# Summary of Research Papers

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Abstract. In this article, the core results of the author's papers are summarized. The author's work has been carried out mainly on the finite field analogues of classical problems in harmonic analysis, geometric measure theory, and combinatorics. In particular, the author has investigated the boundedness of extension operators and the Erdős-Falconer type distance problems in the finite field setting. The author has also studied the connection between these two topics.

## 1. Recent developments in extension problems for finite fields

Let $V$ be a subset of $\mathbb{R}^d, d \geq 2$, and $d\sigma$ a positive measure supported on $V$. Then, one may ask that for which values of $p$ and $r$ does the estimate

$$\|\widehat{fd\sigma}\|_{L^r(\mathbb{R}^d)} \leq C_{p,r,d}\|f\|_{L^p(V,d\sigma)} \quad \text{for all} \quad f \in L^p(V, d\sigma)$$

hold? This problem is known as the extension theorem which has not been completely solved in higher dimensions. Mockenhaupt and Tao ([8]) first addressed and studied the finite field analogue of the extension problem. Let us review what is the extension problem in the finite field setting. We denote by $(\mathbb{F}_q^d, dx)$ a $d$-dimensional vector space over the finite field $\mathbb{F}_q$, where we endow the space with a normalized counting measure $dx$. On the other hand, $(\mathbb{F}_q^d, dm)$ denotes the dual space with a counting measure $dm$. Let $(V, d\sigma)$ be an algebraic variety in $(\mathbb{F}_q^d, dx)$, where we endow the variety $V$ with a normalized surface measure $d\sigma$. Namely, we have the following formula

$$d\sigma(x) = \frac{|\mathbb{F}_q^d|}{|V|} V(x),$$

where $|V|$ denotes a cardinality of $V \subset \mathbb{F}_q^d$ and we identify the set $V$ with the characteristic function $\chi_V$ on $V$. For each $1 \leq p, r \leq \infty$ we define $R^*(p \to r)$ to be the smallest constant such that the extension estimate

$$\|\widehat{fd\sigma}\|_{L^r(\mathbb{F}_q^d, dm)} \leq R^*(p \to r)\|f\|_{L^p(V,d\sigma)}$$

holds for all functions $f$ on $V$. Notice that $R^*(p \to r)$ is always a finite number and it may depend on the underlying finite field $\mathbb{F}_q$. The extension problem in finite fields is to determine possibly small exponents, $1 \leq p, r \leq \infty$ such that

$$R^*(p \to r) \leq C_{p,r,d} < \infty,$$

where the main point is that the constant $C_{p,r,d}$ is independent of the size of the underlying finite field $\mathbb{F}_q$. In [8], Mockenhaupt and Tao developed useful tools to study the extension problem in finite fields for various algebraic varieties $V$, but their work was mostly restricted

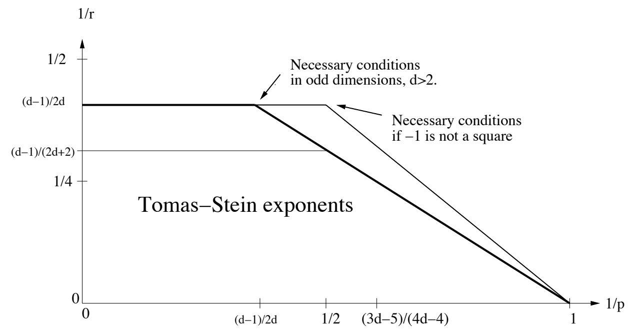


FIGURE 1. In Odd Dimensions $d \geq 3$, the Necessary Conditions for Spheres or Paraboloids

