a contradiction. Thus we must have that $\mathbf{x} \cap B \neq \emptyset$. Let β denote a component of that intersection. Suppose $\beta \subset (x \cap B)$ for some $x \in \mathbf{x}$. As observed earlier, β must be an properly embedded arc in B that is parallel to β_i, $i = 1, 2$. Notice that β_i intersects each of the arcs $\alpha_{1,2}$ at least once; thus β intersects each of $\alpha_{1,2}$ at least once. Since each curve in $\mathbf{x}$ intersects γ at exactly two points (with zero algebraic intersection number), this case can only happen if β intersects each of $\alpha_{1,2}$ exactly once. In that case it follows that $x \cap B = \beta$ and that the only intersections of x and γ are those between β and γ. But since each of $\beta_{1,2}$ intersects z exactly once, the curve $x' := x$, satisfies the properties: $(x' \cap \gamma) \subset B$, $|x' \cap z| = 1$ and $|\gamma' \cap x'| = 0$. See Figure 3.

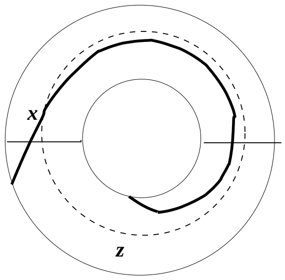


FIGURE 3. The intersection $x \cap B$ when $x \cap \gamma \subset B$ and $|x \cap z| = 1$.

To prove that conclusion (b) holds, in this case, we now show that for every $x_i \in \mathbf{x}$, with $x_i \neq x$, we have $|x_i \cap z| = 0$; this will finish the proof of the lemma in this case. Suppose, on the contrary, that there is $x_i \in \mathbf{x}$, with $x_i \neq x$, and such that $|x_i \cap z| \neq 0$. Since $|x_i \cap x| = 0$, $x_i \cap B$ must consist of arcs parallel to β in B. But since x_i intersects γ exactly twice there can be exactly one such arc that can, moreover, be isotoped to β in B. We can isotope x_i and x on Σ so that their parts in the annulus

B are identical; that is $x_i \cap B = x \cap B = \beta$. Now, $\gamma := (x_i \setminus \beta) \cup (x \setminus \beta)$ is a closed curve on $\Sigma \setminus \gamma$ that is homotopically trivial in N. Since $\Sigma \setminus \gamma$ is incompressible in N, γ is homotopically trivial on Σ. It follows that x_i and x are isotopic on Σ which contradicts the assumption that $\mathbf{x}$ is a super-complete system of curves. This contradiction finishes the proof of this case.

Case 2: Suppose that $|y \cap \mathbf{x}| \neq 0$. For $i = 1, \ldots, n$, let D_i denote the disc in $\mathbf{D}$ bounded by x_i and let D a meridian disc of N bounded by y. By assumption, there is $1 \leq i \leq n$, such that $|x_i \cap y| \neq 0$. After an isotopy in the interiors of the discs in $\mathbf{D}$ and of the interior of D we can arrange so that the intersection $\mathbf{D} \cap D$ consists of a collection of disjoint, properly embedded arcs on D. Suppose for a moment that $\mathbf{x} \cap B = \emptyset$ and thus $z \cap \mathbf{x} = \emptyset$. Then, by Lemma 3.2, z bounds a disc in N which, as stated earlier, contradicts the assumption that γ' is not disc-busting. Hence, we can assume that $\mathbf{x} \cap B$ is non-empty. Note that if $\mathbf{x}$ doesn't intersect y inside B then, $\mathbf{x} \cap B$ must be a collection of arcs each of which is parallel to $\beta_{1,2}$ in B and we are reduced to Case 1. Thus, from now on we may assume that $\mathbf{x} \cap B$ is a collection of arcs each of which is parallel to $\alpha_{1,2}$ in B. Then, using the fact that y was chosen so that $|y \cap \mathbf{x}|$ is minimized, we will show:

Claim: We have $|y \cap \gamma| = 2$. Furthermore, for every $D_i \in \mathbf{x}$, either $D \cap D_i$ is empty or it consists of a single arc, say η_i, so that the two points in $y \cap \gamma$ (resp. $x_i \cap \gamma$) lie in different components of $D \setminus \eta_i$ (resp. $D_i \setminus \eta_i$).

Proof of Claim: The Claim will be seen to follow from the next two subclaims and the fact that $\mathbf{x} \cap B$ consists of arcs parallel to $\alpha_{1,2}$ in B.

Subclaim 1: Each component, say a, of $D_i \cap D \neq \emptyset$ must separate the points $x_i \cap \gamma$ on D_i. That is the two points in $x_i \cap \gamma$ lie in different components of $D_i \setminus a$.

Proof of Subclaim 1: Suppose, on the contrary, that there is an arc $a \subset x_i \cap y$ such that the points of $x_i \cap \gamma$ lie in the same component of $D_i \setminus a$. Then we can find such an arc a that its outermost on x_i; it cuts off a disc $E \subset D_i$ whose interior is disjoint from γ and $\partial E = a \cup w$,where

w is an arc on x_i that is disjoint from γ and from y. Now, a cuts y into arcs, say y', y'' as shown in Figure 4.

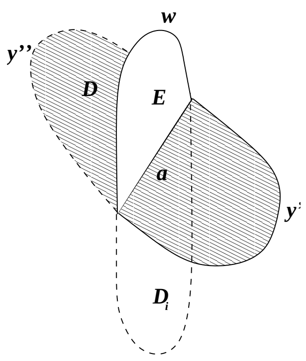


FIGURE 4. Part of the intersection $D_i \cap D$.

Note that both of y', y'' must intersect γ. For if one of them , say, y' is disjoint from γ then the curve $w \cup y'$ would bound a disc on $\partial N \setminus \gamma$ which, in turn, would result to an isotopy reducing $|\mathbf{x} \cap y|$; a contradiction. Next, let us observe that the curve $y' \cup w$ is disjoint from γ': Since $y' \subset y$, by assumption, y' is disjoint from γ'. Thus, the curve $y' \cup w$ can intersect γ' only along w. Furthermore, since $w \cap \gamma = \emptyset$ and γ' agrees with γ outside B, the arc w can intersect γ' only inside B. On the other hand, since $y' \cap \gamma \neq \emptyset$ the intersection $y' \cap B$ lies on arcs that run parallel to $\beta_{1,2}$ in B and intersect $\gamma \cap B = \alpha_1 \cup \alpha_2$. But then, since the intersection $w \cap B$ is arcs parallel to $\alpha_{1,2}$, we conclude that w intersects one of y', y'' inside B. But this is clearly absurd since w was chosen so that it contains no further intersections with y. This contradiction implies that the curve $y' \cup w$ is disjoint from γ'. Now by considering a parallel copy of $y' \cup \eta$ on ∂N (compare, Lemma 1 of [RR]) we obtain a simple closed curve y_1 that bounds a disc in N, $y \cap \gamma' = \emptyset$ and $|y_1 \cap \mathbf{x}| < |y \cap \mathbf{x}|$. Since this contradicts our minimality assumption Subclaim 1 is proved.

Subclaim 2: If $D_i \cap D \neq \emptyset$, then it consists of a single arc.

Proof of Subclaim 2: By Subclaim 1, $D_i \cap D$ consists of a collection of arcs that are parallel parallel in D_i and each separates the points in $x_i \cap \gamma$ on D_i. If $|D_i \cap D| > 1$ then we can pick a pair of components that are innermost on x_i; they cut off a a disc $E \subset D_i$ whose interior is disjoint from y and its boundary disjoint from γ. Then, arguing as in Subclaim 1, we can replace a suitable subdisc of D by E, to obtain a simple closed curve y_1 with the following properties: y_1 bounds a disc in N, $y_1 \cap \gamma' = \emptyset$ and $|y_1 \cap \mathbf{x}| < |y \cap \mathbf{x}|$; a contradiction. The details of the argument are similar to this of Subclaim 1.

To finish the proof of the Claim it remains to show that $|y \cap \gamma| = 2$. For, the fact that the points in $y \cap \gamma$ must lie on different components of $D \setminus \eta_i$ is immediate from Subclaim 1 since γ is separating. Let D_i such that $D_i \cap D \neq \emptyset$. By Subclaim 2, $|x_i \cap y| = 2$ so the intersection of y with each component of $x_i \cap B$ is at most two points. We claim that, for some $1 \leq i \leq n$, we have a component $w \subset x_i \cap B$, such that $|w \cap y| = 1$. For, otherwise y will intersect each component of $\mathbf{x} \cap B$ twice geometrically and with zero algebraic number. Then using an outermost argument and Lemma 3.2 we would be able to reduce $|\mathbf{x} \cap y|$ by isotopy of y on Σ; a contradiction. Now the condition $|w \cap y| = 1$, implies that $y \cap B$ must be an β that intersects each of $\alpha_{1,2}$ exactly once (see Figure 3); thus $|y \cap \gamma| = 2$.

