RESEARCH STATEMENT

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While I am interested in harmonic analysis in Euclidean spaces, most of my current work has been carried out mainly in the finite field settings. It seems that the classical tools pertaining just to harmonic analysis are not enough to settle down some of the deepest open problems in the field. In order to find a good source for new approaches to these problems, analysts have recently studied the finite field analogues of the problems in classical analysis. Finite fields have a particular structure which may be useful to find some connection between our analysis problems with the problems in the well-studied other fields such as combinatorics, number theory, or discrete geometry. This is the main reason why people deserve to study the finite field analogues of the problems in Euclidean spaces, because we may learn how to apply useful techniques from other fields to attack our analysis problems. Moreover, finite field problems exhibit some interesting, new features forced upon the problems by number theoretic issues.

For last few years, I has investigated the boundedness of extension operators and the Erdös-Falconer type distance problems in finite fields. In [16], Mockenhaupt and Tao first addressed and studied the extension problems in finite fields and I was deeply impressed by their work. I am also interested in studying the Erdös-Falconer type distance problems, mainly because these problems are closely related to extension problems. After the authors' work in [1], Iosevich and Rudnev ([10]) established the finite analogues of the Erdös-Falconer type distance problems. Working with my coworkers, I have recently found new interesting results for aforementioned problems. The main purpose of this article is to introduce what I have done. Furthermore, I hope that many mathematicians become interested in these new projects and they shed insight on these problems.

1. Extension problems in the finite field setting

Let V be a subset of $\mathbb{R}^d$, $d \geq 2$, and $d\sigma$ a positive measure supported on V. Then, one may ask that for which values of p and r does the estimate

$$\|\widehat{fd\sigma}\|_{L^r(\mathbb{R}^d)} \leq C_{p,r,d}\, \|f\|_{L^p(V,d\sigma)} \quad \text{for all} \quad f \in L^p(V, d\sigma)$$

hold? This problem is known as the extension problem in the Euclidean spaces. When the set V is the paraboloid, sphere, or cone, the extension problem has received much attention, but the problem is still open in higher dimensions. See, for example, [18],[19], [20], [4], [22], and [17] and the references contained therein. Mockenhaupt and Tao ([16]) first addressed and studied the finite field analogue of the extension problem. Let us review what is the extension problem in the finite field setting. We denote by $(\mathbb{F}_q^d, dx)$ a d-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements, where we endow the space with a normalized counting measure dx. On the other hand, we denote by $(\mathbb{F}_q^d, dm)$ the dual space with a counting measure dm. Let $(V, d\sigma)$ be an algebraic variety in $(\mathbb{F}_q^d, dx)$, where we endow the variety V with a normalized surface measure $d\sigma$. Namely, we have the following formula

$$d\sigma(x) = \frac{|\mathbb{F}_q^d|}{|V|}\chi_V(x),$$

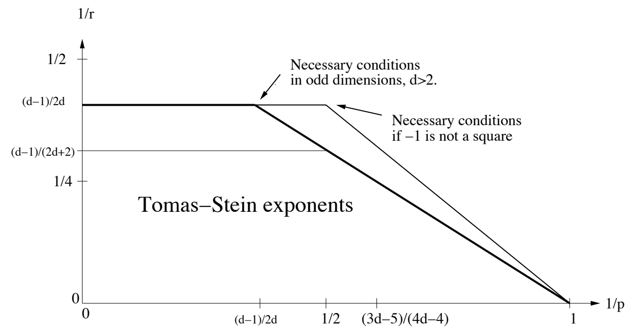

FIGURE 1. In Odd Dimensions $d \geq 3$, the Necessary Conditions for Spheres or Paraboloids

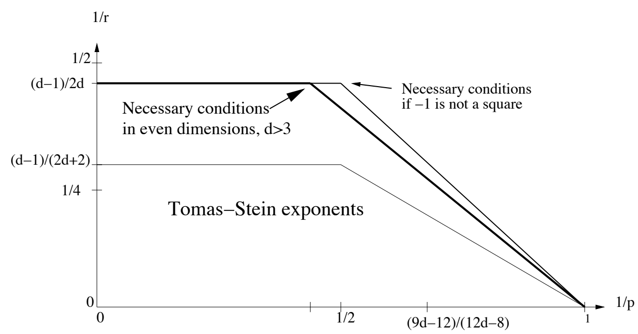

FIGURE 2. In Even Dimensions $d \geq 4$, the Necessary Conditions for Spheres or Paraboloids