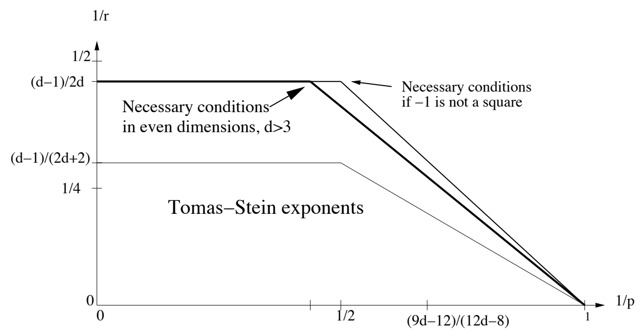


FIGURE 2. In Even Dimensions $d \geq 4$, the Necessary Conditions for Spheres or Paraboloids

to cones and paraboloids. In two and three dimensions, they obtained reasonably good results but it was not clear whether such good results can be obtained in higher dimensions. In last few years, the author has made an effort to develop aforementioned authors' work. Before summarizing the author's main results for the extension problems, it would be helpful to remark that the extension problem for the finite field case has analogous in many respects to its Euclidean case, but it exhibits some interesting new features forced upon the problem by number theoretic issues. For example, the necessary conditions for $R^*(p \to r) \lesssim 1$ can be determined depending on whether $-1$ is a square number or not in the underlying finite field $\mathbb{F}_q$. Moreover, the extension estimates in even dimensions can be better than those in odd dimensions. In fact, if $V \subset \mathbb{F}_q^d, d \geq 2$ is a sphere or a paraboloid, then it turns out that the necessary conditions for $R^*(p \to r) \lesssim 1$ are given as in Figure 1 and Figure 2.

In the following subsections, we state the titles of the author's papers for extension problems and give a brief summary of main results in the papers.

**1.1. A.Iosevich and D.Koh, Extension theorems for the Fourier transform associated with nondegenerate quadratic surfaces in vector spaces over finite fields, Illinois J. Math. 52 (2008), no. 2, 611–628.** In this paper, the authors initially study the extension theorems related to the nondegenerate quadratic surfaces $S_j$ which is given by

$$S_j = \{x \in \mathbb{F}_q^d : Q(x_1, \cdots, x_d) = j\}$$

where $j$ is not zero and $Q(x)$ is a nondegenerate quadratic form over $\mathbb{F}_q$. In dimension two, the authors completely settle down the extension problems for nondegenerate quadratic curves by showing that $R^*(2 \to 4) \lesssim 1$. In higher dimensions, $d \geq 3$, we obtain the standard

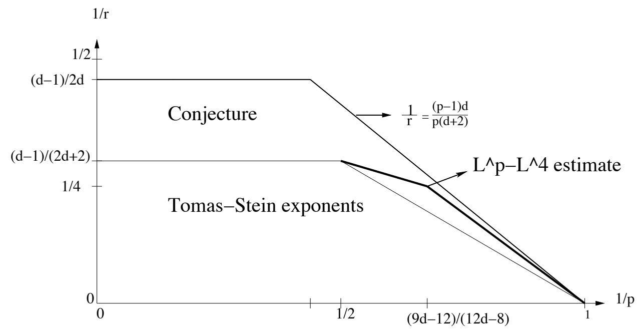


FIGURE 3. In Even Dimensions $d \geq 4$, the improved $L^p - L^4$ extension estimates for Spheres

Tomas-Stein exponents, that is $R^*(2 \to 2(d+1)/(d-1)) \lesssim 1$. As usual, this result follows from the sharp decay estimate of Fourier transform on nondegenerate quadratic surfaces. In order to obtain the sharp decay, authors shows that the estimate of the Fourier decay is closely related to the classical Kloosterman sums. Moreover, the authors study the weak-type extension estimates and give a partial evidence for the conjecture on the extension problems.

**1.2. A.Iosevich and D.Koh, Extension theorems for spheres in the finite field setting, Forum Math. 22 (2010), no.3, 457–483.** In this paper, we improve upon our previous result for extension theorems related to nondegenerate quadratic surfaces in the specific case when the nondegenerate quadratic surfaces are spheres $S_j, j \neq 0$, defined by

$$S_j = \{x \in \mathbb{F}_q^d : x_1^2 + \cdots + x_d^2 = j\}.$$

In higher even dimensions $d \geq 4$, the authors obtain the "$p$" index improvement of the Tomas-Stein $L^p - L^4$ exponents by showing that

$$R^*\left((12d-8)/(9d-12) \to 4\right) \lessapprox 1.$$

See Figure 3. The proof is based on an elaborate effort to estimate several kinds of incidence theorems. Moreover, we show that if the dimension $d \geq 3$ is odd and $-1 \in \mathbb{F}_q$ is a square number, then we can not expect the "$p$" index improvement of the Tomas-Stein $L^p - L^4$ exponents (see Figure 1).