Now we finish the proof of Case 2 and that of the lemma: Suppose, without loss of generality, that y intersects the curves $x_1, \ldots, x_s$ and is disjoint from $x_{s+1}, \ldots, x_n$. By the Claim, $|y \cap \gamma| = 2$ and, for $i = 1, \ldots, s$, $y \cap x_i$ consists of a single arc that separates the components of $y \cap \gamma$ (resp. $x_i \cap \gamma$) on D (resp. D_i). Now we apply the process in the proof of Lemma 1 of [RR] to obtain a super-complete system $\mathbf{x}'$ of N that contains y and $x_{s+1}, \ldots, x_n$. By the intersection conditions imposed earlier, $\mathbf{x}'$ is admissible for γ and the curve $x' := y$ has the properties of (b) in the statement of the lemma. $\square$

4. Uniqueness properties of knot fibrations

Here we recall some terminology and known results about fibered knots. Suppose that K is a fibered knot and let S be a minimum genus Seifert surface for K. Then, as it is argued in the proof of Theorem 5.1 in [BZ], the complement $S^3 \setminus K$ admits a fibration over S^1 with fiber S. More specifically, it is shown that the complement $S^3 \setminus K$ cut along S is homeomorphic to $S \times [-1, 1]$. Thus, there is an orientation preserving homeomorphism $h : S \longrightarrow S$ such that $S^3 \setminus K$

is obtained from $S \times [-1, 1]$ by identifying $S \times \{-1\}$ with $S \times \{1\}$ so that $(x, -1) = (h(x), 1)$. The map h is called the monordormy of the fibration. Following the notation of Section 5 of [BZ], we write

$$S^3 \setminus K = S \times J/h,$$

where $J := [-1, 1]$. In particular, the proof of Theorem 5.1 of [BZ] shows the following:

Lemma 4.1. *Let $M = S^3 \setminus K = S \times J/h$ be an oriented, fibered knot complement. Given a minimum genus Seifert surface S_2 of K, there exists an orientation preserving homeomorphism of M that is fixed on K and brings S_2 to the fiber $S_1 := S \times \{1\} = S \times \{-1\}$.*

We also need to recall the following.

Proposition 4.2. [Proposition 5.10, [BZ]] *Suppose that $M := S \times J/h$ and $M' := S' \times J/h'$ are fibered, oriented knot complements. Then, there exists an orientation preserving homeomorphism $F : M \longrightarrow M'$, with $F(\partial S) = \partial S'$, if and only if there exists an orientation preserving surface homeomorphism $f : (S, \partial S) \longrightarrow (S', \partial S')$ such that fhf^{-1} and h' are equal up to isotopy on S'.*

5. Handlebodies in knot complements

5.1. **Definitions.** In this section we will consider decompositions of knot complements that arise from Heegaard splittings of S^3. We begin by recalling some familiar terminology about Heegaard splittings: A compression body N is an oriented 3-manifold obtained from a product $\Sigma \times [0, 1]$, where Σ is a closed, connected, oriented surface, by attaching 2-handles along a collection of disjoint, simple closed curves on $\Sigma \times \{1\}$ and capping off the newly produced 2-sphere boundary components with 3-cells. Let $\partial_+ N := \Sigma \times \{0\}$ and $\partial_- N := \partial N \setminus \partial_+ N$. If $\partial_- N = \emptyset$ then N is a handlebody. A trivial compression body is a product $\Sigma \times [0, 1]$. A compression body N can be reduced to a trivial compression body (or to a 3-ball if $\partial_- N = \emptyset$) by cutting along a complete disc system $(\mathcal{D}, \partial \mathcal{D}) \subset (N, \partial_+ N)$.

Definition 5.1. *A Heegaard splitting of a compact, oriented 3-manifold M, is a decomposition $M = N_1 \cup N_2$, where N_1 and N_2 are compression bodies and $Q := N_1 \cap N_2 = \partial_+ N_1 = \partial_+ N_2$ is an embedded orientable surface. The surface Q is called the Heegaard surface of the decomposition.*

We will be particularly interested in handlebodies of even genus. Let P denote a model genus k compact, connected, oriented surface with one boundary component. It will be convenient for us to think of the model oriented handlebody of genus $2k$ as a product $N := P \times I$.

Definition 5.2. *A simple closed curve K on the boundary of a $2k$ genus handlebody N_1 is called a preferred curve if there exists a homeomorphism $N_1 \longrightarrow N$ that takes K onto the curve $\partial P \times \{\frac{1}{2}\}$ of ∂N.*

An H-body is a 3-manifold H_1 that is homeomorphic to the complement of a knot in a handlebody. That is given a handlebody N_1 and a knot $K \subset N_1$, let $\eta(K)$ be a tubular neighborhood of K in N_1. Then, $H_1 := \overline{N_1 \setminus \eta(K)}$ is an H-body. If K can be isotoped on ∂N_1 so that it is a preferred curve of N_1, then H_1 is called a preferred H-body for K. We will use $\partial_+ H_1$ to denote the non-torus component of ∂H_1.

Definition 5.3. *An HN-splitting of a knot complement $M := S^3 \setminus K$ is a decomposition $M = H_1 \cup N_2$, where N_2 is a handlebody and H_1 is an H-body. The closed oriented surface $Q := H_1 \cap N_2 = \partial_+ H_1 = \partial N_2$ is called an HN-surface of M. The genus of Q is called the genus of the HN-splitting.*

Remark 5.4. Note that an HN-splitting $M = H_1 \cup N_2$ of a knot complement $M = S^3 \setminus K$ is not a Heegaard splitting of M. For, an H-body is not, in general, a compression body. Note, however, that $(H_1 \cup \eta(K)) \cup N_2$ is a Heegaard splitting of S^3. Throughout the paper we will refer to this Heegaard splitting as the splitting of S^3 corresponding to $H_1 \cup N_2$. Conversely, if $N_1 \cup N_2$ is a Heegaard splitting of S^3 and $K \subset \text{int} N_1$ is a knot, then $\overline{N_1 \setminus \eta(K)} \cup N_2$ is an HN-splitting of $S^3 \setminus K$.

Let $(-N)$ denote N with the opposite orientation and fix i an orientation reversing involution of N; we can of i as a map $i : N \longrightarrow (-N)$.

Given an H-body $H \subset N$ with $\partial_+ H = \partial N$ and an orientation preserving homeomorphism $g : \partial_+ H \longrightarrow \partial N$, the quotient space

$$H \cup_g (-N) := H \sqcup (-N)/\{y \sim ig(y) \mid y \in \partial_+ H\},$$

is an oriented knot complement. Conversely, let $M = H_1 \cup N_2$ be an HN-splitting of a knot complement $M = S^3 \backslash K$ and let $S^3 = N_1 U N_2 = (H_1 \cup \eta(K)) \cup N_2$ be the corresponding Heegaard splitting of S^3. For $k = 1, 2$, we choose a homeomorphism $m_k : N_k \longrightarrow N$. The image $H := m_1(H_1) = \overline{N \setminus m_1((\eta(K))}$ is an H-body with $H \subset N$ and $\partial_+ H = \partial N$. The composition $g := m_2 m_1^{-1} | \partial_+ H : \partial_+ H \longrightarrow \partial N$ is an orientation preserving homeomorphism and $M = S^3 \setminus K$ is homeomorphic to the quotient $H \cup_g (-N)$. Now $g : \Sigma \longrightarrow \Sigma$ defines an element in the mapping class group $\Gamma := \Gamma(\Sigma)$, where $\Sigma := \partial_+ H = \partial N$. As in Section 2, we will also use g to denote the mapping class of g.

Definition 5.5. *The pair $(\Sigma,\ g)$ is called an HN-model of the HN-splitting $M = H_1 \cup N_2$.*

By the discussion above, obtaining an HN-model (Σ, g) of an HN-splitting $M := S^3 \setminus K = H_1 \cup N_2$ involves the choice of homeomorphisms $m_2 : N_2 \longrightarrow N$ and $m_1 : H_1 \longrightarrow H \subset N$ so that we have a homeomorphism

$$m : M \longrightarrow H \cup_g (-N), \quad \text{with} \quad m|H_1 = m_1 \quad \text{and} \quad m|N_2 = im_2. \tag{5.1}$$

With this in mind, we will often work with the model of an HN-splitting rather than the splitting itself.