where $|V|$ denotes a cardinality of $V \subset \mathbb{F}_q^d$, and χ_V is the usual characteristic function on V. For each $1 \leq p, r \leq \infty$ we define $R^*(p \to r)$ to be the smallest constant such that the extension estimate

$$\|\widehat{f d\sigma}\|_{L^r(\mathbb{F}_q^d, dm)} \leq R^*(p \to r)\|f\|_{L^p(V, d\sigma)}$$

holds for all functions f on V. Notice that $R^*(p \to r)$ is always a finite number and it may depend on the underlying finite field $\mathbb{F}_q$. The extension problem in finite fields is to determine possible exponents, $1 \leq p, r \leq \infty$ such that

$$R^*(p \to r) \leq C_{p,r,d} < \infty,$$

where the main point is that the constant $C_{p,r,d}$ is independent of the size of the underlying finite field $\mathbb{F}_q$. For more information about the definition of the extension theorem, see [9] or [16]. In [16], Mockenhaupt and Tao developed useful tools to study the extension problem in finite fields for various algebraic varieties V, but their work was mostly restricted to cones and paraboloids. In two and three dimensions, they obtained reasonably good results but it was not clear whether such good results can be obtained in higher dimensions. In last few years, I has made an effort to develop aforementioned authors' work. Before stating such results for the extension problems, it would be helpful to remark that the extension problem for the finite field case is analogous in many respects to its Euclidean case, but it also presents unique and independently interesting features. For example, the necessary conditions for $R^*(p \to r) \lesssim 1$ may depend on whether -1 is a square number or not in the underlying finite field $\mathbb{F}_q$. Moreover, the extension estimates in even dimensions can be better than those in odd dimensions. In fact, if $V \subset \mathbb{F}_q^d, d \geq 2$ is a sphere or a paraboloid, then it turns out that the necessary conditions for $R^*(p \to r) \lesssim 1$ are given as in Figure 1 and Figure 2.

1.1. **Complete solution for extension problems in two dimension for arbitrary algebraic curves.** In [16], Mockenhaupt and Tao proved that $L^2 - L^4$ estimate implies the complete answer to the extension problem for the parabola in two dimension. It was also shown in [7] that the same result holds for the nondegenerate quadratic curves in two dimensional vector spaces over finite fields. In [12], Shen and I recently found the sufficient and necessary conditions for $L^2 - L^4$ extension estimate related to arbitrary algebraic curves. More precisely, we proved the following theorem.

Theorem 1.1. *Suppose that $P(x) \in \mathbb{F}_q[x_1, x_2]$ is non-zero polynomial. Define an algebraic variety $V \subset \mathbb{F}_q^2$ by*

$$V = \{x \in \mathbb{F}_q^2 : P(x) = 0\}.$$

Then, $R^*(2 \to 4) \lesssim 1$ if and only if $|V| \sim q$ and the polynomial $P(x)$ does not have any linear factor.

Here and throughout the paper, $|V| \sim q$ means that there exist $C, c > 0$ depending only on the degree of the polynomial $P(x)$ such that $cq \leq |V| \leq Cq$. We also notice that the norm of the extension operator depends only on the degree of V and on the ratio $|V|/q$. In addition, we assume that the characteristic of the underlying finite field $\mathbb{F}_q$ is greater than the degree of V.

Remark 1.2. Our Theorem 1.1 presents an interesting fact that there exist some differences between the finite field case and the Euclidean case. For example, let us consider a set V consisting of all zeros of $x_1^4 + x_2^4 - 1 = 0$. In the Euclidean case, the extension estimate for this variety V is much worse than that for the circle variety, $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, because V is a curve with a vanishing Gaussian curvature. However, Theorem 1.1 says that the circle and the variety V yield the same extension estimate in finite fields. Another difference is that the curve in the finite field case yields much better extension estimate than its counterpart of the Euclidean case. For instance, the $L^4 - L^4$ estimate gives the critical exponents up to the endpoints for the circular extension estimate in $\mathbb{R}^2$ (see [19]). However, this best possible result is much worse than the $L^2 - L^4$ extension estimate which yields sharp exponents in finite fields case.