**1.3. A. Iosevich and D. Koh, Extension theorems for paraboloids in the finite field setting. Mathematische Zeitschrift, 266, (2010) no.2 471-487.** Mockenhaupt and Tao ([8]) studied the extension theorems for paraboloids in the finite field setting. In two dimension, they obtained the sharp result that $R^*(2 \to 4) \lesssim 1$. In general higher dimensions $d \geq 3$, they obtained the standard Tomas-Stein exponents, that is $R^*(2 \to 2(d+1)/(d-1)) \lesssim 1$. In this paper, the authors improves upon the standard Tomas-Stein exponents in the case when the dimension $d \geq 4$ is even. In fact, the authors prove that $R^*(4d/(3d-2) \to 4) \lessapprox 1$ and $R^*(2 \to 2d^2/(d^2-2d+2) \lessapprox 1$. In particular, our $L^p - L^4$ result is sharp up to end points unless we have the restriction on $\mathbb{F}_q$ that $-1 \in \mathbb{F}_q$ is not a square number (see Figure 4). The proof relies on getting good upper bound of the number of additive quadruples $(x, y, z, w)$ with $x + y = z + w$, where $x, y, z$ and $w$ are elements of the subset $E$ of the paraboloid $P$, which is defined as

$$P = \{x \in \mathbb{F}_q^d : x' \cdot x' = x_d\},$$

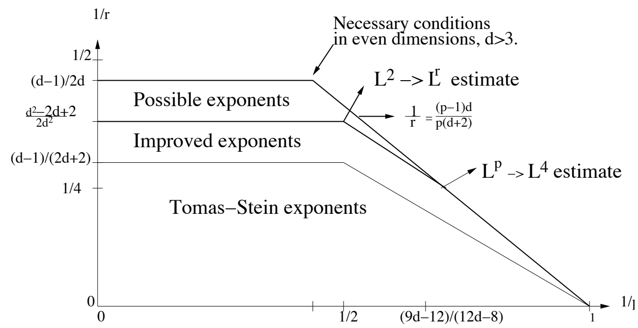


FIGURE 4. In Even Dimensions $d \geq 4$, the improved extension estimates for Paraboloids

where $x = (x', x_d)$ and $x' \cdot x'$ is usual dot product. However, if $d$ is odd, then the standard Tomas-Stein exponents give in general the sharp $L^2 - L^r$ estimate and we can not improve the exponents. On the other hand, if we assume that $-1$ is not a square number in the underlying finite field $\mathbb{F}_q$, and the dimension $d$ is odd, then one can improve upon the standard Tomas-Stein exponents. In fact, Mochenhaupt and Tao in [8] improved upon the Tomas-Stein exponents in the case when $d = 3$ and $-1$ is not a square number in the underlying finite field $\mathbb{F}_q$. In this paper, we extend their work to certain higher odd dimensions, using the explicit Gauss sum estimates. Our results say that if the dimension $d = 4k + 3$ for some $k \in \mathbb{N}$ and $-1$ is not a square number, then it follows that $R^*(4d/(3d-2) \to 4) \lessapprox 1$ and $R^*(2 \to 2d^2/(d^2 - 2d + 2)) \lessapprox 1$, which are better than the result given by Tomas-Stein exponents, $R^*(2 \to 2(d+1)/(d-1)) \lesssim 1$.

**Remark 1.1.** Working in the bilinear setting, the authors in [7] recently improved our results for paraboloids by showing that the endpoint estimates hold.

