COSMETIC CROSSING CHANGES OF FIBERED KNOTS

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ABSTRACT. We prove that a crossing circle L of a fibered knot K bounds a disc in the complement of K, if and only if there is a crossing change supported on L that doesn't change the isotopy class of K. We also sow that if a knot K is n-adjacent to a fibered knot K', for some $n > 1$, then either the genus of K is larger that of K' or K is isotopic to K'. This statement, which strengthens a result of [9], leads to criteria for detecting non-fibered knots and it has some applications in the theory of finite type 3-manifold invariants ([10]).

Keywords: crossing change, commutator length of a Dehn twist, fibered knot, mapping class group, Heegaard splitting, Thurston norm.

1. INTRODUCTION

An open question in classical knot theory is the question of when a crossing change on a knot changes the isotopy class of the knot. The purpose of this paper is to answer this question for fibered knots.

A crossing disc for a knot $K \subset S^3$ is an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ twice with zero algebraic intersection number. A crossing change on K can be achieved by twisting D or equivalently by performing appropriate Dehn surgery of S^3 along the crossing circle ∂D. The crossing is called nugatory if and only if ∂D bounds an embedded disc in the complement of K. This disc and D form a 2-sphere that decomposes K into a connected sum, where some of the summands may be trivial. Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. Problem 1.58 of [8] asks whether the converse is true (see also [19] for related conjectures): That is, if a crossing change on a knot K yields a knot isotopic to K is the crossing nugatory?

In the case that K is the trivial knot an affirmative answer follows from work of Gabai [7]. An affirmative answer is also known in the case of 2-bridge knots [19]. In this paper we will show the following.

Theorem 1.1. *Let K be a fibered knot. A crossing change on K yields a knot isotopic to K if and only if the crossing is nugatory.*

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To give a brief outline of the proof, let K be a fibered knot such that a crossing change on K gives a knot K' that is isotopic to K. The complement of K can be fibered over S^1 with fiber, say S; a minimum genus Seifert surface of K. A result of Gabai implies that the crossing change from K to K' can be achieved along an arc that is properly embedded on S. Equivalently, the crossing change can be achieved by twisting K along a meridian disc D of a handlebody neighborhood, say $N = N(S)$, of the fiber. If this twist extends to a homeomorphism of the knot complements then a result of McCullough [14] and Oertel [15] implies that ∂D bounds a disc in the complement of K. In the general case, using a result of Scharlemann-Thompson [18] and Ni [16], and properties of fibered knot complements, the problem reduces to the question of whether a power of a Dehn twist on the surface ∂N along the curve ∂D, can be written as a single commutator in the mapping class group of the surface. In the recent years, this question has arisen in the study of Lefschetz fibrations and the theory of symplectic 4-manifolds and has been studied in this context [5], [3], [11], [12]. In particular, a result of Kotschick implies that a product of Dehn twists of the same sign, along a collection of disjoint, homotopically essential curves on an orientable surface cannot be written as a single commutator in the mapping class group of the surface. Using this result, we show that the assumption that K is isotopic to K' implies that ∂D bounds a disc in the complement of K.

Theorem 1.1 says that an essential crossing change always changes the isotopy class of a fibered knot. It is natural to ask whether the crossing change produces a simpler or more complicated knot with respect to some knot complexity. A complexity function whose interplay with crossing changes has been studied using the theory of taut foliations and sutured 3-manifolds is the knot genus. For example [17] studies the interplay of link genus and Conway skein moves. Simple examples show that a single crossing change may decrease or increase the genus of a knot even if one stays within the class of fibered knots. However there are interesting consequences if one replacing a crossing change by the more refined notion of knot adjacency [9], [10] : We recall that K is called 2-adjacent to K' if K admits a projection that contains two crossings such that changing any of them or both of them simultaneously, transforms K to K'.

Theorem 1.2. *Let $g(K)$ and $g(K')$ denote the genera of K and K', respectively. Suppose that K' is a fibered knot and that K is 2-adjacent to K'. Then, either $g(K) > g(K')$ or K is isotopic to K'.*

It is known that the Alexander polynomial can be used to detect non-fibered knots. Namely, the Alexander polynomial of a fibered knot is monic. Theorem 1.2 can be used to obtain criteria for detecting non-fibered knots when the Alexander polynomial provides inconclusive evidence. This direction is explored in [10] where we also provide applications in the theory of finite type invariants of 3-manifolds.

We organize the paper as follows: In Section 2 we summarize the mapping class group results that we need for the proof of Theorem 1.1 and in in Section 3 we summarize know properties of fibered knot complements. In Section 4 we discuss a setting relating fibrations of knot complements and Heegaard splittings of S^3, from the point of view needed in the rest of the paper. In Section 5, we study nugatory crossings of fibered knots and we prove Theorem 1.1. In Section 6 we study adjacency to fibered knots and prove Theorem 1.2.

Throughout the paper we work in the PL or the smooth category.

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2. COMMUTATOR LENGTHS AND DEHN TWISTS

2.1. **Commutators in the mapping class group.** Let Σ_k denote a closed oriented surface of genus k and let Γ_k denote the mapping class group of Σ_k. That is Γ_k is the group of isotopy classes of orientation preserving homeomorphisms $\Sigma_k \longrightarrow \Sigma_k$. Let $\Gamma'_k := [\Gamma_k,\ \Gamma_k]$ denote the commutator subgroup of Γ_k. An element $f \in \Gamma'_k$ is written as a product of commutators. The commutator length of f, denoted by $c(f)$, is the minimum number of factors needed to express f as a product of commutators. In the recent years, the growth of the commutator length of Dehn twists has been studied using methods from the theory of symplectic four-manifolds [5], [3], [11], [12]. In this paper, we will need a result of D. Kotschick which we recall below.

For a simple closed curve $a \subset \Sigma_k$ let T_a denote the right hand Dehn twist about a; then the left hand Dehn twist about a is T_a^{-1}.

Theorem 2.1. [Theorem 7, [11]] *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geq 2$. Suppose that $a_1, \ldots, a_m \subset \Sigma_k$ are homotopically essential, disjoint, simple closed curves on Σ_k. Let $f := T_{a_1} \cdot T_{a_2} \cdot \ldots \cdot T_{a_n}$ denote the product of right-handed Dehn twists along $a_1, \ldots, a_m$. Suppose that for some $q > 0$ we have $f^q = T_{a_1}^q \cdot T_{a_2}^q \cdot \ldots \cdot T_{a_n}^q \in \Gamma'_k$. Then, we have*

$$c(f^q) \geq 1 + \frac{qm}{18k-6}.$$

We will need the following corollary of Theorem 2.1.

Corollary 2.2. *Let Γ_k be the mapping class group of a closed oriented surface Σ_k of genus $k \geq 2$. Let $a \subset \Sigma_k$ be a simple closed curve. Suppose that there exist $g, h \in \Gamma_k$ such that*

$$T_a^q = [g, h] = ghg^{-1}h^{-1},$$

for some $q \neq 0$. Then a is homotopically trivial on Σ_k.

The proof of Theorem 2.1 given in [11] relies on the theory of Lefschetz fibrations, which, as the author points out, is sensitive to the chirality of

Dehn twists. In fact, the argument of [11] breaks down if one allows f to be a product of right-handed Dehn twists and their inverses and, as the following example shows, Theorem 2.1 is not true in this case. In subsequent sections we will discuss how this situation is reflected when one tries to apply Theorem 2.1 to the study of crossing changes that do not alter the isotopy class of fibered knots (see Example 5.10).