1.2. **Generalized conical extension problems.** In the finite field setting, the cone $C \subset \mathbb{F}_q^d$ is defined as the set

$$C = \{x : x_{d-1}x_d = x_1^2 + \cdots + x_{d-2}^2\}.$$

Based on a geometric observation, Mockenhaupt and Tao ([16]) gave the complete answer to the conical extension problems in three dimension. In fact, they proved that $L^2 - L^4$ estimate implies the complete solution for the conical extension problems in three dimensional vector spaces over finite fields. In order to generalize the conical extension problems, Shen and I ([13]) considered the homogeneous varieties $H \subset \mathbb{F}_q^3$ in three dimension:

$$H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$$

where $P(rx) = r^s P(x)$ for all $r \in \mathbb{F}_q$ and for any fixed integer $s \geq 2$. We made a new observation that the variety H such as a cone in $\mathbb{F}_q^3$ yields a good Fourier decay. This is a different phenomenon between the finite field and the Euclidean case. As a consequence, we obtained a generalized result on the conical extension problems in three dimension.

Theorem 1.3. *For each homogeneous polynomial $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$, let $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$. Suppose that $|H| \sim q^2$ and the homogeneous variety H does not contain any plane passing through the origin. Then, we have the following extension estimate on H:*

$$R^*(2 \to 4) \lesssim 1.$$

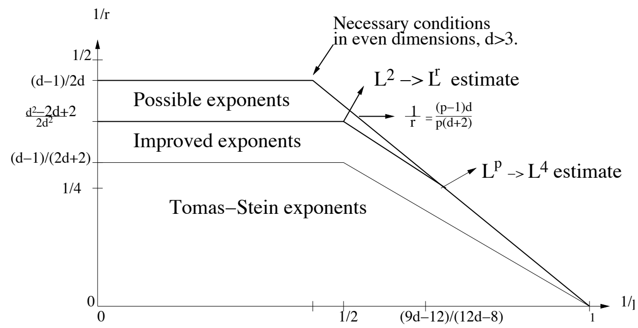

FIGURE 3. In Even Dimensions $d \geq 4$, the improved extension estimates for Paraboloids

In [11], I also observed that if the dimension $d \geq 3$ is odd and the variety H is a cone in $\mathbb{F}_q^d$, we have the Tomas-Stein exponents, that is $R^*(2 \to 2(d+1)/(d-1)) \lesssim 1$. On the other hand, the dimension $d \geq 2$ is even, the Tomas-Stein exponents could not be obtained in general.

1.3. **Results on extension problems for paraboloids in finite fields.** In the finite field setting, the paraboloid $P \subset \mathbb{F}_q^d$ is defined by

$$P = \{x \in \mathbb{F}_q^d : x' \cdot x' = x_d\}, \tag{1.1}$$

where $x = (x', x_d)$ and $x' \cdot x'$ is the usual dot product. Estimating the Fourier transform of the surface measure on the paraboloid $P \subset \mathbb{F}_q^d$, Mockenhaupt and Tao ([16]) showed that the Tomas-Stein inequality

$$\|\widehat{fd\sigma}\|_{L^{2(d+1)/(d-1)}(\mathbb{F}_q^d, dm)} \lesssim \|f\|_{L^2(P, d\sigma)}$$

holds. In the case when the dimension $d \geq 4$ is even, Iosevich and I ([9]) improved on the Tomas-Stein inequality. More precisely, we proved the following (see Figure 3).

Theorem 1.4. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as in (1.1). If $d \geq 4$ is even, then we have*

$$R^*(p \to 4) \lessapprox 1 \quad \textit{for} \quad p = \frac{4d}{3d-2}.$$

and

$$R^*(2 \to r) \lessapprox 1 \quad \textit{for} \quad r = \frac{2d^2}{d^2 - 2d + 2}.$$

In even dimensions, if we consider arbitrary finite fields $\mathbb{F}_q$, the $L^p - L^4$ result in Theorem 1.4 is sharp up to logarithmic factors as is shown in Figure 2 and 3. However, if we just consider the finite fields $\mathbb{F}_q$ such that $-1 \in \mathbb{F}_q$ is not a square number, then it might be possible to improve the $L^p - L^4$ result in Theorem 1.4. Here, the main point is that if the dimension d is even, we can always improve the Tomas-Stine exponents under any assumption on the underlying finite field $\mathbb{F}_q$. An interesting thing happens if the dimension $d \geq 3$ is odd. In this case, it is known that we can improve the Tomas-Stein exponents only if we have a restriction that $-1 \in \mathbb{F}_q$ is not a square. For example, when $d = 3$ and -1 is not a square number in the underlying finite fields $\mathbb{F}_q$, Mockenhaupt and Tao in [16] improved the Tomas-Stein exponents by showing that $R^*(8/5 \to 4) \lessapprox 1$ and $R^*(2 \to 18/5) \lessapprox 1$. In [2], the logarithmic factor was eliminated by Bennett, Carbery, Garrigós, and Wright. In [9], Iosevich and I extended their result to higher specific odd dimensions. Namely, we obtained the following.