**1.4. Sharp extension theorems and Falconer distance problems for algebraic curves in two dimensional vector spaces over finite fields, joint work with Chun-Yen Shen, Submitted for publication (2010).** The authors study extension theorems associated with general varieties in two dimensional vector spaces over finite fields. Applying Bezout's theorem, we obtain the sufficient and necessary conditions on general curves where sharp $L^p - L^r$ extension estimates hold. Our main result can be considered as a nice generalization of works by Mochenhaupt and Tao in [8] and Iosevich and Koh in [3]. More precisely, the authors prove the following theorem.

**Theorem 1.2.** *Suppose that $P(x) \in \mathbb{F}_q[x_1, x_2]$ is non-zero polynomial. Define an algebraic variety $V \subset \mathbb{F}_q^2$ by*
$$V = \{x \in \mathbb{F}_q^2 : P(x) = 0\}.$$
*Then, $R^*(2 \to 4) \lesssim 1$ if and only if $|V| \sim q$ and the polynomial $P(x)$ do not have any linear factor.*

**1.5. D.Koh, Extension and averaging operators for finite fields, submitted for publication, (2010), www.arxiv.org.** The author investigates $L^p - L^r$ estimates of both extension operators and averaging operators associated with the algebraic variety $V = \{x \in \mathbb{F}_q^d : Q(x) = 0\}$ where $Q(x)$ is a nondegenerate quadratic form over the finite field $\mathbb{F}_q$ with $q$ elements. This variety $V$ is considered as a general form of a cone in finite fields. Using a geometric observation on a cone in three dimension, Mockenhaupt and Tao [8] showed that $L^2 - L^4$ conical extension estimate holds and it gives a complete answer for the conical extension problems in three dimension. In this paper, the author observes that if the dimension $d \geq 3$ is odd, then the decay of the Fourier transform on $V$ is good enough to obtain the Tomas-Stein exponents, that is $R^*(2 \to 2(d+1)/(d-1)) \lesssim 1$. As a consequence, the author recovers and extends the work by Mockenhaupt and Tao into higher odd dimensions. In addition, the author shows that if the dimension $d \geq 2$ is even, then the Tomas-Stein exponents can not be obtained in general, because the sharp decay of the Fourier transform on $V$ in even dimension is worser than that in odd dimension. On the other hand, the author proves that the sharp Fourier decay on $V$ yields the sharp $L^2 - L^r$ extension estimate in the case when $V$ contains $d/2$ dimensional subspace. Moreover, this extension result is applied to obtain the almost complete answer for the boundedness of averaging operators over an algebraic variety $V$ where $V$ contains a large dimensional subspace. This approach for the boundedness of averaging operators is quite new and it is very interesting that there is some connection between extension problems and averaging problems. In the finite field case, there were no useful tools to study the boundedness of averaging operators over a variety $V$ which contains a large dimensional subspace. For more detail information about the averaging problems, see [1].

**1.6. Harmonic analysis related to homogeneous varieties in three dimensional vector spaces over finite fields , (joint work with Chun-Yen Shen), submitted for publication, (2010), www.arxiv.org.** In this paper the authors study extension problems, averaging problems, and generalized Erdős-Falconer distance problems associated with arbitrary homogeneous varieties in three dimensional vector space over finite fields. In the case when homogeneous varieties in three dimension do not contain any plane passing through the origin, we obtain the best possible results on aforementioned three problems. In particular, our results on extension problems recover and generalize the work due to Mockenhaupt and Tao [8] who settled down the conical extension problems in three dimension. Investigating the decay of the Fourier transform on homogeneous varieties, we give the complete mapping properties of averaging operators over homogeneous varieties in three dimension. In addition, studying the generalized Erdős-Falconer distance problems related to homogeneous varieties in three dimensions, we improve the cardinality condition on sets where the size of distance sets is nontrivial.