5.2. ***HN*-splittings and knot fibrations.** Given a knot fibration $M := S^3 \setminus K = S \times J/h$ there is a natural way to obtain an HN-splitting of M and an associated HN-model in terms of the monodromy of the fibration. We now describe this process: To begin, we set $N_1 := S \times [0, 1]$ and $N_2 := S \times [-1, 0]$. Also we set $A := \partial S \times (0, 1)$ and $A' := \partial S \times (-1, 0)$. We have $\partial N_1 = (S \times \{0\}) \cup A \cup (S \times \{1\})$. Similarly, we have $\partial N_2 = (S \times \{-1\}) \cup A' \cup (S \times \{0\})$. We will assume that $K := \partial S \times \{\frac{1}{2}\}$ on ∂N_1; thus K is the preferred curve of N_1. Define $g : \partial N_1 \longrightarrow \partial N_1$ by

$$g(x, 0) = (x, 0), \quad \text{for} \quad x \in S, \tag{5.2}$$

$$g(x,t) = (x,t), \quad \text{for} \;\; x \in \partial S \;\; \text{and} \;\; 0 < t < 1, \tag{5.3}$$

$$g(x,1) = (h(x),1), \quad \text{for} \;\; x \in S. \tag{5.4}$$

Consider the homeomorphism $rg : \partial N_1 \longrightarrow \partial N_2$, where $r : N_1 \longrightarrow N_2$ is defined by $(x,t) \to (x,-t)$. We obtain a Heegaard splitting

$$S^3 = N_1 \cup_g N_2 := N_1 \sqcup N_2 / \{y \sim rg(y) \mid y \in \partial N_1\}, \tag{5.5}$$

such that K is a preferred curve on N_1 and N_2. To pass to an HN-splitting we push K on $S \times \{\frac{1}{2}\}$ slightly in the interior of N_1 and then we take $A(K)$ to be an annulus neighborhood of K on $S \times \{\frac{1}{2}\}$. If we remove a tubular neighborhood of K, say $\eta(K) := A(K) \times (\{\frac{1}{2}\} - \epsilon,\ \{\frac{1}{2}\} + \epsilon)$, from $\text{int}(N_1)$ we obtain an HN-splitting of genus $l := 2\text{genus}(K)$ for M. Let P denote the genus l model surface within the homeomorphism class of S. To obtain an HN-model for our splitting, we pick an orientation preserving homeomorphism $m : (S, \partial S) \longrightarrow (P, \partial P)$ and define

$$m_1 : N_1 \longrightarrow N, \quad (x,t) \to (m(x),t), \tag{5.6}$$

and

$$m_2 := m_1 r^{-1} : N_2 \longrightarrow N, \quad (x,t) \to (m(x),-t). \tag{5.7}$$

The restriction $m_1|H_1$ sends H_1 to an H-body $H \subset N$ and the restriction $m_2 rg m_1^{-1}|\partial_+ H : \partial_+ H \longrightarrow \partial N$ is an orientation preserving automorphism of Σ. Now $(\Sigma,\ m_2 rg m_1^{-1})$ is an HN-model for $M = H_1 \cup N_2$.

Remark 5.6. Often, it will be convenient for us to abuse the conventions of (5.6)-(5.7) and take $N := N_1 = S \times [0,1]$ so that N_2 is identified to $(-N)$ via r^{-1}. Note that by the construction of the HN-splitting associated to the fibration $S^3 \backslash K = S \times J/h$, we have a surface $S_1 \subset S \times \{\frac{1}{2}\}$ that is disjoint from the corresponding HN-surface. Furthermore S_1 and $S \times \{\frac{1}{2}\}$ differ by an annulus and after the construction of $H_1 \cup N_2$ we have $K = \partial S_1$. Thus, we may think of this HN-surface as sitting in the original fibration $S^3 \setminus K = S \times J/h$ and the surface S_1 is a level surface of the fibration.

Definition 5.7. *The pair* $(\Sigma,\ g)$ *is called the* HN*-model associated to the fibration* $M = S \times J/h$.

5.3. **Equivalence relations for HN-models.** The problem of equivalences between Heegaard splittings from the point of view of models is formalized by Birman in [Bi]. Here, we only deal with the case of Heegaard splittings of S^3 corresponding to fibrations of knot complements. Let K' be a fibered knot. Suppose that we have HN-models $(\Sigma,\ g_2)$ and $(\Sigma,\ g_1)$ that correspond to two fibrations of $S^3 \setminus K'$. With the conventions of Remark 5.6 we will consider the corresponding Heegaard splittings of S^3 as the quotient spaces $S^3 = N_1 \cup_{g_2} N_2$ and $S^3 = N_1 \cup_{g_1} N_2$, respectively (see (5.5)). Let $\Delta := \Delta(\Sigma)$ denote the subgroup of the mapping class $\Gamma(\partial N_1) = \Gamma(\Sigma)$ defined in §2.2. It will also be convenient for us to consider $r^{-1} : N_2 \longrightarrow N_1$ as the composition ip where

$$p : N_2 \longrightarrow N_1, \quad (x,t) \to (x, 1+t)$$

$$i : N_1 \longrightarrow N_1, \quad (x,t) \to (x, 1-t),$$

where N_2 is considered with the orientation that inherits from N_1 under p^{-1} and i is the orientation reversing involution on N_1. In this setting, the formalization of Theorem 1 of [Bi] translates as follows:

Lemma 5.8. *a) There is an orientation preserving homeomorphism $f : S^3 \longrightarrow S^3$ such that $f(N_1) = N_1$ and $f(N_2) = N_2$ if and only if there exist $F_1, F_2 \in \Delta$ such that in Γ we have*

$$F_1 g_2 F_2 = g_1. \tag{5.8}$$

b) There is an orientation preserving homeomorphism $f : S^3 \longrightarrow S^3$ such that $f(N_1) = N_2$ and $f(N_2) = N_1$ if and only if there exist $F_1, F_2 \in \Delta$ such that

$$F_1(i^{-1} g_2^{-1} i) F_2 = g_1, \tag{5.9}$$

where, $i : N_1 \longrightarrow N_1$ is the orientation reversing involution fixed above.

Proof: We show part (b); part(a) is similar. Assume that there is an orientation preserving homeomorphism $f : S^3 \longrightarrow S^3$ such that $f(N_1) = N_2$ and $f(N_2) = N_1$. If we let $f_i := f|\partial N_i$, $i = 1, 2$, the following diagram is commutative.

$$\begin{array}{ccc} \partial N_1 & \xrightarrow{rg_1} & \partial N_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ \partial N_2 & \xrightarrow{g_2^{-1}r^{-1}} & \partial N_1 \end{array}$$

Thus we have

$$f_2 r g_1 = g_2^{-1} r^{-1} f_1. \tag{5.10}$$

Now rewriting $r^{-1} = ip$, where p, i are defined before the statement of the lemma, we have

$$g_1 = (ipf_2^{-1}i)i^{-1}g_2^{-1}i(pf_1). \tag{5.11}$$

Since $(ipf_2^{-1}i)$, (pf_1) and $i^{-1}g_2^{-1}i$ are orientation preserving homeomorphisms of ∂N_1 they represent elements in Γ. Note, furthermore, that $(ipf_2^{-1}i)$, (pf_1) extend to N_1. Thus letting $F_1, F_2 \in \Delta$ denote the mapping classes of $(ipf_2^{-1}i)$ and (pf_1), respectively, we see that (5.9) is satisfied. Conversely, starting from (5.9) we can easily reverse (5.11), (5.10) to obtain f_1, f_2 that can be used to define f. $\square$

Next we compare HN-splittings of a fibered knot K' under the assumption that the homeomorphism $F : S^3 \longrightarrow S^3$ that preserves the HN-surfaces satisfies certain technical conditions. In this case, we show that the model maps are conjugate to each other in the mapping class group. This is important for creating the suitable environment in which Corollary 2.4 can be utilized. We note that the relations between the model maps given by Lemma 5.8 (a) are not enough for this purpose. We also note that it is important that the conjugation is in the mapping class group and not the extended mapping class group (compare, Example 2.8).

According to the conventions adapted in Remark 5.6, the HN-surface, say Q, of the splitting corresponding to a fibration $M' := S^3 \setminus K' = S' \times J/h'$, is disjoint from from a level surface S'_1 of the fibration. To facilitate the exposition in the proof of the next lemma we will assume, without loss of generality, that $S'_1 = S' \times \{\frac{1}{2}\}$. Now Q lifts to a separating surface in the handlebody $R' := \overline{M' \setminus \eta(S'_1)}$; we will also use

Q to denote the lifted surface. We note that Q bounds a handlebody, say N', on one side in R'; this handlebody is the lift of the handlebody part $N_2 \subset M'$ of the HN-splitting.

Lemma 5.9. *Let $M' := S^3 \setminus K' = S' \times J/h'$ be an oriented fibered knot complement. Let $(\Sigma,\ g')$ denote the HN-model associated to the fibration and let Q denote the HN-surface of the corresponding HN-splitting of M'. As in Remark 5.6, we will think of Q sitting in the fibration so that $S'_1 = S' \times \{\frac{1}{2}\}$ is disjoint from Q and we will think of N as $S' \times [0,1]$. Let $(\Sigma,\ g'')$ be a second HN-model of M' and let Q' denote the HN-surface of the corresponding HN-splitting. Let $M' = H_1 \cup N_2$ and $M' = H'_1 \cup N'_2$ denote the HN-splittings of M' corresponding to Q and Q', respectively. Suppose that we can choose homeomorphisms $m : H \cup_{g'} (-N) \longrightarrow H_1 \cup N_2$ and $j : H \cup_{g''} (-N) \longrightarrow H'_1 \cup N'_2$, as in (5.1), so that the following are satisfied:*

(1) We have $S'_1 \subset \mathrm{int} H'_1$. In particular, S'_1 is also disjoint from Q'.

(2) With the convention of Remark 5.6, we have $j|S'_1 = m|S'_1 = \mathrm{id}$.