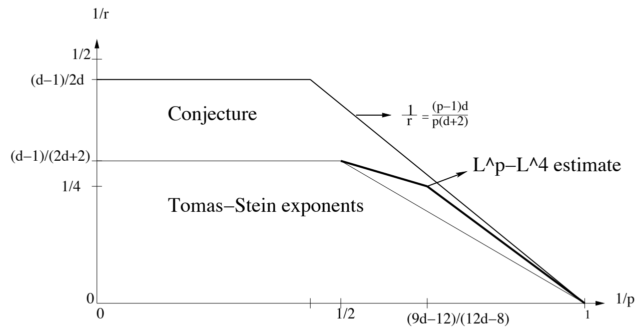

FIGURE 4. In Even Dimensions $d \geq 4$, the improved $L^p - L^4$ extension estimates for Spheres

Theorem 1.5. *Let P be the paraboloid in $\mathbb{F}_q^d$ defined as before. If -1 is not a square number in $\mathbb{F}_q$ and $d = 4k + 3$ for some $k \in \mathbb{N}$, then the conclusions in Theorem 1.4 hold.*

Remark 1.6. Working in the bilinear setting, the authors in [15] recently improved our results (Theorem 1.4 and 1.5) by showing that the endpoint estimates hold.

1.4. **Results on spherical extension problems in the finite field setting.** In the finite field setting, the extension problems for spheres is more difficult than those of paraboloids, because the Gauss sums, used to study the Fourier transform of the discrete paraboloid are no longer adequate for the task. We define the sphere as the set

$$S_j = \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_d^2 = j \neq 0\}. \tag{1.2}$$

Working with Iosevich, we first studied the extension problem for spheres. In [7], we observed that the estimate of the Fourier decay on spheres is closely related to the classical Kloosterman sums. As a consequence, we proved that the Tomas-Stein inequality holds for spherical extension problem in all dimensions. In even dimension $d \geq 4$, an improved $L^p - L^4$ estimate was also achieved in [8] by showing that $R^*(\frac{12d-8}{9d-12} \to 4) \lessapprox 1$. The proof is based on an elaborate effort to estimate several kinds of incidence theorems. See Figure 4 for the improved $L^p - L^4$ estimate.

2. Erdös-Falconer distance problem in finite fields

The Erdös-Falconer distance problem, in a generalized sense, is a question of how many distances are determined by a set of points. Let $E \subset \mathbb{F}_q^d$, $d \geq 2$, the d-dimensional vector space over the finite field $\mathbb{F}_q$ whose characteristic is greater than two. For each $x \in \mathbb{F}_q^d$ and n a positive integer ≥ 2, we define $\|x\|_n = x_1^n + \ldots + x_d^n$. We consider a distance set $\Delta_n(E) = \{\|x - y\|_n : x, y \in E\}$, viewed as a subset of $\mathbb{F}_q$. The Erdös-Falconer distance problem in this context asks for the smallest number s_0 such that the distance set $\Delta_n(E)$ contains a positive proportion of the elements of $\mathbb{F}_q$ provided that $|E| \geq Cq^{s_0}$. Iosevich and Rudnev ([10]) conjectured that if $|E| \geq Cq^{d/2}$ with C sufficiently large, then $|\Delta_2(E)| \gtrapprox q$. In addition, they proved that if $|E| \geq 2q^{(d+1)/2}$, then $|\Delta_2(E)| = q$. However, it turned out that the conjecture is not true if the dimension d is odd. In [5], we constructed arithmetic examples which show that the exponent $(d + 1)/2$ is sharp. However, if the dimension d is even, then it may be possible that the conjecture is true. As an evidence of the belief, we proved the following theorem in [3].

Theorem 2.1. *Let $E \subset \mathbb{F}_q^2$ of cardinality $\geq Cq^{4/3}$, with C sufficiently large. Then we have*

$$|\Delta_2(E)| \gtrsim q.$$

If the dimension d is two, then the exponent $4/3$ in Theorem 2.1 is better than the exponent $(d+1)/2$ which gives a sharp exponent in odd dimensions. Theorem 2.1 was obtained by applying our sharp result on the extension problem for the circle. The Erdös-Falconer distance problem on spheres was also studied by me along with D. Hart, A. Iosevich, and M. Rudnev. We showed in [5] that if E is a subset of the sphere S_j, then we always get a positive proportion of all the distances if $|E| \geq Cq^{d/2}$ with C sufficiently large.