## 2. Results on Erdős-Falconer type distance problems

**2.1. A.Iosevich and D.Koh, The Erdos-Falconer distance problem, exponential sums, and Fourier analytic approach to incidence theorems in vector spaces over finite fields, SIAM J. Discrete Math. 23 (2008/09), no. 1, 123–135.** The Erdös-Falconer distance problem, in a generalized sense, is a question of how many distances are determined by a set of points. In this paper, authors study the Erdös-Falconer distance problem in vector spaces over finite fields with respect to the cubic metric. Let $E \subset \mathbb{F}_q^d, d \geq 2$, the $d$-dimensional vector space over the finite fields $\mathbb{F}_q$ whose characteristic is greater than two. For each $x \in \mathbb{F}_q^d$ and $n$ a positive integer$\geq 2$, we define $\|x\|_n = x_1^n + \ldots + x_d^n$. The distance set $\Delta_n(E)$ is defined by $\Delta_n(E) = \{\|x-y\|_n : x, y \in E\}$, viewed as a subset of $\mathbb{F}_q$. Then the Erdös-Falconer distance problem in this context asks for the smallest number $s_0$ such that $\Delta_n(E)$ contains a positive proportion of the elements of $\mathbb{F}_q$ provided that $|E| \geq Cq^{s_0}$ with $C$ sufficiently large. Using the estimates for discrete Airy sums and Adolphson-Sperber estimates for exponential sums in terms of Newton polyhedra, the authors obtain a nontrivial range of exponents that appear to be difficult to attain using combinatorial methods. The main result is as follows.

**Theorem 2.1.** *Suppose that $q$ is a prime number congruent to 1 modulo 3. Let $E \subset \mathbb{F}_q^d$, such that $|E| \geq Cq^{(d+1)/2}$. Then if $C$ is sufficiently large, then $\Delta_3(E)$ contains every elements in $\mathbb{F}_q$. Moreover, in dimension $d = 2$, if $|E| \geq Cq^{3/2}$ for $C$ sufficiently large, then $\Delta_n(E), n \geq 2$, contains every element of $\mathbb{F}_q$.*

In this paper, the authors also study the number of incidences between points and surfaces in vector spaces over finite fields. It is proved by the authors that for each $E, F \subset \mathbb{F}_q^d, d \geq 2$, and $j \neq 0$,

$$|\{(x,y) \in E \times F : \|x-y\|_n = j\}| \lesssim |E||F|q^{-1} + q^{(d-1)/2}|E|^{1/2}|F|^{1/2},$$

where $n$ is two or three. In particular, this result says that if $|E| \sim |F| \sim q^{(d+1)/2}$, then the number of incidences between points and "spheres", quadratic or cubic, centered at elements of $F$ is $\lesssim q^d$. In two dimensions this says that the number of incidences between $N$ points and $N$ circles is $\lesssim N^{4/3}$, provided that $N \sim q^{3/2}$, matching in this setting the exponent in the classical incidence theorem due to Szemerédi and Trotter in the Euclidean plane.

**2.2. D.Hart, A.Iosevich, D.Koh, and M.Rudnev, Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdös-Falconer distance conjecture, TAMS, in press, 2010.** A. Iosevich and M. Rudnev [5] studied the Erdös-Falconer distance problems related to the quadratic distance in the finite field setting and they conjectured that if the number of elements of the subset $E$ of $\mathbb{F}_q^d$ is greater than $Cq^{d/2}$ with C sufficiently large, then we have $|\Delta_2(E)| = q$, where $\Delta_2(E)$ is a distance set defined by

$$\Delta_2(E) = \{(x_1-y_1)^2 + \cdots + (x_d-y_d)^2 \in \mathbb{F}_q : x, y \in E\}.$$

Moreover, they showed that if $|E| \geq Cq^{(d+1)/2}$ with $C$ sufficiently large, then all distances can be achieved. In this paper, the authors proves that if the dimension $d$ is odd, then the conjecture can not be true. In fact, arithmetic examples constructed by the authors show that the exponent $(d+1)/2$ is sharp. The Erdös-Falconer distance problem on spheres $S_j$ is also studied by the authors. We show that if $E$ is a subset of the sphere $S_j$, then we always get a positive proportion of all distances whenever $|E| \geq Cq^{d/2}$ with $C$ sufficiently large. In addition, if the dimension $d$ is even, then all distances can be achieved under the same assumption. This is geometrically analogous to the general case, for since the sphere $S_j$ is $(d-1)/2$-dimensional variety in $\mathbb{F}_q^d$, it makes sense that the sharp index should be $\frac{(d-1)+1}{2} = d/2$.