Example 2.3. [Example 9, [11]] Suppose that $a \subset \Sigma_k$ is an essential simple closed loop on a closed oriented surface of genus at least two. Let $g : \Sigma_k \longrightarrow \Sigma_k$ be an orientation preserving homeomorphism such that $a \cap g(a) = \emptyset$. We will also use g to denote the mapping class of g. Set $b := g(a)$ and set $f := T_a T_b^{-1}$. In the mapping class group Γ_k we have $gT_a g^{-1} = T_{g(a)}$ or equivalently $T_a = g^{-1}T_b g$. Since a, b are disjoint we also have $T_a T_b^{-1} = T_b^{-1}T_a$. Thus

$$f^q = (T_a T_b^{-1})^q = T_a^q T_b^{-q} = (g^{-1}T_b g)^q T_b^{-q} = [g^{-1}, T_b^q],$$

for all $q > 0$. Hence we have $c(f^q) = 1$ showing that Theorem 2.1 is not true in this case.

2.2. When is T_a^q trivial?

It is known that if a is homotopically essential on Σ then no non-trivial power T_a^q ($0 \neq q \in \mathbf{Z}$) is isotopic to the identity on Σ_k. This statement is well known to researchers working on mapping class groups: It is for example asserted in [2] when the authors state that the kernel of the reduction homomorphism corresponding to an essential simple closed curve a is the free abelian group generated by Dehn twists along a. Below we include a proof that uses properties of intersection numbers stemming from Thurston's study of surface homeomorphisms [6].

Corollary 2.4. *Suppose that $T_a^q = 1$ in the mapping class group $\Gamma := \Gamma(\Sigma_k)$, $k > 0$. Then, either a is homotopically trivial on Σ_k or $q = 0$.*

Proof: Suppose that the curve a is not homotopically trivial on Σ and that $T_a^q = 1$ in the mapping class group $\Gamma(\Sigma)$. We will argue that $q = 0$. First suppose that a is a non-separating loop on Σ. Then, we can find an embedded loop b that intersects a exactly once. Orient a, b so that the algebraic intersection of a, b is 1; that is $< a, b >= 1$. In $H_1(\Sigma)$ we have

$$T_a^q(b) = b + q < a, b > a = b + qa.$$

Thus we have $< T_a^q(b), b >=< b, b > +q < a, b >$ which, since $T_a^q(b) = b$, gives $q = 0$ as desired. If a is separating, we appeal to the geometric intersection number. For b a simple closed loop on Σ let $i(a, b)$ denote the intersection number; the minimum number of intersections in the isotopy classes of a and b. Since we assumed that a is homotopically essential on Σ, we can find b so that $i(a, b) \neq 0$. Proposition 1, Exposé 4 [6] implies the following:

$$i(T_a^q(b),\ b) = |q|\ (i(a,\ b))^2.$$

Since $T_a^q = 1$, we have $0 = i(b,\ b) = i(T_a^q(b),\ b)$. Thus we obtain $|q|\ (i(a,\ b))^2 = 0$; which implies that $q = 0$. ☐

Suppose that Σ_k is the boundary of a handlebody N_k of genus k. Clearly, if the loop a bounds an embedded disc $D \subset N_k$ then T_a^q extends to a Dehn twist of N_k along D. The following result shows that the converse is also true:

Theorem 2.5. [14, 15] *Let a be a simple loop on $\Sigma_k = \partial N_k$ and let T_a denote the right hand side Dehn twist along a. Suppose that there is a homeomorphism $G : N_k \longrightarrow N_k$, such that the restriction $G|\Sigma_k$ is isotopic to T_a^q, for some $q \neq 0$. Then, a bounds a disc in N_k.*

Notation. To simplify our notation, throughout the paper, we will use $\Sigma := \Sigma_k$ to denote an oriented surface of any genus $k \geq 0$ and $\Gamma := \Gamma_k$ to denote the mapping class group of Σ. Also, as we've done in this section, we will use the same symbol to denote a homeomorphism of Σ and its class in Γ.

3. UNIQUENESS PROPERTIES OF KNOT FIBRATIONS

Here we recall some terminology and known results about fibered knots. Suppose that K is a fibered knot and let S be a minimum genus Seifert surface for K. Let $\eta(K)$ denote a tubular neighborhood of K. Then, as it is argued in the proof of Theorem 5.1 in [4], the complement $\overline{S^3 \setminus \eta(K)}$ admits a fibration over S^1 with fiber S. More specifically, it is shown that the complement $\overline{S^3 \setminus \eta(K)}$ cut along S is homeomorphic to $S \times [-1, 1]$. Thus, there is an orientation preserving homeomorphism $h : S \longrightarrow S$ such that $\overline{S^3 \setminus \eta(K)}$ is obtained from $S \times [-1, 1]$ by identifying $S \times \{-1\}$ with $S \times \{1\}$ so that $(x, -1) = (h(x), 1)$. The map h is called the monodromy of the fibration. We write

$$\overline{S^3 \setminus \eta(K)} = S \times J/h,$$

where $J := [-1, 1]$. The following are proved in Section 5 of [4].

Proposition 3.1. *a) Let $M := \overline{S^3 \setminus \eta(K)} = S \times J/h$ be an oriented, fibered knot complement. Given a minimum genus Seifert surface S_2 of K, there exists an orientation preserving homeomorphism of M that is fixed on ∂M and brings S_2 to the fiber $S_1 := S \times \{1\} = S \times \{-1\}$.*

b) Let $M := S \times J/h$ and $M' := S' \times J/h'$ be fibered, oriented knot complements. Then, there exists an orientation preserving homeomorphism $F : M \longrightarrow M'$, with $F(\partial S \times \{j\}) = \partial S' \times \{j\}$ $(j \in J)$ if and only if there exists an orientation preserving surface homeomorphism $f : (S, \partial S) \longrightarrow (S', \partial S')$ such that fhf^{-1} and h' are equal up to isotopy on S'.

4. HANDLEBODIES IN KNOT COMPLEMENTS

4.1. **Definitions.** In this section we will consider decompositions of knot complements that arise from Heegaard splittings of S^3. We begin by recalling some familiar terminology about Heegaard splittings: A compression body N is an oriented 3-manifold obtained from a product $\Sigma \times [0,1]$, where Σ is a closed, connected, oriented surface, by attaching 2-handles along a collection of disjoint, simple closed curves on $\Sigma \times \{1\}$ and capping off the newly produced 2-sphere boundary components with 3-cells. Let $\partial_+ N := \Sigma \times \{0\}$ and $\partial_- N := \partial N \setminus \partial_+ N$. If $\partial_- N = \emptyset$ then N is a handlebody. A trivial compression body is one homeomorphic to product $\Sigma \times [0,1]$. A compression body N can be reduced to a trivial compression body (or to a 3-ball if $\partial_- N = \emptyset$) by cutting along a complete disc system $(\mathcal{D}, \partial\mathcal{D}) \subset (N, \partial_+ N)$.

Definition 4.1. *A Heegaard splitting of a compact, oriented 3-manifold M, is a decomposition $M = N_1 \cup N_2$, where N_1 and N_2 are compression bodies and $Q := N_1 \cap N_2 = \partial_+ N_1 = \partial_+ N_2$ is an embedded orientable surface. The surface Q is called the Heegaard surface of the decomposition.*

We will be particularly interested in handlebodies of even genus. Let P denote a model genus k compact, connected, oriented surface with one boundary component. It will be convenient for us to think of the model oriented handlebody of genus $2k$ as a product $N := P \times I$.