(3) If we set $R' := \overline{M \setminus \eta(S'_1)}$ and also use Q and Q' to denote the lifts of Q and Q' in the handlebody R', there is an orientation preserving homeomorphism $F : R' \longrightarrow R'$ such that

$$F(W) = W', \quad F(Q) = Q' \quad \text{and} \quad F|\partial R' = \mathrm{id}, \tag{5.12}$$

where $W := \overline{R' \setminus N_2}$ and $W' := \overline{R' \setminus N'_2}$. Then, there is an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ such that in the mapping class group $\Gamma = \Gamma(\Sigma)$ we have

$$g'' = f g' f^{-1}. \tag{5.13}$$

Proof: We will consider R' as an interval bundle $R' = S'_1 \times J$, where $J = [0, \frac{1}{2}] \cup [-1, 0] \cup [1/2, 1]$ and the sub-bundle pieces corresponding to these sub-intervals are glued together according to (5.2)-(5.5). Furthermore, we may consider N_2 as a product $N_2 = S'' \times [-1, 0]$ such that S'' lies on $S' := S' \times \{-\frac{1}{2}\}$ and $E := \overline{S' \setminus S''}$ is a spanning annulus in W. Thus, for $t \in [-1, 0]$, Q intersects each fiber $S'_t = S' \times \{t\}$ at a simple curve that is parallel to $\partial S'_t$ on S'_t.

Let F the homeomorphism given in (3) of the statement of the lemma. We claim that, up to an isotopy relative to $\partial R'$, F can be assumed to also be level-preserving; that is $F(S'_1 \times \{x\}) = S'_1 \times \{x\}$, for all $x \in J$.

This claim that F is isotopic to a level preserving homeomorphism relative to $\partial R'$ follows from Lemma 3.5 of [W] and in particular from the argument given in the proof of Case 1 of that lemma. To outline the argument let $p : R' = S'_1 \times J \longrightarrow S'_1$ denote the J-bundle projection and let F be as in (5.11). Pick $\mathbf{D}$ to be a complete system of vertical discs in $R' = S'_1 \times J$; so that S'_1 split along the set of arcs $p(\mathbf{D})$ is disc. Now $F(\mathbf{D})$ is a collection of discs in R'. Let D be a component of $\mathbf{D}$. By (5.12), $F|\partial D = \text{id}$ and $Q' \cap F(D)$ is a simple closed curve on Q'. Now $E' := F(D) \cap W'$ is a spanning annulus in W'. Let $D_1 := \overline{D \setminus E'}$. As in Lemma 3.4 of [W], we may change F in $D \setminus (D \cap Q)$ so that we have $F(D) = D$. Then, as in the proof of Lemma 3.5 of [W], we deduce that F can be assumed to be level preserving, up to an isotopy relative to $\partial R'$.

We may extend this homeomorphism $F : R' \longrightarrow R'$ to a homeomorphism $F : M' \longrightarrow M'$ such that $F|S'_1 = \text{id}$. The existence of this later homeomorphism F implies that Q' is the HN-surface corresponding to a fibration of $M' = S^3 \setminus K'$ with fiber S'_1. Let (Σ, g_1) denote the model corresponding to this HN-splitting. Proposition 4.2 implies that there is an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ so that in the mapping class group $\Gamma(\Sigma)$ we have $g_1 = fg'f^{-1}$. In fact, this f is closely related with F: There is a fiber preserving homeomorphism $F_1 : H \cup_{g'} (-N) \longrightarrow H \cup_{g_1} (-N)$, with $F_1|S'_1 = \text{id}$, and a homeomorphism $l : H \cup_{g_1} (-N) \longrightarrow M'$ satisfying (5.1), with $l|S'_1 = \text{id}$, such that the diagram

$$\begin{array}{ccc} H \cup_{g'} (-N) & \xrightarrow{m} & M' \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F} \\ H \cup_{g_1} (-N) & \xrightarrow{l} & M' \end{array}$$

is commutative.

Let j_1 (resp. l_1) denote the restriction of j (resp. l) on H and let j_2 (resp. l_2) denote the restriction of j (resp. l) on N. Now consider the composition $e := jl^{-1} : M' = H'_1 \cup N'_2 \longrightarrow M' = H'_1 \cup N'_2$. By condition (2) we have $e(R') = R'$. We have $e(Q') = Q'$ and $e|\partial R' = \text{id}$. We have the following commutative diagram

$$\begin{array}{ccc}
\partial H_1' & \xrightarrow{l_2 g_1 l_1^{-1}} & \partial N_2' \\
{\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle e_2} \\
\partial H_1' & \xrightarrow{j_2 g'' j_1^{-1}} & \partial N_2'
\end{array}$$

where $e_1 := e|H_1'$ and $e_2 := e|N_2'$. By an argument similar to that applied for F earlier, after an isotopy relatively $\partial R'$, we can make $e|R'$ level preserving. But then, since $e|\partial R' = \text{id}$ and $e(Q') = Q'$, we will conclude that e_1 and e_2 are isotopic to the identity in H_1' and N_2', respectively. In the view of the commutative diagram above, this implies that in $\Gamma(\Sigma)$ we have $g'' = g_1 = fg'f^{-1}$. $\square$

Remark 5.10. The fact that e_1, e_2 are isotopic to the identity is very important here. If this is not the case then we conclude

$$g'' = e_2 g_1 e_1^{-1} = e_2 f g' f^{-1} e_1^{-1},$$

which a relation of the type discussed in Lemma 5.8(a). However, as discussed earlier, this is not enough for our purposes in this paper.

6. Crossings changes and Dehn twists

In this section we study the question of when a crossing change in a fibered knot leaves the isotopy class of the knot unaltered and we will prove Theorem 1.1. In fact we will work in a more general context as we will consider "generalized crossing changes".

6.1. **Nugatory crossing changes in fibered knots.** Let K be a knot in $\mathbf{S}^3$ and let $q \in \mathbf{Z}$. A generalized crossing of order q on a projection of K is a set C of $|q|$ twist crossings on two strings that inherit opposite orientations from any orientation of K. If K' is obtained from K by changing all the crossings in C simultaneously, we will say that K' is obtained from K by a generalized crossing change of order q (see Figure 5). Notice that if $|q| = 1$, K and K' differ by an ordinary crossing change while if $q = 0$ we have $K = K'$. A crossing disc for K is an embedded disc $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic number. Performing $\frac{1}{-q}$-surgery on $L := \partial D$, for $q \in \mathbf{Z}$, changes K to another knot $K^{'} \subset S^3$. Clearly $K^{'}$ is obtained from K

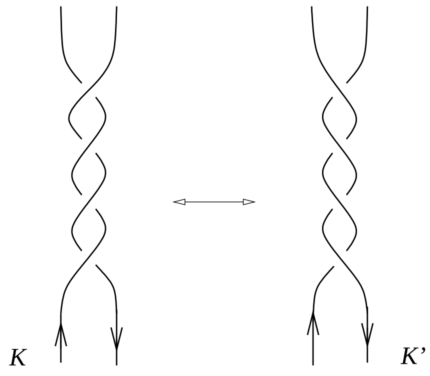


FIGURE 5. The knots K and K' differ by a generalized crossing change of order $q = -4$.

by a generalized crossing change of order q. The boundary $L := \partial D$ is called a crossing circle supporting the generalized crossing change.

Definition 6.1. *A generalized crossing supported on a crossing circle L of a knot K is called nugatory if and only if $L := \partial D$ bounds an embedded disc in the complement of K. This disc and D form an embedded 2-sphere that decomposes K into a connected sum where some of the summands may be trivial.*

Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. It is an open question is whether, in general, the converse is true.

Question 6.2. (Problem 1.58, [GT]) *If a crossing change in a knot K yields a knot isotopic to K, is the crossing nugatory?*

The answer is known to be *yes* in the case when K is the unknot ([ScT]) and when K is a 2-bridge knot ([To]). In [To], I. Torisu also reduces Question 6.2 to the analogous question for prime knots and he conjectures that the answer is always *yes*. To these we add the following theorem.

Theorem 6.3. *Let K be a fibered knot and let K' a knot obtained from K by a generalized crossing change. If K' is isotopic to K then a crossing circle L supporting this crossing change bounds an embedded disc in the complement of K.*

An immediate consequence of Theorem 6.3 is the following corollary which answers Question 6.2 for fibered knots.

Corollary 6.4. *Let K be a fibered knot. A generalized crossing change in K yields a knot isotopic to K if and only if the crossing is nugatory.*

Clearly, Theorem 1.1 stated in the Introduction is a special case of Corollary 6.4. Next we recall the following theorem that was proved by Torisu in [To].

Theorem 6.5. (Theorem 2.1, [To]) *Let $K := K_1 \# K_2$ be a composite knot and K' a knot that is obtained from K by a crossing change supported on a crossing circle L. If K' is isotopic to K then either L bounds a disc in the complement of K or the crossing change occurs within one of K_1 and K_2.*

Now combining Theorem 6.5 with Corollary 6.4 and the positive answer to Question 6.2 for 2-bridge knots mentioned earlier, we obtain the following corollary.

Corollary 6.6. *Let $\mathbf{U}$ denote the set of knots K that are 2-bridge knots or fibered knots or their connect sums. A crossing change in a knot $K \in \mathbf{U}$ yields a knot isotopic to K if and only if the crossing is nugatory.*

Remark 6.7. We should point out that the arguments of [ScT] and [To] go through if one replaces an ordinary crossing by a generalized crossing. Thus, actually, Corollary 6.6 holds if we replace ordinary crossings by generalized crossings.

Remark 6.8. In the view of Theorem 6.5, it is enough to prove Theorem 6.3 for prime fibered knots. However, here we will not relay on Theorem 6.5 as the proof we will give works for all fibered knots

Remark 6.9. For a knot K, a crossing circle L and an integer r, let $K(r)$ denote the knot obtained from K by a generalized crossing of order r along L. In [KL] we show that for a given K, up to isotopy in the complement of K, there are finitely many choices for L and r so that $K(r)$ is isotopic to K.