**2.3. Pinned distance sets, k-simplices, Wolff's exponent in finite fields and sum-product estimates, (joint work with J.Chapman, M.B.Erdogan, D.Hart,and A.Iosevich),(2009).** An analog of the Falconer distance problem in vector spaces over finite fields asks for the threshold $\alpha > 0$ such that $|\Delta(E)| \gtrsim q$ whenever $|E| \gtrsim q^\alpha$, where $E \subset \mathbb{F}_q^d$, the $d$-dimensional vector space over a finite field with $q$ elements (not necessarily prime). Here $\Delta(E) = \{(x_1-y_1)^2 + \cdots + (x_d-y_d)^2 : x, y \in E\}$. A. Iosevich and M. Rudnev ([5]) established the threshold $\frac{d+1}{2}$, and the authors in [6] proved that this exponent is sharp in odd dimensions. In two dimensions we improve the exponent to $\frac{4}{3}$, consistent with the corresponding exponent in Euclidean space obtained by Wolff ([11]).

The pinned distance set $\Delta_y(E) = \{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 : x \in E\}$ for a pin $y \in E$ has been studied in the Euclidean setting. Peres and Schlag ([9]) showed that if the Hausdorff dimension of a set $E$ is greater than $\frac{d+1}{2}$ then the Lebesgue measure of $\Delta_y(E)$ is positive for almost every pin $y$. In this paper we obtain the analogous result in the finite field setting. In addition, the same result is shown to be true for the pinned dot product set $\Pi_y(E) = \{x \cdot y : x \in E\}$. Under the additional assumption that the set $E$ has cartesian product structure we improve the pinned threshold for both distances and dot products to $\frac{d^2}{2d-1}$.

The pinned dot product result for cartesian products implies the following sum-product result. Let $A \subset \mathbb{F}_q$ and $z \in \mathbb{F}_q^*$. If $|A| \geq q^{\frac{d}{2d-1}}$ then there exists a subset $A' \subset A$ with $|A'| \gtrsim |A|$ such that for every $a_1, \ldots, a_{d-1}$ in $A'$ one has $|a_1 A + a_2 A + \cdots + a_{d-1} A + zA| > \frac{q}{2}$, where $a_j A = \{a_j a : a \in A\}, j = 1, \ldots, d-1$.

A generalization of the Falconer distance problem is determine the minimal $\alpha > 0$ such that $E$ contains a congruent copy of every $k$ dimensional simplex whenever $|E| \gtrsim q^\alpha$. Here the authors improve on known results (for $k > 3$) using Fourier analytic methods, showing that $\alpha$ may be taken to be $\frac{d+k}{2}$.

**2.4. The generalized Erdos-Falconer distance problems in vector spaces over finite fields, (joint work with Chun-Yen Shen), Submitted for publication.** In this paper the authors study the generalized Erdös-Falconer distance problems in the finite field setting. The generalized distances are defined in terms of polynomials, and various formulas for sizes of distance sets are obtained. Given a polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ and $E, F \subset \mathbb{F}_q^d$, one may define a generalized distance set $\Delta_P(E, F)$ by the set

$$\Delta_P(E, F) = \{P(x - y) \in \mathbb{F}_q : x \in E, y \in F\}.$$

As a generalized version of spherical distance problems in [5] and cubic distance problems in [4], we have the following theorem.

**Theorem 2.2.** *Let $P(x) = \sum_{j=1}^d a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ for $s \geq 2$ integer and $a_j \neq 0$. Suppose that the characteristic of $\mathbb{F}_q$ is sufficiently large. If $|E||F| \geq Cq^{d+1}$ for $E, F \subset \mathbb{F}_q^d$, then $|\Delta_P(E, F)| = q - 1$, where $C > 0$ is a sufficiently large constant.*

In addition, the authors set up and study the generalized pinned distance problems in finite fields. As a consequence, the authors obtain the following theorem which sharpens and generalizes the Vu's result [10].