Definition 4.2. *A simple closed curve K on the boundary of a $2k$ genus handlebody N_1 is called a preferred curve if there exists a homeomorphism $N_1 \longrightarrow N$ that takes K onto the curve $\partial P \times \{\frac{1}{2}\}$ of ∂N.*

An H-body is a 3-manifold H_1 that is homeomorphic to the complement of a knot in a handlebody. That is given a handlebody N_1 and a knot $K \subset N_1$, let $\eta(K)$ be a tubular neighborhood of K in N_1. Then, $H_1 := \overline{N_1 \setminus \eta(K)}$ is an H-body. If K can be isotoped on ∂N_1 so that it is a preferred curve of N_1, then H_1 is called a preferred H-body for K. We will use $\partial_+ H_1$ to denote the non-torus component of ∂H_1.

Definition 4.3. *An HN-splitting of a knot complement $M := \overline{S^3 \setminus \eta(K)}$ is a decomposition $M = H_1 \cup N_2$, where N_2 is a handlebody and H_1 is an H-body. The closed oriented surface $Q := H_1 \cap N_2 = \partial_+ H_1 = \partial N_2$ is called an HN-surface of M. The genus of Q is called the genus of the HN-splitting.*

Remark 4.4. Note that an HN-splitting $M = H_1 \cup N_2$ of a knot complement $M = \overline{S^3 \setminus \eta(K)}$ is not a Heegaard splitting of M. For, an H-body is not, in general, a compression body. Note, however, that $(H_1 \cup \eta(K)) \cup N_2$ is a Heegaard splitting of S^3. Throughout the paper we will refer to this Heegaard splitting as the splitting of S^3 corresponding to $H_1 \cup N_2$. Conversely, if $N_1 \cup N_2$ is a Heegaard splitting of S^3 and $K \subset \text{int} N_1$ is a knot, then $\overline{N_1 \setminus \eta(K)} \cup N_2$ is an HN-splitting of $\overline{S^3 \setminus \eta(K)}$.

Let $(-N)$ denote N with the opposite orientation and fix i an orientation reversing involution of N; we can think of i as a map $i : N \longrightarrow (-N)$.

Given an H-body $H \subset N$ with $\partial_+ H = \partial N$ and an orientation preserving homeomorphism $g : \partial_+ H \longrightarrow \partial N$, the quotient space

$$H \cup_g (-N) := H \sqcup (-N)/\{y \sim ig(y) \mid y \in \partial_+ H\},$$

is an oriented knot complement. Conversely, let $M = H_1 \cup N_2$ be an HN-splitting of a knot complement $M = \overline{S^3 \setminus \eta(K)}$ and let $S^3 = N_1 \cup N_2 = (H_1 \cup \eta(K)) \cup N_2$ be the corresponding Heegaard splitting of S^3. For $k = 1, 2$, we choose a homeomorphism $m_k : N_k \longrightarrow N$. The image $H := m_1(H_1) = \overline{N \setminus m_1((\eta(K))}$ is an H-body with $H \subset N$ and $\partial_+ H = \partial N$. The composition $g := m_2 m_1^{-1} | \partial_+ H : \partial_+ H \longrightarrow \partial N$ is an orientation preserving homeomorphism and $M = \overline{S^3 \setminus \eta(K)}$ is homeomorphic to the quotient $H \cup_g (-N)$. Now $g : \Sigma \longrightarrow \Sigma$ defines an element in the mapping class group $\Gamma := \Gamma(\Sigma)$, where $\Sigma := \partial_+ H = \partial N$. As in Section 2, we will also use g to denote the mapping class of g.

Definition 4.5. *The pair* $(\Sigma,\ g)$ *is called an* HN*-model of the* HN*-splitting* $M = H_1 \cup N_2$.

By the discussion above, obtaining an HN-model $(\Sigma,\ g)$ of an HN-splitting $M := \overline{S^3 \setminus \eta(K)} = H_1 \cup N_2$ involves the choice of homeomorphisms $m_2 : N_2 \longrightarrow N$ and $m_1 : H_1 \longrightarrow H \subset N$ so that we have a homeomorphism

$$m : M \longrightarrow H \cup_g (-N), \quad \text{with} \quad m|H_1 = m_1 \quad \text{and} \quad m|N_2 = im_2.$$

This implies that we have a commutative diagram

$$\begin{tikzcd}
\partial H_1 \arrow[r, "\mathrm{id}"] \arrow[d, "m_1|\partial H_+"'] & \partial N_2 \arrow[d, "m_2|\partial N"] \\
\partial H_+ \arrow[r, "g"] & \partial N
\end{tikzcd} \tag{4.1}$$

or

$$g := m_2 m_1^{-1} | \partial_+ H : \partial_+ H \longrightarrow \partial N$$

With this in mind, we will often work with the model of an HN-splitting rather than the splitting itself.

4.2. *HN***-splittings and knot fibrations.** Given a knot fibration $M := \overline{S^3 \setminus \eta(K)} = S \times J/h$ there is a natural way to obtain an HN-splitting of M and an associated HN-model in terms of the monodromy of the fibration. We now describe this process: To begin, we set $N_1 := S \times [0, 1]$ and $N_2 := S \times [-1, 0]$. Also we set $E := \partial S \times (0, 1)$ and $E' := \partial S \times (-1, 0)$. We have $\partial N_1 = (S \times \{0\}) \cup E \cup (S \times \{1\})$. Similarly, we have $\partial N_2 = (S \times \{-1\}) \cup E' \cup (S \times \{0\})$. We will assume that $K := \partial S \times \{\frac{1}{2}\}$ on ∂N_1; thus K is the preferred curve of N_1. Define $h^* : \partial N_1 \longrightarrow \partial N_1$ by

$$h^*(x, 0) = (x, 0), \quad \text{for} \quad x \in S, \tag{4.2}$$

$$h^*(x, t) = (x, t), \quad \text{for} \quad x \in \partial S \quad \text{and} \quad 0 < t < 1, \tag{4.3}$$

$$h^*(x, 1) = (h(x), 1), \quad \text{for} \quad x \in S. \tag{4.4}$$

Consider the homeomorphism $rh^* : \partial N_1 \longrightarrow \partial N_2$, where $r : N_1 \longrightarrow N_2$ is defined by $(x, t) \to (x, -t)$. We obtain a Heegaard splitting

$$S^3 = N_1 \cup_g N_2 := N_1 \sqcup N_2 / \{y \sim rh^*(y) \mid y \in \partial N_1\}, \tag{4.5}$$

such that K is a preferred curve on N_1 and N_2. To pass to an HN-splitting we push K on $S \times \{\frac{1}{2}\}$ slightly in the interior of N_1 and then we take $A(K)$ to be an annulus neighborhood of K on $S \times \{\frac{1}{2}\}$. If we remove a tubular neighborhood of K, say $\eta(K) := A(K) \times (\{\frac{1}{2}\} - \epsilon,\ \{\frac{1}{2}\} + \epsilon)$, from $\mathrm{int}(N_1)$ we obtain an HN-splitting of genus $l := 2\mathrm{genus}(K)$ for M. Let P denote the genus l model surface within the homeomorphism class of S. To obtain an HN-model for our splitting, we pick an orientation preserving homeomorphism $m : (S, \partial S) \longrightarrow (P, \partial P)$ and define

$$m_1 : N_1 \longrightarrow N, \quad (x, t) \to (m(x), t), \tag{4.6}$$

and

$$m_2 := m_1 r^{-1} : N_2 \longrightarrow N, \quad (x, t) \to (m(x), -t). \tag{4.7}$$

The restriction $m_1|H_1$ sends H_1 to an H-body $H \subset N$ and the restriction $m_2 rgh^* m_1^{-1}|\partial_+ H : \partial_+ H \longrightarrow \partial N$ is an orientation preserving automorphism of Σ. Now $(\Sigma,\ m_2 rh^* m_1^{-1}) = (\Sigma,\ m_1 h^* m_1^{-1})$ is an HN-model for $M = H_1 \cup N_2$.