6.2. **Preliminaries and conventions.** Let C be a generalized crossing of order $q \neq 0$ of a fibered knot K. Let K' denote the knot obtained from K by changing C and let D be a crossing disc for C with $L := \partial D$. Since the linking number of L and K is zero, K is homologically trivial in the complement of L and thus it bounds a Seifert surface disjoint from L. Let S be a Seifert surface that is of minimum genus among all such Seifert surfaces. Since S is incompressible, after an isotopy we can arrange so that the closed components of $S \cap D$ are homotopically essential in $D \setminus K$. But then each such component is parallel to L on D and by further isotopy of S we can arrange so that $S \cap D$ is an arc that is properly embedded on S. The surface S gives rise to Seifert surfaces S and S' of K and K', respectively. We set $M := S^3 \setminus K$ and $M' := S^3 \setminus K'$.

Convention 1. Without loss of generality we will assume that we have $E := S \cap B^3 = S' \cap B^3$, where B^3 is a 3-ball and E is a disc, disjoint from D, and such that $E \cap K = E \cap K'$. It is clear that S and S' share a common spine, say G, based at a point $p \in E$.

By the discussion so far, we have:

Assumption 1: $M' = S^3 \setminus K'$ is obtained from $M = S^3 \setminus K$ by performing $\frac{1}{-q}$-surgery on $L := \partial D$.

Next we restrict to fibered knots and state the assumptions that we have to work with from the statement of Theorem 6.3:

Assumption 2: K and K' are fibered knots that are isotopic.

Assumptions 1 and 2, combined with Corollary 2.4 of [Ga], give that S and S' are minimum genus Seifert surfaces of K and K', respectively. Thus, in our setting, we may and will assume the following:

Assumption 3: S and S' are minimum genus Seifert surfaces, for K and K', respectively.

With the notation of Section 4, there is a fibration $M := S \times J/h$ with monodromy $h : S \longrightarrow S$. As in §5.1, let P denote the model surface within the homeomorphism class of S and S', and let $N := P \times [0, 1]$ and $\Sigma := \partial N$. Choose a homeomorphism $m : S \longrightarrow P$, as in (5.5)-(5.6), to obtain an HN-model (Σ, g) corresponding to the fibration of M.

Convention 2. From now on we will work with this fixed model (Σ, g) for $M = S^3 \setminus K$. Throughout the section, we will abuse the conventions as discussed in Remark 5.6. We will think of N as an I-bundle $S \times I$, $I := [0,1]$. We will assume that the crossing disc D is a meridian disc of N that is vertical with respect to the I-bundle structure of N: We have a properly embedded arc λ' on S so that $D = \lambda' \times I$. Thus $L := \partial D$ consists of two copies of λ' and two vertical arcs on $\partial S \times I$. By (5.2)-(5.4), $g(L)$ and L are identical everywhere on Σ except on the top level surface $S_1 := S \times \{1\}$ where

$$L \cap S_1 = \lambda_1 := \lambda' \times \{1\} \ \text{ and } \ g(L) \cap S_1 = g(\lambda_1). \tag{6.1}$$

Each of Assumptions 1 and 2 provides a way to construct an HN-model for $M' := S^3 \setminus K'$ out of the fixed model (Σ, g) of M; this is done next.

6.3. **Construction of two HN-models for M'.** Let $\tau : N \longrightarrow N$ denote the right-handed Dehn twist of N along the meridional disc D and let $T_L := \tau|\Sigma$, where $L = \partial D$. We have $\tau^{-q}(S) = S'$ and $\tau^{-q}(K) = K'$. Recall that (Σ, g) is the HN-model of $M = S^3 \setminus K$ corresponding to the fibration $M = S \times J/h$ and that $M' := S^3 \setminus K'$ is obtained from M by $\frac{1}{-q}$-surgery on L. The first construction of an HN-model for M' is based on that information. The proof of Lemma 6.10 follows the well known process of passing between gluing maps of Heegaard splittings and Dehn surgeries of 3-manifolds (compare, for example, pp. 86-87 of [AM]).

With the conventions of Remark 5.6 will think of the Heegaard splitting of S^3 corresponding to the fibration $M = S \times J/h$ as the quotient

$$N \cup_g (-N) := N \sqcup (-N)/\{y \sim ig(y), \mid y \in \Sigma\}, \tag{6.2}$$

where i is the involution defined in §5.3.

Lemma 6.10. $(\Sigma,\ gT_L^{-q})$ *is an HN-model for $M' = S^3 \setminus K'$.*

Proof: By assumption (Σ, g) is an HN-model corresponding to the fibration $M = S \times J/h$. As before, let A denote an annulus on Σ that supports T_L and let $B := g(A)$. We will think of this HN-splitting of M as the quotient

$$H \cup_g (-N) := H \sqcup (-N)/\{y \sim ig(y), \mid y \in \Sigma\}, \tag{6.3}$$

where $H \subset N$. We consider the complement $M_L := S^3 \setminus (K \cup L)$ as the pre-quotient space

$$H \cup_{g^1} (-N) \quad \text{where} \quad g^1 := g|(\Sigma \setminus A) : \Sigma \setminus A \longrightarrow \Sigma \setminus B. \tag{6.4}$$

Thus we can think of the torus $\mathcal{T} := A \cup B$ as the boundary torus of a tubular neighborhood of L. Let α be an arc that is properly embedded and essential on A such that it intersects L exactly once and let $\beta := g(\alpha)$. Now $\mu := \alpha \cup \beta$ is the meridian of $\mathcal{T}$ and $\lambda := L$ is the longitude which we will orient so that their algebraic intersection number on $\mathcal{T}$, denoted by $< \lambda, \mu >$, is one. Since T_L is supported in A it can be considered as a Dehn twist on $\mathcal{T}$. We have

$$T_L^q(\mu) = \mu - q\lambda = T_L^q(\alpha) \cup \beta,$$

where

$$\beta = gT_L^{-q}(\alpha') \quad \text{and} \quad \alpha' := T_L^q(\alpha).$$

(Recall that, in general, if a, b are simple closed curves on $\mathcal{T}$, we have $T_a(b) = b + < a, b > a$. Since $< \lambda, \mu >= 1$, we have $T_L^{-1}(\mu) = \mu + \lambda$, which explains the change of sign between the power T_L^q and the coefficient of λ in $T_L^q(\mu)$ in the equations above.

Now if we set $\mu' := \alpha' \cup \beta$ we have

$$\mu' := \alpha' \cup \beta = \mu - q\lambda.$$

Let $M_L(q)$ denote the 3-manifold obtained from $M_L := S^3 \setminus (K \cup L)$ by $\frac{1}{-q}$ Dehn filling on $\mathcal{T}$. From the discussion above, in order to obtain $M_L(q)$ one needs to attach a solid torus to $\mathcal{T}$ in such away so that the meridian is attached along the curve μ. It follows that $H \cup_{gT_L^{-q}} (-N)$ is an HN-splitting for $M_L(q)$. But since by assumption we have $M_L(q) = S^3 \setminus K' = M'$, it follows that $(\Sigma,\ gT_L^{-q})$ is a HN-model for M'. $\square$

The following will be useful to us later in the section.

Lemma 6.11. *The homeomorphism $\tau^{-q} : N \longrightarrow N$ is isotopic in N to a level preserving homeomorphism $\psi : N = S \times I \longrightarrow N = S' \times I$.*

Proof: Recall the spine G of Convention 1. An element $x \in \pi_1(S,p)$ is represented by a loop on G which also represents an element $x' \in \pi_1(S',p)$. The assignment $x \longrightarrow x'$, defines an isomorphism $\pi_1(S,p) \longrightarrow \pi_1(S',p)$ which, by Nielsen's theorem ([ZVC], Theorem 5.7.1), is induced by a surface homeomorphism $\phi : S \longrightarrow S'$. We extend ϕ to $\psi : S\times I \longrightarrow S'\times I$ by $\psi(x,t) = (\phi(x),t)$. Now $\tau^q\psi : S\times I \longrightarrow S\times I$ induces the identity on $\pi_1(S \times I, p)$ and it follows that $\psi = \tau^{-q}$ up to isotopy in N. $\square$

Next, we use Assumptions 2 and 3 to construct a second HN-model for $M' = S^3 \setminus K'$. By Assumption 3, S' is a minimum genus Seifert surface of K'. Thus M' can be fibered with fiber S'. Since, by Assumption 2, K' is isotopic to K Proposition 4.2 gives an orientation preserving homeomorphism $f_1 : S \longrightarrow S'$, with

$$f_1 h f_1^{-1} = h', \tag{6.5}$$

up to isotopy on S' and such that we have $M' = S' \times J/h'$. Consider the HN-model (Σ, g') corresponding to the fibration $M' = S' \times J/h'$, with g' defined from h' as in (5.2)-(5.5). Now f_1 gives rise to an orientation preserving homeomorphism $f_2 : \Sigma \longrightarrow \Sigma$ such that $f_2 g f_2^{-1} = g'$ up to isotopy on Σ. Thus we have:

Lemma 6.12. *There is a fibration $M' = S' \times J/h'$ with HN-model (Σ, g') and an orientation preserving homeomorphism $f_2 : \Sigma \longrightarrow \Sigma$ such that in $\Gamma(\Sigma)$ we have*

$$f_2 g f_2^{-1} = g'. \tag{6.6}$$

6.4. **The strategy of the proof of Theorem 6.3.** Having the HN-models (Σ, gT_L^{-q}) and (Σ, g') of M' we proceed as follows:

1. By assumption, $q \neq 0$. By Corollary 2.6, if $T_L^{-q} = 1$ in Γ then, L is homotopically trivial on Σ. Then, clearly L bounds a disc in the complement of K and the conclusion of Theorem 6.3 follows. Thus, we may assume that $T_L^{-q} \neq 1$ in Γ.