**Theorem 2.3.** *Let $P(x) \in \mathbb{F}_q[x_1, x_2]$ be a non-degenerate polynomial. If $|E||F| \geq Cq^3$ for $E, F \subset \mathbb{F}_q^2$ and $C > 0$ sufficiently large, then there exists a subset $F_0$ of $F$ with $|F_0| \sim |F|$ such that*

$$|\Delta_P(E, y)| \gtrsim q \quad \textit{for all } y \in F_0.$$

**2.5. Additive energy and the Falconer distance problem in finite fields, (joint work with Chun-Yen Shen), Submitted for publication (2010).** The authors study the number of the vectors determined by two sets in $d$-dimensional vector spaces over finite fields. We observe that the lower bound of cardinality for the set of vectors can be given in view of an additive energy or the decay of the Fourier transform on given sets. As an application of our observation, we find sufficient conditions on sets where the Falconer distance conjecture for finite fields holds in two dimension. Moreover, we give an alternative proof of the theorem, due to Iosevich and Rudnev, that any Salem set satisfies the Falconer distance conjecture for finite fields.

**2.6. D.Covert, D.Hart, A.Iosevich, D.Koh, and M.Rudnev, Generalized incidence theorems, homogeneous forms and sum-product estimates in finite fields, European J. Combin. 31 (2010), no. 1, 306–319.** In recent years, sum-product estimates in Euclidean space and finite fields have been studied using a variety of combinatorial, number theoretic and analytic methods. Erdös type problems involving the distribution of distances, areas and volumes have also received much attention. In this paper we prove a relatively straightforward function version of an incidence results for points and planes previously established in [2] and [6]. As a consequence of our methods, we obtain sharp or near sharp results on the distribution of volumes determined by subsets of vector spaces. More precisely, we prove that if $E = A \times \cdots \times A$ is a product set in $\mathbb{F}_q^d, d \geq 4$, the d-dimensional vector space over a finite field $\mathbb{F}_q$, such that the size $|E|$ of $E$ exceeds $q^{\frac{d}{2}}$ (i.e. the size of the generating set $A$ exceeds $\sqrt{q}$) then the set of volumes of $d$- dimensional parallelepipeds determined by $E$ covers $\mathbb{F}_q$. This result is sharp as can be seen by taking $A = \mathbb{F}_p$, a prime sub-field of its quadratic extension $\mathbb{F}_q$, with $q = p^2$. For in three dimensions, however, we are able to establish the same result only if $|E| \gtrsim q^{\frac{15}{8}}$ (i.e., $|A| \geq Cq^{\frac{5}{8}}$, for some $C$; in fact, the $q^{\frac{15}{8}}$ bound can be justified for a slightly wider class of "Cartesian product-like" sets), and this uses Fourier methods. Yet we do prove a weaker near-optimal result in three dimensions : that the set of volumes generated by a product set $E = A \times A \times A$ covers a positive proportion of $\mathbb{F}_q$ if $|E| > q^{\frac{3}{2}}$ (so $|A| > \sqrt{q}$). Besides, without any assumptions on the structure of $E$, we show that in three dimensions the set of volumes covers a positive proportion of $\mathbb{F}_q$ if $|E| \geq Cq^2$, which is again sharp up to the constant $C$, as taking $E$ to be a 2-plane through the origin shows.