Remark 4.6. It will be convenient for us to abuse the conventions of (4.6)-(4.7) and take $N := N_1 = S \times [0, 1]$ so that N_2 is identified to $(-N)$ via r^{-1}. Thus we will often take $m_1 = \mathrm{id}$. By the construction of the HN-splitting associated to the fibration $\overline{S^3 \setminus \eta(K)} = S \times J/h$, we have a surface $S_1 \subset S \times \{\frac{1}{2}\}$ that is disjoint from the corresponding HN-surface. Furthermore S_1 and $S \times \{\frac{1}{2}\}$ differ by an annulus and after the construction of $H_1 \cup N_2$ we have $K = \partial S_1$. Thus, we may think of this HN-surface as sitting in the original fibration $\overline{S^3 \setminus \eta(K)} = S \times J/h$ and the surface S_1 is a level surface of the fibration.

Definition 4.7. *Let $g := m_1 h^* m_1^{-1}$. The pair $(\Sigma,\ g)$ is called the HN-model associated to the fibration $M = S \times J/h$. Note that, by (4.1)-(4.7), g is the identity on $\Sigma \setminus P$, where $P := S \times \{1\}$.*

4.3. **Equivalence relations for HN-models.** According to the conventions adapted in Remark 4.6, the HN-surface, say Q, of the splitting corresponding to a fibration $M' := \overline{S^3 \setminus \eta(K')} = S' \times J/h'$, is disjoint from from a level surface S'_1 of the fibration. To facilitate the exposition in the proof of the next lemma we will assume, without loss of generality, that $S'_1 = S' \times \{\frac{1}{2}\}$.

Lemma 4.8. *Let $M' := \overline{S^3 \setminus \eta(K')} = S' \times J/h'$ be an oriented fibered knot complement. Let $(\Sigma,\ g')$ denote the HN-model associated to the fibration and let Q denote the HN-surface of the corresponding HN-splitting of M'. As in Remark 4.6, we will think of Q sitting in the fibration so that $S'_1 :=$*

$S' \times \{\frac{1}{2}\}$ is disjoint from Q and we will think of N as $S' \times [0,1]$. Let $(\Sigma,\ g'')$ be a second HN-model of M' and let Q' denote the HN-surface of the corresponding HN-splitting. Let $M' = H_1 \cup N_2$ and $M' = H_1' \cup N_2'$ denote the HN-splittings of M' corresponding to Q and Q', respectively. Suppose we've chosen homeomorphisms $m : H_1 \cup N_2 \longrightarrow H \cup_{g'} (-N)$ and $j : H_1' \cup N_2' \longrightarrow H \cup_{g''} (-N)$, as in (4.1), such that:

(1) We have $S_1' \subset \text{int} H_1'$. In particular, S_1' is also disjoint from Q'.

(2) We have $j|S_1' = m|S_1'$.

(3) We have an orientation preserving homeomorphism $F : M' \longrightarrow M'$ such that

$$F(S_1') = S_1' \quad \text{and} \quad F(Q) = Q'. \tag{4.8}$$

Assume, moreover, that F is level preserving; that is $F(S' \times \{x\}) = S' \times \{x\}$, for all $x \in J$. Then, there is an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ such that in the mapping class group $\Gamma = \Gamma(\Sigma)$ we have

$$g'' = fg'f^{-1}. \tag{4.9}$$

Proof: The existence of the homeomorphism $F : M' \longrightarrow M'$ in (4.8) implies that Q' is the HN-surface corresponding to a fibration of M' with fiber S_1'. We will now discuss a model of this fibration: If we let f_1 denote the restriction of F on the fiber S_1' then the monodromy of our second fibration should be a conjugate of h' by f_1 (Proposition 3.1). That is the monodromy of the fibration in which Q' is the corresponding HN-surface is $h_1 := f_1 h f_1^{-1}$ (where, recall, the equality is understood up to isotopy on the fiber.) Recall the homeomorphism $m : H_1 \cup N_2 \longrightarrow H \cup_{g'} (-N)$. By (4.1)-(4.7) and Definition 4.7, g' is the identity on $\Sigma \setminus P$, where $P := S' \times \{1\}$. Furthermore, $g' = m_1 h' m_1^{-1}$ on P, where m_1 is the restriction of m on H_1. Now consider the homeomorphism $j|S_1' : S_1' \longrightarrow P_1 := j(S_1')$; it is a homeomorphism from the fiber onto the model surface of genus 2genus(S_1'). Using this homeomorphism and following the process in (4.2)-(4.7), we can find an model $(\Sigma,\ g_1)$ corresponding to the fibration $S' \times J/h'$ of M'. More specifically, we can find a homeomorphism $l : H_1' \cup N_2' \longrightarrow H \cup_{g_1} (-N)$ with $l|S_1' = j|S_1'$ and such, if we let l_1 denote the restriction of l^{-1} on H, then $g_1 = l_1 h_1 l_1^{-1}$ on P and g_1 is the identity on $\Sigma \setminus P$. Now we define $f : \Sigma \longrightarrow \Sigma$ by $f|P = l_1 f_1 m_1^{-1}$ and $f|(\Sigma \setminus p) = l_1 m_1^{-1}$. Now on P we have

$$g_1 = l_1 h_1 l_1^{-1} = l_1 f_1 h f_1^{-1} l_1^{-1} = (l_1 f_1 m_1^{-1}) m_1 h' m_1^{-1} (m_1 f_1^{-1} l_1^{-1}) = fg'f^{-1}.$$

Similarly, we have $g_1 = fg'f^{-1}$ on $\Sigma \setminus P$. Thus, up to isotopy on Σ we have, $g_1 = fg'f^{-1}$.