2. Set $g'' := gT_L^{-q}$ and let Q' and Q denote the HN-surfaces of M' corresponding to the HN-models (Σ, g'') and (Σ, g'), respectively. First, we use the setting developed in §3 and Lemma 5.9 to show that g'' is conjugate in Γ to an HN- model map corresponding to a fibration of M' with fiber S' (Lemma 6.15). Then, to prove Theorem 6.3, we further compare the two models by means of Lemmas 5.8 and 5.9:

Case (A): If Q, Q' satisfy the hypothesis of Lemma 5.9, we obtain an orientation preserving homeomorphism $H : \Sigma \longrightarrow \Sigma$ such that in Γ we have

$$gT_L^{-q} = HgH^{-1}.$$

Since we assumed $T_L^{-q} \neq 1$, we conclude that $H \neq 1$ showing that T_L^{-q} can be written as a single commutator in Γ.

Case (B): If Q, Q' do not satisfy the hypothesis of Lemma 5.9, we show that the corresponding Heegaard splittings of S^3 are equivalent in the sense of Lemma 5.8(b) (see Lemma 6.16). In this case, we use Theorem 2.7 to conclude that $g(L)$ bounds a disc in N (Lemma 6.17.)

In Case (A), Corollary 2.4 applies to conclude that L is homotopically trivial on Σ; a contradiction. In Case (B), the disc $g(L)$ bounds in N is easily seen to lead to a disc bounded by L in the complement of K and Theorem 6.3 follows.

6.5. **Understanding the HN-model (Σ, g'').** In this subsection we use the setting and the results of Section 3 to gain a better understanding of the model (Σ, g'') constructed in Lemma 6.10.

In the view of Lemma 6.11, Lemma 3.3, and the conventions adapted earlier, both of K and K' are disc-busting curves for the handlebody N. Let A an annulus on Σ supporting T_L so that the core of A is L and the intersection $A \cap K$ consists of two arcs, say α_1, α_2, that are parallel in A and each of which intersects L exactly once. We can choose a super-complete system of discs, say $\mathbf{D}$, for N that consists of crossing discs for K, K' and such that the crossing disc D, fixed in §6.2, lies in $\mathbf{D}$. Let $\mathbf{x}$ denote the super-complete system of curves on Σ corresponding to $\mathbf{D}$. By assumption, $L \in \mathbf{x}$ and $\mathbf{x}$ is admissible for K (see Definition 3.6). We set, $B := g(A)$ $\gamma := g(K)$, $\gamma' := g(K')$ and $z := g(L)$. By construction, we have $g|K = \mathrm{id}$. Thus, $g(K) = K$,

$B\cap\gamma=\alpha_1\cup\alpha_2$ and γ is a disc-busting curve for N. The curve γ' is the result of $\gamma := g(K) = K$ under a non-trivial power of the right-hand Dehn twist along $z := g(L)$ supported on B. Thus we are in the setting of Lemma 3.7 which leads to the following:

Lemma 6.13. *One of the following is true:*

(a) The curve $g(K')$ is disc-busting for N.

(b) We have $|g(L)\cap L| = 1$ and $|g(K')\cap L| = 0$. Furthermore, there exists a super-complete system of curves, say $\mathbf{x}'$, that is admissible for $\gamma := g(K) = K$, it contains L and for every $x_i \in \mathbf{x}'$, with $x_i \neq L$, we have $x_i \cap B = \emptyset$.

Proof: We will apply Lemma 3.7 for $\gamma := g(K) = K$, $\gamma' := g(K')$ and $z := g(L)$. As explained above, we can find a system of curves $\mathbf{x} \subset \Sigma$ that contains L and is admissible for γ. Suppose that γ' is not disc-busting in N. Then, there is a super-complete system $\mathbf{x}'$ and a curve $x' \in \mathbf{x}'$ with the properties stated in Lemma 3.7 (b): Thus, the intersection $x'\cap\gamma$ lies B and we have $|x'\cap z| = 1$ and $|\gamma'\cap x'| = 0$. Furthermore, no other curve in $\mathbf{x}'$, except for x', intersects B. Since x' intersects $\gamma = K$ only inside B we have $x'\cap\gamma = x'\cap(\alpha_1\cup\alpha_2)$, where $\alpha_{1,2}$ are the two components of $\gamma\cap B$. Thus x' is a crossing disc for K and we have $K\cap x' = K\cap L$. From this and the argument in the proof of Case 1 of Lemma 3.7 it follows that we can take $x' = L$. $\square$

We will think of the HN-splitting of $M' = S^3\setminus K'$ corresponding to the model $(\Sigma,\ gT_L^{-q})$ as the quotient

$$M' = H\cup(-N)/\{y\sim igT_L^{-q}(y),\ \mid y\in\Sigma\}, \tag{6.7}$$

and we will identify the corresponding Heegaard splitting of S^3 with the quotient

$$N\sqcup(-N)/\{y\sim igT_L^{-q}(y),\ \mid y\in\Sigma\}. \tag{6.8}$$

Next we show that Case (b) of Lemma 6.13 does not occur in our setting. Thus we have:

Lemma 6.14. *Let the notation and the setting be as in the statement of Lemma 6.13. Then, $g(K')$ is disc-busting on N.*

Proof: Assume, on the contrary, that $g(K')$ is not disc-busting for N; thus option (b) of 6.13 holds. Then, the pair of curves $(L, g(L))$ defines a stabilization of the Heegaard splitting of (6.2). Thus there is a connect sum decomposition $N := N' \# V$, such that V is a solid torus with meridian $a := L$, and longitude $b := g(L)$ and such that the pair $(\partial V, g|\partial V)$ is a model for the standard genus one Heegaard splitting of S^3. Let $\bar{T} \subset \partial V$ denote the punctured torus defined by the decomposition $N := N' \# V$. Then we have $\bar{g} := g|\bar{T} \longrightarrow \bar{T}$, with $\bar{g}|\partial\bar{T} = \mathrm{id}$, $\bar{g}(a) = b$ and $\bar{g}(b) = a$. Since $|g(K') \cap a| = 0$ (Lemma 6.13), $\bar{g}(\bar{T} \cap K')$ consists of properly embedded arcs on $\bar{T}$ that are disjoint from $a := \bar{g}(b)$. Thus $\bar{T} \cap K'$ is disjoint from $b := \bar{g}^{-1}(a)$, which implies that $|K' \cap b| = 0$. But this, in turn, implies that the curve $b = g(L)$ bounds a compression disc for $\Sigma \setminus K'$ in N; a contradiction. □

Lemma 6.15. *There exists a fibration of M', with fiber S', and corresponding HN-model (Σ, g_1), and an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ such that in Γ the following relation holds:*

$$g'' := gT_L^{-q} = fg_1f^{-1}. \tag{6.9}$$

Proof: Recall that we used Q' to denote the HN-surface of (6.7). By Lemma 6.11, $T_L^{-q} : \Sigma \longrightarrow \Sigma$ is the restriction on ∂N of a level preserving homeomorphism $S \times I \longrightarrow S' \times I$. From this, and the fact that the HN-surface of $H \cup_g (-N)$ is disjoint from a level surface of the fibration $S \times J/h$, it follows that Q' is disjoint from a copy of $S' \subset \mathrm{int}(H)$ say S'_1. Let S_2 denote a second copy of S' that lies in a neighborhood of S'_1 in $\mathrm{int}(H)$. Since S'_1 has minimum genus, we have a fibration $M' = S'_1 \times J/h_1$, with monodromy $h_1 : S'_1 \longrightarrow S'_1$ and such that S_2 is a level surface in the fibration. The monodromy givers rise to an HN-model (Σ, g_1). Now we argue that Lemma 5.9 applies to the models (Σ, g_1), $(\Sigma,\ g'')$. To that end, we let $R' := M' \setminus S_2$. By construction, conditions (1), (2) of Lemma 5.9 are satisfied. Now Q' lifts to a separating surface in R' which bounds a handlebody $N'' \subset R'$ on one side. This handlebody N'' is the lift of the handlebody part of the HN-splitting of (6.7); let $W' := \overline{R' \setminus N''}$. Note that R' is also the quotient space

$$W' \sqcup (-N)/\{y \sim ig''(y),\ \mid y \in \Sigma\}. \tag{6.10}$$

The boundary $\partial W'$ has two components one of which is Σ. Since W' is H cut along the surface S_2, ∂S_2 and the curve $K' \subset \Sigma \subset \partial H'$ cobound a spanning annulus $E \subset W'$. Since $K' \subset \Sigma$ is disc-busting for N, $\Sigma \setminus K'$ is incompressible in W'. On the other hand, the fact that $g(K')$ is disc-busting for N (lemma 6.14) implies that in R' the curve $K'' := g(K')$ is disc-busting for the handlebody N''. Thus Lemma 3.5 applies to obtain $F : R' \longrightarrow R'$ such that $F|\partial R' = \mathrm{id}$ and F maps the HN-surface of (Σ, g_1) to Q'. This shows that condition (3) of Lemma 5.9 is also satisfied. It follows that there is f as claimed above. To finish the proof, note that by construction S_1' is a copy of S'. $\square$