**2.7. Distance graphs in vector spaces over finite fields, coloring, pseudo-randomness and arithmetic progressions, (joint work with D.Hart, A. Iosevich, S.Senger,and I.Uriarte-Tuero), (2009).** Let $\mathbb{F}_q$ be the finite field with $q$ elements. We assume that the characteristic of $\mathbb{F}_q$ is greater than two. For each $r \in \mathbb{F}_q^*$, the multiplicative group of $\mathbb{F}_q$, the distance graph $G_q^{\Delta}(r)$ in $\mathbb{F}_q^d$ is obtained by taking $\mathbb{F}_q^d$ and connecting two vertices corresponding to $x, y \in \mathbb{F}_q^d$ by an edge if $||x - y|| = r$ where

$$||x|| = x_1^2 + x_2^2 + \cdots + x_d^2.$$

We also consider the set of colors $L_q = \{c_q^r : r \in \mathbb{F}_q^*\}$ corresponding to elements of $\mathbb{F}_q^*$. We connect two vertices corresponding to points $x, y \in \mathbb{F}_q^d$ by a $c_q^r$-colored edge if $||x-y|| = r$. We denote by $G_q^{\Delta}$ the resulting almost complete graph with the implied edges and the coloring set $L_q$. When $q$ runs over powers of odd primes, we obtain a family of the almost complete distance graphs $\{G_q^{\Delta}\}$. For each $r \in \mathbb{F}_q^*$, the single-colored distance graph $G_q^{\Delta}(r)$ can be considered as a sub-graph of the almost complete distance graph $G_q^{\Delta}$ with $q-1$ colors. Consider the almost complete distance graph $G_q^{\Delta}$ with the coloring set $L_q = \{c_q^r : r \in \mathbb{F}_q^*\}$ defined as before. For each $r \in \mathbb{F}_q^*$, we also consider the $c_q^r$-colored distance graph $G_q^{\Delta}(r)$ defined as before. Given a fixed color $c_q^r$ in $L_q$, we define the diameter of the $c_q^r$-colored distance graph $G_q^{\Delta}(r)$ as follows. Given vertices $x$, $y$ in $G_q^{\Delta}(r)$, define a *path* of length $k$ from $x$ to $y$ to be a sequences $\{x^1, \dots, x^{k+1}\}$, where $x^j$s are distinct, $x^1 = x$, $x^{k+1} = y$, each $x^j$ is a vertex in $G_q^{\Delta}(r)$ and $x^i$ is connected to $x^{i+1}$ by a $c_q^r$-colored edge for every $1 \leq i \leq k$. We say that a path from $x$ to $y$ is optimal if it is a path and its length is as small as possible. Define the *diameter* of $G_q^\Delta(r)$, denoted by $\mathbb{D}(G_q^\Delta(r))$, to be the largest length of the optimal path between any two vertices in $G_q^\Delta(r)$. We also define the diameter of the almost complete distance graph $G_q^\Delta$, denoted by $\mathbb{D}(G_q^\Delta)$, as follows:

$$\mathbb{D}(G_q^\Delta) = \max_{r\in\mathbb{F}_q^*}\left(\mathbb{D}(G_q^\Delta(r))\right).$$

Using the finite Fourier machinery, we obtain the following theorem.

**Theorem 2.4.** 1)*If $d\geq 4$, then $\mathbb{D}(G_q^\Delta)=2$ for all $q\geq 3$.*
2) *Suppose that $d=3$ and $\psi$ is the quadratic character of $\mathbb{F}_q$. For each $r\in\mathbb{F}_q^*$ $q\geq 3$, we have*

$$\mathbb{D}(G_q^\Delta(r)) = \begin{cases} 2 & \text{if} \quad \psi(-r)=1 \\ 3 & \text{if} \quad \psi(-r)=-1. \end{cases}$$

*Namely, $\mathbb{D}(G_q^\Delta)=3$.*
3) *If $d=2$, then we have*

$$\mathbb{D}(G_q^\Delta) = \begin{cases} 2 & \text{for} \quad q=3 \\ 3 & \text{for} \quad q\neq 3,5,9,13. \end{cases}$$

**Remark 2.5.** The author recently observed that if $d=2$, then

$$\mathbb{D}(G_q^\Delta) = \begin{cases} 4 & \text{if} \quad q=5,9 \\ 3 & \text{if} \quad q=13. \end{cases}$$

This observation and Theorem 2.4 give the complete answer for the diameter problem in finite fields.

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