2.1. **The generalized Erdös-Falconer distance problem in finite fields.** The generalized distances can be defined in terms of polynomials. Given a polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ and $E, F \subset \mathbb{F}_q^d$, one may define a generalized distance set $\Delta_P(E, F)$ by the set

$$\Delta_P(E, F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}.$$

In [14], Shen and I proved the following theorem which generalizes the spherical distance problems in [10] and cubic distance problems in [6].

Theorem 2.2. *Let $P(x) = \sum_{j=1}^d a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ for $s \geq 2$ integer and $a_j \neq 0$. Suppose that the characteristic of $\mathbb{F}_q$ is sufficiently large. If $|E||F| \geq Cq^{d+1}$ for $E, F \subset \mathbb{F}_q^d$, then $\mathbb{F}_q \setminus \{0\} \subset \Delta_P(E, F)$, where $C > 0$ is a sufficiently large constant.*

In addition, we also set up and study the generalized pinned distance problems in finite fields. In [14], we obtain the following theorem which sharpens and generalizes the Vu's result in [21].

Theorem 2.3. *Let $P(x) \in \mathbb{F}_q[x_1, x_2]$ be a non-degenerate polynomial. If $|E||F| \geq Cq^3$ for $E, F \subset \mathbb{F}_q^2$ and $C > 0$ sufficiently large, then there exists a subset F_0 of F with $|F_0| \sim |F|$ such that*

$$|\Delta_P(E, y)| := |\{P(x-y) \in \mathbb{F}_q : x \in E\}| \gtrsim q \quad \textit{for all } y \in F_0.$$

3. Plan of research

3.1. **Improving our results on the Erdös-Falconer distance problem in even dimensions.** Iosevich and Rudnev ([10]) showed that if $E \subset \mathbb{F}_q^d$ has a cardinality $\geq Cq^{(d+1)/2}$ with C sufficiently large, then $|\Delta_2(E)| \gtrsim q$. If the dimension d is odd, then this result is sharp, which was proved by the authors in [5]. However, one may improve the result in the case when d is even. I propose to improve the Erdös-Falconer distance problem in even dimensional vector spaces over finite fields via the extension theorems for spheres. Let us observe the connection between the extension problem and the Erdös-Falconer distance problem in finite fields. One can obtain the following formula for the Erdös-Falconer distance problem in the finite field setting.

Theorem 3.1. *Let $E \subset \mathbb{F}_q^d, d \geq 2$. Suppose that $|E| \geq Cq^{\frac{d}{2}}$ with C sufficiently large. Then*

$$|\Delta_2(E)| \gtrsim \min\left\{q, \frac{q}{\mathbb{M}_E(q)}\right\},$$

where $\mathbb{M}_E(q)$ is given by the formula

$$\mathbb{M}_E(q) = \frac{q^{3d+1}}{|E|^4} \sum_{t \in \mathbb{F}_q^*} \sigma_E^2(t) \quad \textit{with} \quad \sigma_E(t) = \sum_{m \in S_t} |\widehat{E}(m)|^2.$$

From Theorem 3.1, we see that the Erdös-Falconer distance problem is directly related to estimating the good upper bound for $\sigma_E(t)$. However, using the duality argument, the upper bound of $\sigma_E(t)$ can be given by $L^2 - L^r$ spherical extension estimates. In conclusion, the extension estimate is a key to improve the results on distance problems.

3.2. **Finding some connection between spherical extension problems and conical extension problems.** In the Euclidean case, the compact subset of a cone in $\mathbb{R}^{d+1}$ and the sphere in $\mathbb{R}^d$ have the same conjectured exponents for extension problems (see [19]). The conjecture for the cone in four dimension was solved by Wolff in [22], but the problem for sphere in three dimension is still open. It might be interesting if we could show that the extension estimate for a cone in $d+1$ dimension is essentially same as that for a sphere in d dimension. This looks true as is shown from the conjectures for the conical and spherical extension problems. I has recently tried to find some connections between the conical extension problem and the spherical extension problem in finite fields. I hope that such connections exist in finite field case and we apply such connections to the Euclidean case.

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