6.6. **Comparison of models.** Next, we compare (Σ, g_1) to the model (Σ, g') given in Lemma 6.12. By construction, (Σ, g') is the model of a fibration $M' = S' \times J/h'$ that M' inherits from the fibration of M fixed in §6.2. Let Q denote the HN-surfaces of the splittings $H \cup_{g'} (-N)$; we will consider Q embedded in the fibration $M' = S' \times J/h'$ as described in Remark 5.6. By Lemma 6.15, (Σ, g_1) also corresponds to a fibration $M' = S' \times J/h_1$ and g'' is conjugate to g_1. Hence we consider the HN-surface Q' of (6.7), embedded in $M' = S' \times J/h_1$ and disjoint from a level surface of that fibration. By Lemma 4.1, there is an orientation preserving homeomorphism $\Phi : M' \longrightarrow M'$ sending $S' \times \{-1\} \cong S' \times \{1\}$ of $S' \times J/h_1$ to the corresponding fiber of $S' \times J/h'$. Viewing Φ as homeomorphism $H \cup_{g_1} (-N) \longrightarrow H \cup_{g'} (-N)$, we have:

Case 1. Φ takes a level surface of $S' \times J/h_1$ disjoint from Q' to a level surface of $S' \times J/h'$ disjoint from Q.

Case 2. Φ takes every level surface of $S' \times J/h_1$ disjoint from Q' to a level surface of $S' \times J/h'$ that intersects Q.

Consider the Heegaard splittings of S^3 corresponding to (Σ, g_1) and (Σ, g') as the quotient spaces

$$\begin{aligned} N_H \cup_{g_1} (-N) &:= N_H \sqcup (-N)/\{y \sim i g_1(y), \mid y \in \Sigma\} \\ N_H \cup_{g'} (-N) &:= N_H \sqcup (-N)/\{y \sim i g'(y), \mid y \in \Sigma\}, \end{aligned} \tag{6.11}$$

where we use N_H to denote the handlebody of the splitting resulting from the H-body of the corresponding HN-splitting (see §5.2).

Lemma 6.16. *One of the following is true:*

a) There is an orientation preserving homeomorphism $h : \Sigma \longrightarrow \Sigma$ so that in Γ we have

$$g' = hg_1h^{-1}. \tag{6.12}$$

b) There are orientation preserving homeomorphisms $F_1, F_2 : \Sigma \longrightarrow \Sigma$ that, up to isotopy on Σ, extend to N and such that in Γ we have

$$F_1(i^{-1}g_2^{-1}i)F_2 = g_1. \tag{6.13}$$

Here $g_2 := g'$ and i is the involution of Lemma 5.8 (b).

Proof: Suppose we are in Case 1 above. Then, with the conventions of Remark 5.6, we may assume that $\Phi(Q') = Q$ and $\Phi(H) = H$. Without loss of generality we can assume that Φ sends the surface $S' \times \{\frac{1}{2}\}$ of $S' \times J/h_1$ to the surface $S' \times \{\frac{1}{2}\}$ of $S' \times J/h'$. Set $S_1 := S' \times \{\frac{1}{2}\}$ and $R' := M' \setminus \eta(S_1)$. Then, Φ induces a homeomorphism $\Phi : R' \longrightarrow R'$, such that $\Phi|\partial R'$ leaves ∂S_1 fixed. After changing g_1 by conjugation if necessary, we can assure that $\Phi|\partial R' = \mathrm{id}$ and conclusion (a) follows from Lemma 5.9. Next suppose we are in Case 2 above: First we extent Φ to S^3 and pass to the corresponding Heegaard splittings given in (6.11). In each case we can isotope K' to lie on the Heegaard surface as described in §5.2. Then Φ gives rise to a homeomorphism $\Phi : N_H \cup_{g'} (-N) \longrightarrow N_H \cup_{g_1} (-N)$ with $\Phi(N_H) = N$ and $\Phi(N) = N_H$. This means that Φ satisfies the hypothesis of Lemma 5.8(b) and (6.13) follows from (5.9). □

Lemma 6.17. *If Lemma 6.16(b) holds, then $g(L)$ bounds a disc in N.*

Proof: By (6.9) we have $g_1 = f^{-1}g''f = f^{-1}gT_L^{-q}f$. Combining this with (6.13) we have

$$F_1(i^{-1}g_2^{-1}i)F_2 = f^{-1}gT_L^{-q}f, \tag{6.14}$$

where by (6.6), we have $g_2^{-1} = f_2g^{-1}f_2^{-1}$. We decompose Σ into surfaces $S_1' := S' \times \{1\}$ and $S_2' := (S' \times \{0\}) \cup (\partial S' \times (0,\ 1))$. Since g and g_2

satisfy (5.2)–(5.4), we have $g_2^{-1}|S_2' = \mathrm{id}$. Thus (6.14) implies

$$gT_L^{-q}|S_1' = G|S_1', \tag{6.15}$$

where $G := fF_1F_2f^{-1}$ is an orientation preserving homeomorphism of Σ that extends to the handlebody. Let $H := G|S_2'$ and let H also denote the extension of H to a homeomorphism of the handlebody. by product. Using [W] and an argument similar to this in the proof of Lemma 5.9, we see that $GH^{-1}|\Sigma$ is isotopic to the identity. By this and (6.15) we conclude that up to isotopy on Σ we have $gT_L^{-q} = G$ or equivalently $g = GT_L^q$. But $gT_L^{-q}g^{-1} = T_{g(L)}^{-q}$. Thus, in the mapping class group of Σ we have

$$G = gT_L^{-q} = T_{g(L)}^{-q}g = T_{g(L)}^{-q}GT_L^q,$$

or equivalently

$$GT_L^qG^{-1} = T_{g(L)}^{-q}.$$

Since the left hand side of the last equation extends to a homeomorphism of N we conclude that $T_{g(L)}^{-q}$ extends to a homeomorphism of N. Thus, by Theorem 2.7, since $q \neq 0$, $g(L)$ bounds a disc in N. $\square$

6.7. **Back on the main road.** We are now ready to show the following proposition which is used proof of Theorem 6.3.

Proposition 6.18. *With the notation of §6.2, suppose that S and S' are minimum genus Seifert surfaces for the fibered knots K and K', respectively. If K is isotopic to K', then L bounds an embedded disc in the complement $S^3 \setminus K$.*

Proof: If K is the unknot then S is a disc and N is a 3-ball. Thus L lies on a 2-sphere and it is clearly homotopically trivial. Hence, for the remaining of this proof, we will assume that K and K' are non-trivial fibered knots. Recall the model (Σ, g) of $M := S^3 \setminus K$ and the model (Σ, g') of $M' := S^3 \setminus K'$ given in Lemma 6.12. By construction, there is an orientation preserving homeomorphism $f_2 : \Sigma \longrightarrow \Sigma$ such that in the mapping class group we have

$$f_2 g f_2^{-1} = g'. \tag{6.16}$$

Next we turn to (Σ, g') and (Σ, g_1) of Lemma 6.16:

Case 1: Suppose the conclusion of Lemma 6.16 (b) holds. By Lemma 6.17, $g(L)$ bounds a disc $D' \subset N$. Now $\Lambda := D \cap D'$, where D is the disc bounded by L in N, is a reduction sphere for for (6.2) that intersects K at exactly two points and $\Sigma \cap \Lambda = L$. Pushing $\Lambda \cap K$ in the interior of D, D' becomes a disc bounded by L in the complement $S^3 \setminus K$.

Case 2: Suppose the conclusion of Lemma 6.16 (a) holds: That is we have $g' = hg_1h^{-1}$; g' and g_1 are conjugate in the mapping class group. By Lemma 6.15 there exists an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ such that

$$fg'f^{-1} = g'' := gT_L^{-q} \tag{6.17}$$

By (6.16),(6.17) we have

$$h^{-1}f^{-1}gT_L^{-q}fh = g' = f_2gf_2^{-1} \tag{6.18}$$

in Γ. This last relation, in turn, gives

$$T_L^{-q} = g^{-1}fhf_2gf_2^{-1}h^{-1}f^{-1} = [g^{-1}, H] \tag{6.19}$$

where $H := fhf_2$ is also in Γ. Suppose that L is not homotopically trivial on ∂N; since otherwise there is nothing to prove. Then, since we assumed $q \neq 0$, by Corollary 2.6, $T_L^{-q} \neq 1$. It follows that $H \neq 1$ in Γ. Since K was assumed to be a non-trivial knot its genus is at least one. Hence, the genus of Σ is at least two. Now Corollary 2.4 applies to conclude that L is homotopically trivial on ∂N, contradicting our earlier assumption. $\square$

Finally, we are ready to complete the proof of Theorem 6.3. The two principal ingredients of the proof are a result of Gabai ([Ga]) and Proposition 6.18.