Let j_1 (resp. l_1) denote the restriction of j^{-1} (resp. l^{-1}) on H and let j_2 (resp. l_2) denote the restriction of j^{-1} (resp. l^{-1}) on N. Consider the composition $e := lj^{-1} : H \cup_{g''} (-N) \longrightarrow H \cup_{g_1} (-N)$. Let R' denote $H \cup_{g_1} (-N)$ cut along $P_1 := j(S_1') = m(S_1')$. Let $Q'' := j(Q') = m(Q')$. By condition (2), and our choice of l, we have $e(R') = R'$, $e(Q'') = Q''$, $e|\partial R' = \text{id}$, $e(H) = H$ and $e(N) = N$. Since e preserves the HN-surfaces we have a commutative diagram

$$\begin{tikzcd}
\partial H_+ \arrow[r, "g_1"] \arrow[d, "e_1|\partial H_+"'] & \partial N \arrow[d, "e_2|\partial N"] \\
\partial H_+ \arrow[r, "g''"] & \partial N
\end{tikzcd}$$

where $e_1 := e|H$ and $e_2 := e|N$. We will consider R' as an interval bundle $R' = P_1 \times J$, where $J = [0, \frac{1}{2}] \cup [-1, 0] \cup [1/2, 1]$ and the sub-bundle pieces corresponding to these sub-intervals are glued together according to (4.2)-(4.5). Furthermore, we may consider N as a product $N = P'' \times [-1, 0]$ such that P'' lies on P_1 and $E := \overline{P_1 \setminus P''}$ is an annulus with one boundary component on Q'' and the second on $\partial R'$. Thus, for $t \in [-1, 0]$, Q'' intersects each fiber $P_t = P \times \{t\}$ at a simple curve that is parallel to ∂P_t on P_t. We claim that, up to an isotopy relative to $\partial R'$, e can be assumed to also be level-preserving; that is $e(P_1 \times \{x\}) = P_1 \times \{x\}$, for all $x \in J$. The claim follows from Lemma 3.5 of [20] and in particular from the argument given in the proof of Case 1 of that lemma: To outline the argument let $p : R' = P_1 \times J \longrightarrow P_1$ denote the J-bundle projection. Pick $\mathbf{D}$ to be a collection of vertical discs in $R' = P_1 \times J$ so that P_1 split along the set of arcs $p(\mathbf{D})$ is disc. Now $e(\mathbf{D})$ is also a collection of discs in R'. Let D be a component of $\mathbf{D}$. By assumption, $e|\partial D = \mathrm{id}$ and $Q'' \cap e(D)$ is a simple closed curve on Q''. Now $E' := e(D) \cap W'$ is a spanning annulus in $W' := \overline{R' \setminus N}$. Let $D_1 := \overline{D \setminus E'}$. As in Lemma 3.4 of [20], we may change e in $D \setminus (D \cap Q'')$ so that we have $e(D) = D$. Then, as in the proof of Lemma 3.5 of [20], we deduce that e can be assumed to be level preserving, up to an isotopy relative to $\partial R'$. Thus, we can find an isotopy $\{\phi_t\}_{t \in [0,\ 1]}$ of R' such that $\phi_t|\partial R' = \mathrm{id}$, $\phi_0 = e$ and $\phi_1|\mathbf{D} = \mathrm{id}$. Now ϕ_1 can be further isotopied, via an isotopy with the same properties, so that it is identity on each of H and N. It follows that e_1 and e_2 are isotopic to the identity in H and N, respectively. In particular $e_1|\Sigma = e_2|\Sigma$ is isotopic to the identity on and thus $e_1|\Sigma = e_2|\Sigma = 1$ in $\Gamma(\Sigma)$. Now the commutative diagram above implies that in $\Gamma(\Sigma)$ we have $g'' = g_1 = f g' f^{-1}$. □

5. CROSSINGS CHANGES AND DEHN TWISTS

In this section we study the question of when a crossing change in a fibered knot leaves the isotopy class of the knot unaltered and we will prove Theorem 1.1. In fact we will work in a more general context as we will consider "generalized crossing changes".

5.1. Nugatory crossing changes in fibered knots.

Let K be a knot in $\mathbf{S}^3$ and let $q \in \mathbf{Z}$. A generalized crossing of order q on a projection of K is a set C of $|q|$ twist crossings on two strings that inherit opposite orientations from any orientation of K. If K' is obtained from K by changing all the crossings in C simultaneously, we will say that K' is obtained from K by a generalized crossing change of order q (see Figure 1). Notice that if $|q| = 1$,

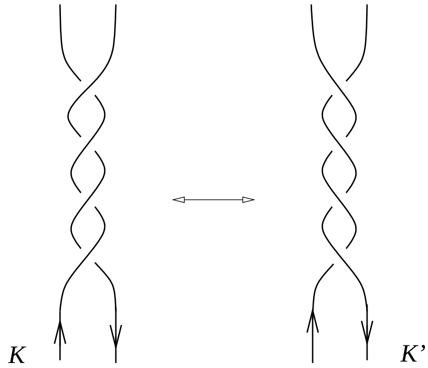

FIGURE 1. The knots K and K' differ by a generalized crossing change of order $q = -4$.

K and K' differ by an ordinary crossing change while if $q = 0$ we have $K = K'$. A crossing disc for K is an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ twice with zero algebraic intersection number. Performing $\frac{1}{-q}$-surgery on $L := \partial D$, for $q \in \mathbf{Z}$, changes K to another knot $K^{'} \subset S^3$. Clearly $K^{'}$ is obtained from K by a generalized crossing change of order q. The boundary $L := \partial D$ is called a crossing circle supporting the generalized crossing change.

Definition 5.1. *A generalized crossing supported on a crossing circle L of a knot K is called nugatory if and only if $L := \partial D$ bounds an embedded disc in the complement of K. This disc and D form an embedded 2-sphere that decomposes K into a connected sum where some of the summands may be trivial.*

Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. It is an open question is whether, in general, the converse is true (Problem 1.58, [8]). The answer is known to be *yes* in the case when K is the unknot [17] and when K is a 2-bridge knot [19]. To these we add the following theorem.

Theorem 5.2. *Let K be a fibered knot and let K' a knot obtained from K by a generalized crossing change. If K' is isotopic to K then a crossing circle L supporting this crossing change bounds an embedded disc in the complement of K.*

5.2. **Preliminaries.** Let C be a generalized crossing of order $q \neq 0$ of a fibered knot K. Let K' denote the knot obtained from K by changing C and let D be a crossing disc for C with $L := \partial D$.

Lemma 5.3. *Suppose that $M_L := \overline{S^3 \setminus (\eta(K) \cup \eta(L))}$ is reducible. Then L bounds a disc in the complement of K. Thus, in particular, the crossing change from K to K' is nugatory.*

Proof: Let Δ be an essential 2-sphere in M_L; $\eta(K)$ and $\eta(L)$ must lie in different components of $M_L \setminus \Delta$. Isotope Δ so that its intersection with D is minimal in M_L. Then $\Delta \cap D$ is a collection of simple closed curves, each parallel to ∂D on D. Let C a curve in this intersection that is innermost on D. Then C bounds discs $E \subset D$ and $E' \subset \Delta$ with their interior disjoint from $\Delta \cap D$. Let D' the disc obtained from D by replacing the interior of E with that of E'. Then $|D' \cap \Delta| < |D \cap \Delta|$ and by an inductive argument we can assume that $|D' \cap \Delta| = 0$. This implies that, in S^3, L bounds disc that is disjoint from Δ. Since Δ separates K from L this disc must necessarily be disjoint from K. $\square$

In the view of Lemma 5.3, we may assume that M_L is irreducible. Since the linking number of L and K is zero, K is homologically trivial in the complement of L. It is known that this implies that K bounds a Seifert surface in the complement of L. Let S be a Seifert surface that is of minimum genus among all such Seifert surfaces. Since S is incompressible, after an isotopy we can arrange so that the closed components of $S \cap D$ are homotopically essential in $D \setminus K$. But then each such component is parallel to ∂D on D and by further modification we can arrange so that $S \cap D$ is an arc that is properly embedded on S. The surface S gives rise to Seifert surfaces S and S' of K and K', respectively.