6.8. **Proof of Theorem 6.3.** Let K, K' be fibered isotopic knots, such that K' is obtained from K by a generalized crossing change supported on a crossing circle L. Let D be a crossing disc with $L := \partial D$. Let S be a Seifert surface of K that is of minimum genus in the complement of L, isotoped so that $S \cap D$ is an arc properly embedded on S. Let S and S' denote the Seifert surfaces for K and K', respectively, obtained from S. Since K and K' are isotopic, by Corollary 2.4 of [Ga], S and S' are minimum genus Seifert surfaces. By Proposition 6.18, L

bounds an embedded disc D' in $S^3 \setminus K$ and thus the crossing change is nugatory. $\square$

7. Adjacency to fibered knots

We begin by recalling from [KL] the following definition.

Definition 7.1. *Let K, K' be knots. We will say that K is n-adjacent to K', for some $n \in \mathbf{N}$, if K admits a projection containing n generalized crossings such that changing any $0 < m \leq n$ of them yields a projection of K'.*

The notion of knot adjacency was studied in [KL] where we showed the following: Given knots K and K' there exists a constant $c = c(K, K')$ such that if K is n-adjacent to K' then either $n \leq c$ or K is isotopic to K'. The constant c was shown to encode information about the relative size of the knot genera $g(K)$, $g(K')$ and the toroidal decompositions of the knot complements. Here, using Theorem 6.3, we will show that if K' is assumed to be fibered, then we can have a much stronger result. As already mentioned in the Introduction, this stronger result leads to criteria for detecting non-fibered knots when the Alexander polynomial gives inconclusive evidence ([KL1]). We now state the main result of this section, which in particular implies Theorem 1.2 stated in the Introduction.

Theorem 7.2. *Suppose that K' is a fibered knot and that K is a knot such that K is n-adjacent to K', for some $n > 1$. Then, either K is isotopic to K' or we have $g(K) > g(K')$.*

Remark 7.3. It is not hard to see that if K is n-adjacent to K', for some $n > 1$, then K is m-adjacent to K', for all $0 < m \leq n$.

Suppose that K is n-adjacent to K' and let L be a collection of n crossing circles supporting the set of generalized crossings that exhibit K as n-adjacent to K'. Since the linking number of K and every component of L is zero, K bounds a Seifert surface S in the complement of L. Define

$$g_n^L(K) := \min \{ \text{genus(S)} \mid S \text{ a Seifert surface of } K \text{ as above} \}.$$

We recall the following.

Theorem 7.4. [Theorem 3.1, [KL]] *We have*

$$g_n^L(K) = \max\{\, g(K), g(K')\,\}$$

where $g(K)$ and $g(K')$ denotes the genera of K and K', respectively.

We now use Theorems 7.4 and 6.3 to prove Theorem 7.2.

Proof: [Proof of Theorem 7.2] Let K' be a fibered knot. In the view of Remark 7.3, it is enough to prove that if K is a knot that is 2-adjacent to K' then either K is isotopic to K' or we have $g(K) > g(K')$. To that end, suppose that K is exhibited as 2-adjacent to K' by a two component crossing link $L := L_1 \cup L_2$. Let D_1, D_2 be crossing discs for L_1, L_2, respectively. Suppose, moreover, that $g(K) \leq g(K')$; otherwise there is nothing to prove. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L. As explained earlier in the paper, we can isotope S so that, for $i = 1, 2$, $S \cap \mathrm{int}(D_i)$ is an arc, say α_i that is properly embedded in S. For $i = 1, 2$, let K_i (resp. S_i) denote the knot (resp. the Seifert surface) obtained from K (resp. S) by changing C_i. Also let K_3 denote the knot obtained by changing C_1 and C_2 simultaneously and let S_3 denote the corresponding surface. By assumption, for $i = 1, 2, 3$, K_i is isotopic to K' and S_i is a Seifert surface for K_i. Since $g(K) \leq g(K')$, Theorem 7.4 implies that S_i is a minimum genus surface for K_i. Observe that K_3 is obtained from K_1 by changing C_2 and that they are fibered isotopic knots. Furthermore, S_3 is obtained from S_1 by twisting along the arc $\alpha_2 \subset S$. By Theorem 6.3, L_2 bounds an embedded disc Δ_2 in the complement of K_1. Since S_3 is incompressible, after an isotopy, we can assume that $\Delta \cap S_3 = \emptyset$. Now let us consider the 2-sphere

$$S^2 := \Delta \cup D_2.$$

By assumption $S^2 \cap S_3$ consists of the arc $\alpha_2 \subset S_3$. Since α_1 and α_2 are disjoint, the arc α_1 is disjoint from S^2. But since K is obtained from K_1 by twisting along α_1, the circle L_2 still bounds an embedded disc in the complement of K. Hence, K is isotopic to K'. $\square$

Remark 7.5. The trefoil knot is 2-adjacent to the unknot. Since the trefoil is a fibered knot Theorem 7.2 implies that the unknot is not 2-adjacent to the trefoil. Thus n-adjacency is not an equivalence relation on the set of knots.

8. Remarks and discussion

Here we give some examples that illustrate the key points in the proof of Theorem 6.3 and point out some difficulties in generalizing the argument to treat multiple crossings.

8.1. **Composite knots.** The proof of Proposition 6.18 has two cases: In Case 1 the conclusion is that the loop L bounds discs on both sides of the Heegaard surface it lies on, but need not bound a disc on the surface itself. The proof relies on the fact that g', g_1 satisfy (6.13) and appeals to Lemma 6.17.

Example 8.1. Consider a connect sum $K := K_1 \# K_2$, where K_1, K_2 are fibered knots with Λ a 2-sphere realizing the connect sum decomposition. We have a fibration $M = S^3 \setminus K = S \times J/h$, where S is the connect sum of surfaces S_1 and S_2 for K_1, K_2, respectively. After an isotopy of S^3, fixing K, we can assume that Λ intersects S in a simple properly embedded arc λ. Let (Σ, g) an HN-model corresponding to the fibration and consider the corresponding Heegaard splitting $S^3 = N \cup_g (-N)$ so that $L := \Lambda \cap \Sigma$ is a crossing circle of K. Up to isotopy on Σ we can take $g(L) = L$. Let D, D' denote the components of $\Lambda \setminus L$. We make a crossing change of order q along L, by twisting along D; let K' denote K after this twist. The model provided by the twisting operation along D (as in Lemma 6.10) is (Σ, gT_L^{-q}). Let S' the surface of K' resulting from S after the twist. (Σ, g) gives rise to a second HN-model for $M' := S^3 \setminus K'$, via the operation of flipping a band of S around a 2-handle of the Heegaard splitting, q-times (belt trick). We have

$$i^{-1}g^{-1}i(i^{-1}T^q_{i^{-1}g^{-1}i(L)}i) = i^{-1}g^{-1}i(i^{-1}T^q_{g(L)}i) = gT^{-q}_{g(L)} = T_L^{-q}g$$

where $i : N \longrightarrow N$ is an orientation reversing involution and the last equation is due to the fact that $g(L) = L$. Since $g(L)$ bounds a disc in

N, $i^{-1}T^{-q}_{g(L)}i$ extends to an orientation preserving homeomorphism of N. Thus the equation above is one of the sort given by (6.13).

8.2. **Multiple crossing changes.** In Case 2 of the proof Proposition 6.18 one concludes that the loop L bounds a disc on the surface Σ. The crucial step here is the application of Corollary 2.4 which is facilitated by Lemma 6.16(a). Since Kotschick's result is not true in the case of twists with mixed signs, this part of the argument breaks down in an attempt to generalize the statement of Proposition 6.18 to multiple crossing changes. But as the following example shows the result is , in fact, not true!

Example 8.2. Let K denote the figure eight knot as boundary of a genus one Seifert surface S obtained by Hopf pluming two once twisted bands B_L and B_R. Consider D_1, D_2 crossing discs of K such that $D_1 \cap B_L$ (resp. $D_2 \cap B_R$) is an essential arc cutting B_L (resp. B_R) into a square. One can perform opposite sign twists of order four along D_1, D_2 to transform S to S' where in S' the Hopf band B_L becomes the Hopf band B_R and vice versa. The knot $K' := \partial S'$ is isotopic to K. Moreover, S and S' are clearly minimum genus Seifert surfaces for K and K', respectively. However, neither of $L_1 := \partial D_1$ or $L_2 := \partial D_2$ is contractible on the boundary $\partial N(S)$ of a neighborhood of S!

8.3. **Where the method fails for multiple crossings.** One possible place is that of Case 2 in the proof of Proposition 6.18: If the argument in there is applied to Example 8.2, one obtains that a product of Dehn twists $T^2_{L_1}T^{-2}_{L_2}$ is isotopic on $\partial N(\Sigma)$ to a simple commutator. But as Example 2.5 shows this is always possible!

We should point out that there is at least one obstacle in generalizing Theorem 6.3 even in the case of multiple crossing changes of the same sign. For, Gabai's result provides no information in the presence of multiple crossing changes. Thus, even with a version of Proposition 6.18 at hand, one would not be able to generalize Theorem 6.3 in the setting of multiple crossings, using the approach of this paper.

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Department of Mathematics, Michigan State University, East Lansing, MI, 48824

E-mail address: `kalfagia@@math.msu.edu`