Proposition 5.4. *Suppose that K is isotopic to K'. Then, S and S' are Seifert surfaces of minimum genus for K and K', respectively.*

Proof: We can consider the surface S properly embedded in M_L so that it is disjoint from the component $\partial\eta(L)$ of ∂M. The assumptions on irreducibility of M_L and on the genus of S imply that the foliation machinery of Gabai [7] applies. In particular, S is taut in the Thurston norm. The manifolds $M := \overline{S^3 \setminus \eta(K)}$ and $M' := \overline{S^3 \setminus \eta(K')}$ are obtained by Dehn filling of M_L along $\partial(\eta(L))$. By Corollary 2.4 of [7], S can fail to remain taut in the Thurston norm (i.e. genus minimizing) in at most one of M and M'. But since K is isotopic to K', M is homeomorphic to M'. Thus S remains taut in both of M and M'. This implies that S and S' are Seifert surfaces of minimum genus for K and K', respectively. $\square$

Next we restrict to fibered knots and recall the assumptions that we have to work with from the statement of Theorem 5.2: K and K' are fibered knots that are isotopic. S and S' are minimum genus Seifert surfaces, for K and K', respectively.

5.3. An HN-model for M' from Dehn surgery.

With the notation of Section 4, there is a fibration $M := \overline{S^3 \setminus \eta(K)} = S \times J/h$ with monodromy $h : S \longrightarrow S$. As in §4.1, let P denote the model surface within the homeomorphism class of S and S', and let $N := P \times [0, 1]$ and $\Sigma := \partial N$. Choose a homeomorphism $m : S \longrightarrow P$, as in (4.5)-(4.6), to obtain an HN-model

(Σ, g) corresponding to the fibration of M. With the conventions of Remark 4.6 we will think of the Heegaard splitting of S^3 corresponding to the fibration $M = S \times J/h$ as the quotient

$$N \cup_g (-N) := N \sqcup (-N)/\{y \sim ig(y), \mid y \in \Sigma\}, \tag{5.1}$$

where i is the involution defined in §4.3. We will further assume that the crossing circle L is embedded on Σ so that D is a meridian disc of N. We will furthermore assume that the embedding of L on Σ is chosen so that, up to isotopy in M, the geometric intersection $|K \cap L|$ is minimum. Note that since we assumed that $M_L := \overline{S^3 \setminus (\eta(K) \cup \eta(L))}$ is irreducible, this minimum intersection must be non-zero. Let $\tau : N \longrightarrow N$ denote the right-handed Dehn twist of N along the meridional disc D and let $T_L := \tau|\Sigma$, where $L = \partial D$. We have $\tau^{-q}(S) = S'$ and $\tau^{-q}(K) = K'$. Recall that $M := S \times J/h$ and that $M' := \overline{S^3 \setminus \eta(K')}$ is obtained from M_L by by Dehn filling along $\partial\eta(L)$ with slope $\frac{1}{-q}$. Next we use that information to construct an HN-model for M'. The proof of Lemma 5.5 follows a known process of passing between gluing maps of Heegaard splittings and Dehn surgeries of 3-manifolds (compare, pp. 86-87 of [1]).

Lemma 5.5. *$(\Sigma,\ gT_L^{-q})$ is an HN-model for $M' = \overline{S^3 \setminus \eta(K')}$.*

Proof: By assumption (Σ, g) is an HN-model corresponding to the fibration $M = S \times J/h$. Let A denote an annulus on Σ that supports T_L and let $B := g(A)$. We will think of this HN-splitting of M as the quotient

$$H \cup_g (-N) := H \sqcup (-N)/\{y \sim ig(y), \mid y \in \Sigma\}, \tag{5.2}$$

where $H \subset N$. We consider the complement $M_L := \overline{S^3 \setminus (\eta(K) \cup \eta(L))}$ as the pre-quotient space

$$H \cup_{g^1} (-N) \quad \text{where} \quad g^1 := g|(\Sigma \setminus A) : \Sigma \setminus A \longrightarrow \Sigma \setminus B. \tag{5.3}$$

Thus we can think of the torus $\mathcal{T} := A \cup B$ as the boundary torus of a tubular neighborhood of L. Let α be an arc that is properly embedded and essential on A such that it intersects L exactly once and let $\beta := g(\alpha)$. Now $\mu := \alpha \cup \beta$ is the meridian of $\mathcal{T}$ and $\lambda := L$ is the longitude which we will orient so that their algebraic intersection number on $\mathcal{T}$, denoted by $< \lambda, \mu >$, is one. Since T_L is supported in A it can be considered as a Dehn twist on $\mathcal{T}$. We have

$$T_L^q(\mu) = \mu - q\lambda = T_L^q(\alpha) \cup \beta,$$

where

$$\beta = gT_L^{-q}(\alpha') \quad \text{and} \quad \alpha' := T_L^q(\alpha).$$

(Recall that, in general, if a, b are simple closed curves on $\mathcal{T}$, we have $T_a(b) = b + < a, b > a$. Since $< \lambda, \mu >= 1$, we have $T_L^{-1}(\mu) = \mu + \lambda$, which explains the change of sign between the power T_L^q and the coefficient of λ in $T_L^q(\mu)$ in the equations above.)

Now if we set $\mu' := \alpha' \cup \beta$ we have

$$\mu' := \alpha' \cup \beta = \mu - q\lambda.$$

Let $M_L(q)$ denote the 3-manifold obtained from M_L by $\frac{1}{-q}$ Dehn filling on $\mathcal{T}$. From the discussion above, in order to obtain $M_L(q)$ one needs to attach a solid torus to $\mathcal{T}$ in such away so that the meridian is attached along the curve μ. It follows that $H \cup_{gT_L^{-q}} (-N)$ is an HN-splitting for $M_L(q)$. But since by assumption we have $M_L(q) = \overline{S^3 \setminus \eta(K')} = M'$, it follows that $(\Sigma,\ gT_L^{-q})$ is a HN-model for M'. □

5.4. **Understanding the HN-model $(\Sigma,\ gT_L^{-q})$.** In the view of the conventions adapted earlier, N is thought as a product $S \times I$ and K is embedded on $\Sigma := \partial N$ as the preferred curve. The Dehn twist $\tau^{-q} : N \longrightarrow N$ changes K to K' and the product structure of N to $S' \times I$. By our assumptions, each of K, K' split Σ into two bounded surfaces that are incompressible in N. Let A be an annulus on Σ supporting the restriction $T_L := \tau|\Sigma$ so that the core of A is L and the intersection $A \cap K$ consists of two properly embedded, disjoint arcs, say α_1, α_2, each of which intersects L exactly once. We set, $B := g(A)$, $\gamma := g(K)$, $\gamma' := g(K')$ and $z := g(L)$. By construction, we have $g|K = \mathrm{id}$. Thus, $g^{-1}(K) = K$, $B \cap \gamma = \alpha_1 \cup \alpha_2$. We have

$$\gamma' := g(K') = g(T_L^{-q}(K)) = g(T_L^{-q}(g^{-1}(K))) = gT_L^{-q}g^{-1}(K) = T_{g(L)}^{-q}(K),$$

where the last equation follows from the fact that in the mapping class group we have $gT_Lg^{-1} = T_{g(L)}$. Thus γ' is the result of $\gamma := g(K) = K$ under a non-trivial power of a Dehn twist along $z := g(L)$ supported on B. We will think of the HN-splitting of $M' = \overline{S^3 \setminus \eta(K')}$ corresponding to the model $(\Sigma,\ gT_L^{-q})$ as the quotient

$$M' = H \cup (-N)/\{y \sim igT_L^{-q}(y),\ \mid y \in \Sigma\}, \tag{5.4}$$

and we will identify the corresponding Heegaard splitting of S^3 with the quotient

$$N \sqcup (-N)/\{y \sim igT_L^{-q}(y),\ \mid y \in \Sigma\}. \tag{5.5}$$

Next we observe that the irreducibility assumption implies that the Dehn twist $T_{g(L)}$ does not extend over the handlebody N.

Lemma 5.6. *Suppose that there exists a homeomorphism $\Phi : N \longrightarrow N$ such that $\Phi|\Sigma = T_{g(L)}$. Then, $g(L)$ bounds an embedded disc in N and M_L is reducible.*

Proof: Theorem 2.5 implies that $g(L)$ must bound a disc, say D', in N. Recall that in the Heegaard splitting of (5.1) $g(L)$ and L are identified and that L bounds a disc D in N. Now the 2-sphere $D \cup D'$ intersects the Heegaard surface of (5.1) at L and the knot K at exactly 2-points. This sphere realizes K as a (possibly trivial) connect sum and it contains L. It follows that L bounds an embedded disc in the complement of K and M_L reducible. □

Lemma 5.7. *If $\overline{N \setminus \eta(g(L))}$ is reducible, then $M_L := \overline{S^3 \setminus \eta(K) \cup \eta(L)}$ is reducible.*

Proof: Since $\overline{N \setminus \eta(g(L))}$ is reducible, $g(L)$ must lie in a 3-ball in N. Hence $g(L)$ is homotopically trivial in N. By Dehn's Lemma it must bound an embedded disc D' in N. Recall that in the Heegaard splitting of (5.1) $g(L)$ and L are identified and that L bounds a disc D in N. Now the 2-sphere $D \cup D'$ intersects the Heegaard surface of (5.1) at L and the knot K at exactly 2-points. We can push these two points away from L in the interior of D. It follows that L bounds an embedded disc in the complement of K and, thus, clearly M_L is reducible. $\square$

In the view of Lemma 5.7 and our earlier assumption that M_L is irreducible we may assume that $\overline{N \setminus \eta(g(L))}$ is irreducible.

For $i = 0, 1$, let $S_i := S \times \{i\}$. The boundary ∂N is the union $S_0 \cup E \cup S_1$, where $E = \partial S \times (0,\ 1)$. Let Σ_0, Σ_1 denote the image of S_0, S_1, respectively, under the Dehn twist $T^{-q}_{g(L)}$. Then, for $i = 0, 1$, $\partial\Sigma_i = \gamma' \times \{i\}$.

Lemma 5.8. *The surfaces Σ_0, Σ_1 are incompressible in N.*

Proof: Suppose, on the contrary that one of Σ_0, Σ_1, say Σ_0 compresses in N. Consider N as a product $S \times I$ with $g(L)$ a knot in N. By assumption Σ_0 compresses in N. Performing the Dehn twist $T^{-q}_{g(L)}$ is equivalent to doing surgery along $g(L)$. Since $q \neq 0$, this surgery is non-trivial (Corollary 2.4). Now Σ_0 is the result of S_0 under this surgery. Thus there is a non-trivial surgery in $S \times I$ such that the surface S_0 compresses in the manifold obtained after surgery. As shown by Scharlemann and Thompson (Theorem 1 of [18]) or by Y. Ni [16], then there is a simple closed homotopically essential curve $L' \subset \Sigma_0$ such that $g(L)$ and L' cobound an embedded annulus in in $N = S \times I$. Furthermore, this annulus determines the slope of the surgery. This implies that $g(L)$ bounds a disc in N. But then, any Dehn twist on ∂N along $g(L)$ extends to a Dehn twist on N; a homeomorphism of N. Since S_0, S_1 are incompressible, their images under any homeomorphism of N are also incompressible in N. This contradicts the assumption that Σ_0 compresses. $\square$

Lemma 5.9. *With the notation and the setting as above, there exists a fibration of M', with fiber S' and corresponding HN-model (Σ, g_1), and an orientation preserving homeomorphism $f : \Sigma \longrightarrow \Sigma$ such that in $\Gamma(\Sigma)$ we have*

$$g'' := gT_L^{-q} = f g_1 f^{-1}. \tag{5.6}$$

Proof: We will use Q' to denote the HN-surface of the HN-splitting in (5.4). We also recall that the Heegaard splitting in (5.5) is the result of the splitting of (5.1) after the Dehn twist τ^{-q} on N. This twist changes the product structure of N from $S \times I$ to $S' \times I$. Since the HN-surface of $H \cup_g (-N)$ is disjoint from a level surface of the fibration $S \times J/h$, Q'

is disjoint from a neighborhood of a copy $S' \subset \operatorname{int}(H)$. Let S'_1 denote a second copy of S' that lies in that neighborhood in $\operatorname{int}(H)$. By Proposition 3.1, M' cut along S' is a product $S' \times J$. Equivalently we have a fibration $M' = S' \times J/h_1$, with monodromy $h_1 : S' \longrightarrow S'$. We will denote by Q the HN-surface corresponding to this fibration. The monodromy gives rise to an HN-model (Σ, g_1). We will apply Lemma 4.8 to (Σ, g_1), $(\Sigma,\ g'')$: By Proposition 3.1, S'_1 can be taken to be a level surface of the fibration $M' = S' \times J/h_1$. Since S'_1 was chosen so it is disjoint from Q, Q', condition (1) of Lemma 4.8 is satisfied. Condition (2) is easy to satisfy from the definitions of models in Section 4. Let R' denote M' cut along S'_1; we have $R' = S'_1 \times J$. Note that R' is also the quotient space

$$W' \sqcup (-N)/\{y \sim ig''(y),\ \mid y \in \Sigma\}, \tag{5.7}$$

where W' is H cut along S'_1. The boundary $\partial W'$ has two components one of which is Σ. Since S' is a minimum genus Seifert surface for K'; $\Sigma \setminus K'$ is incompressible in W'. By Lemma 5.8, $\Sigma \setminus g(K')$ is incompressible in N. We conclude that $Q' = \Sigma_0 \cup E' \cup \Sigma_1$, where

(i) Σ_0, Σ_1 are incompressible;

(ii) $E' = g(K') \times [0,\ 1]$ is an annulus such that for every $0 \leq t \leq 1$, $g(K') \times \{t\}$ and a curve $\partial S' \times \{j\}$ on the second component of $\partial W'$, cobound an annulus E_t in W'.

We consider the surfaces $\Sigma'_i := E_i \cup \Sigma_i$, $i = 0, 1$. They are properly embedded and incompressible in R'. They can be isotopied in R' so that each becomes parallel to S'_1 and the fiber S' (Proposition 3.1 of [20]). There is a homeomorphism of M' that takes Σ'_1 to the fiber S'; in particular M' cut along Σ'_1 is a product $\Sigma'_1 \times J$. We can now find a a level preserving homeomorphism $F : M' \longrightarrow M'$ such that and F maps Q to Q' and so that $F(S'_1) = S'_1$. Thus Lemma 4.8 applies to give the desired conclusion. □

5.5. **Proof of Theorem 5.2.** Let K, K' be fibered isotopic knots, such that K' is obtained from K by a generalized crossing change, of order $q \neq 0$, supported on a crossing circle L. Let D be a crossing disc with $L := \partial D$. We will consider the Heegaard splittings of (5.1) and (5.5) so that the crossing circle L is embedded on Σ and D is a meridian disc of N. Recall that the crossing change from K to K' is now achieved by the Dehn twist τ^{-q} of N along D.

We will assume that L is homotopically essential on Σ since otherwise the crossing change from K to K' is obviously nugatory.

If $M_L := \overline{S^3 \setminus \eta(K) \cup \eta(L)}$ is reducible, then we are done by Lemma 5.3. We will assume that $M_L := \overline{S^3 \setminus \eta(K) \cup \eta(L)}$ is irreducible. Then, by Lemma 5.7 we may assume that $\overline{N \setminus \eta(g(L))}$ is irreducible. By Proposition 5.4, S and S' are of minimum genus for K and K', respectively. By Lemmas 5.8, and 5.9, there is an HN-model (Σ, g_1) that corresponds to a fibration $M' = S' \times J/h_1$ and $f : \Sigma \longrightarrow \Sigma$ so that (5.6) is satisfied. Equivalently, we have $f^{-1} g T_L^{-q} f = g_1$. Since K and K' are isotopic knots there is an orientation preserving homeomorphism, say Φ, of S^3 that brings K to K'. (Note that the restriction of Φ on H cannot agree with τ; so, in particular, $\Phi|\partial H \neq T_L^{-q}$. For, otherwise, if we let $\Phi_1 := \Phi|\partial(-N)$ we must have $gT_L^{-q} = \Phi_1 g$ or equivalently $gT_L^{-q}g^{-1} = \Phi_1$. Since $T_{g(L)}^{-q} = gT_L^{-q}g^{-1}$, we have $T_{g(L)}^{-q} = \Phi_1$. Since Φ_1 extends on N we obtain that $T_{g(L)}^{-q}$ does. But, by Lemma 5.6, this contadicts the assumption that M_L is irreducible).

Now we have two equivalent fibered knot complements; $M' = S' \times J/h_1$ and $M = S \times J/h$. Via Proposition 3.1, Φ gives rise a homeomorphism $\phi : \Sigma \longrightarrow \Sigma$ such that

$$gT_L^{-q} = \phi g \phi^{-1} \quad \text{or} \quad T_L^{-q} = g^{-1}\phi g \phi^{-1}. \tag{5.8}$$

Now (5.8) realizes T_L^{-q} as a commutator of length one in Γ. By Corollary 2.2, L must be homotopically trivial on Σ which contradicts the assumption that M_L is irreducible. ☐

Since Kotschick's result is not true in the case of twists with mixed signs, the argument above breaks down in an attempt to generalize the statement of Theorem 5.2. to multiple crossing changes. But as the following example shows the result is , in fact, not true!

Example 5.10. Let K denote the figure eight knot as boundary of a genus one Seifert surface S obtained by Hopf pluming two once twisted bands B_L and B_R. Consider D_1, D_2 crossing discs of K such that $D_1 \cap B_L$ (resp. $D_2 \cap B_R$) is an essential arc cutting B_L (resp. B_R) into a square. One can perform opposite sign twists of order four along D_1, D_2 to transform S to S' where in S' the Hopf band B_L becomes the Hopf band B_R and vice versa. The knot $K' := \partial S'$ is isotopic to K. Moreover, S and S' are clearly minimum genus Seifert surfaces for K and K', respectively. However, one can check that neither of $L_1 := \partial D_1$ or $L_2 := \partial D_2$ bounds disc in the complement of K.

6. Adjacency to fibered knots

We begin by recalling from [9] the following definition.

Definition 6.1. *Let K, K' be knots. We will say that K is n-adjacent to K', for some $n \in \mathbf{N}$, if K admits a projection containing n generalized crossings such that changing any $0 < m \leq n$ of them yields a projection of K'.*

In [9] we showed the following: Given knots K and K' there exists a constant $c = c(K, K')$ such that if K is n-adjacent to K' then either $n \leq c$ or K is isotopic to K'. Here, using Theorem 5.2, we will show that if K' is assumed to be fibered, then we can have a much stronger result.

Theorem 6.2. *Suppose that K' is a fibered knot and that K is a knot such that K is n-adjacent to K', for some $n > 1$. Then, either K is isotopic to K' or we have $g(K) > g(K')$.*

Remark 6.3. It is not hard to see that if K is n-adjacent to K', for some $n > 1$, then K is m-adjacent to K', for all $0 < m \leq n$.

Suppose that K is n-adjacent to K' and let L be a collection of n crossing circles supporting the set of generalized crossings that exhibit K as n-adjacent to K'. Since the linking number of K and every component of L is zero, K bounds a Seifert surface S in the complement of L. Define

$$g_n^L(K) := \min\{\,\text{genus(S)}\,|\,S\,\text{a Seifert surface of}\,K\,\text{as above}\,\}.$$

We recall the following.

Theorem 6.4. [Theorem 3.1, [9]] *We have*

$$g_n^L(K) = \max\{\,g(K), g(K')\,\}$$

where $g(K)$ and $g(K')$ denotes the genera of K and K', respectively.

Proof: [Proof of Theorem 6.2] Let K' be a fibered knot. In the view of Remark 6.3, it is enough to prove that if K is a knot that is 2-adjacent to K' then either K is isotopic to K' or we have $g(K) > g(K')$. To that end, suppose that K is exhibited as 2-adjacent to K' by a two component crossing link $L := L_1 \cup L_2$. Let D_1, D_2 be crossing discs for L_1, L_2, respectively. Suppose, moreover, that $g(K) \leq g(K')$; otherwise there is nothing to prove. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L. As explained earlier in the paper, we can isotope S so that, for $i = 1, 2$, $S \cap \text{int}(D_i)$ is an arc, say α_i that is properly embedded in S. For $i = 1, 2$, let K_i (resp. S_i) denote the knot (resp. the Seifert surface) obtained from K (resp. S) by changing C_i. Also let K_3 denote the knot obtained by changing C_1 and C_2 simultaneously and let S_3 denote the corresponding surface. By assumption, for $i = 1, 2, 3$, K_i is isotopic to K' and S_i is a Seifert surface for K_i. Since $g(K) \leq g(K')$, Theorem 6.4 implies that S_i is a minimum genus surface for K_i. Observe that K_3 is obtained from K_1 by changing C_2 and that they are fibered isotopic knots. Furthermore, S_3 is obtained from S_1 by twisting along the arc $\alpha_2 \subset S$. By Theorem 5.2, L_2 bounds an embedded disc Δ_2 in the complement of K_1. Since S_3 is incompressible, after an isotopy, we can assume that $\Delta \cap S_3 = \emptyset$. Now let us consider the 2-sphere

$$S^2 := \Delta \cup D_2.$$

By assumption $S^2 \cap S_3$ consists of the arc $\alpha_2 \subset S_3$. Since α_1 and α_2 are disjoint, the arc α_1 is disjoint from S^2. But since K is obtained from K_1 by twisting along α_1, the circle L_2 still bounds an embedded disc in the complement of K. Hence, K is isotopic to K'. □

Remark 6.5. The trefoil knot is 2-adjacent to the unknot. Since the trefoil is a fibered knot Theorem 6.2 implies that the unknot is not 2-adjacent to the trefoil. Thus n-adjacency is not an equivalence relation on the set of knots.

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