NONLINEAR STABILITY OF OSCILLATORY PULSES IN THE PARAMETRIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We extend the renormalization group method, developed for the study of pulse interaction in damped wave equations, to the study of oscillatory motion of super-critical pulses in the parametrically forced nonlinear Schrödinger equation (PNLS). We develop a global manifold which asymptotically attracts the flow into an $\mathcal{O}(r^4)$ neighborhood in the H^1 norm, where r is the amplitude of the internal oscillations. Moreover we rigorously recover the oscillatory and translational dynamics of the pulses as a finite-dimensional flow on the manifold. The normal form for the projected dynamics of the oscillatory pulse show that it is created in a supercritical Poincaré-Hopf bifurcation.

1. Introduction . The parametrically forced nonlinear Schrödinger equation (PNLS) describes a wide variety of physical phenomena, including the optical parametric oscillator in the large pump-detuning limit, Faraday resonance in water, spin waves and magnetic solitons in ferromagnets, and phase-sensitive parametric amplification of solitons in optical fibers [2, 3, 17]. In the context of the optical parametric oscillator a pump field impinges upon a cavity filled with χ^2 material, generating a signal field. When the pump field is detuned, the signal is the sole resonating field. In this limit, investigated in more detail in [17], the signal field evolves according to (2.1), with the characteristic parameters being the signal detuning, a, and the pump strength, γ.

For sufficiently strong parametric excitation the cavity supports two branches of pulse-like solutions created in a saddle-node bifurcation. The lower branch is unstable, while the upper branch has been proven to be asymptotically stable, see [17], for a range of parameter values. For yet stronger parametric excitation numerical simulations [3] and [4] have shown that the stable pulse losses stability due to the generation of internal modes of oscillation. It has been shown, [6], that this instability is associated with a complex conjugate pair of Hopf eigenvalues of the linearized operator crossing into the right-half complex plane.

In [5] the supercritical dynamics of the oscillatory pulse were addressed formally. The oscillatory solution was expressed as a perturbation expansion about the pulse solution and putative reduced amplitude equations for the internal modes were obtained. We expand the renormalization group approach developed in [16], constructing a finite dimensional manifold of oscillatory pulse solutions, rigorously capturing the translational and oscillatory motion of the pulses by projecting their evolution onto the tangent plane of the manifold. In particular we show that the pulse solution of the PNLS equation losses its stability in a super-critical Poincaré-Hopf bifurcation, generating a stable oscillatory pulse. Initial data for the PNLS equation which is sufficiently close to the manifold approaches to within $\mathcal{O}(r^4)$ of the manifold, where $r \ll 1$ is the amplitude of the internal oscillations. This work extends the renormalization group approach since the pulse dynamics on the tangent plan include the fast pulse oscillation modes within the projected evolution equations, see (3.30). To develop the normal form for the Poincaré-Hopf bifurcation from the projected equations, the manifold itself must be modified to account for the fast dynamics, see (2.47-2.48). Only after accounting for the impact of the fast oscillation through the damped modes of the linear operator can the bifurcation analysis be conducted on the projected equations. We rigorously show that a super-critical Poincaré-Hopf bifurcation occurs in the projected equations, and the resulting oscillatory pulses are asymptotically stable up to $\mathcal{O}([\Re\lambda_1]^2)$ in the H^1 norm.

Our analysis is only valid for the damped case which in our scaling corresponds to finite a. The authors in [2] formally investigated the stability problem for the Hamiltonian case, $a \to \infty$, for which the the pulse instability is due to the "Hamilton-Hopf" bifurcation, characterized by the collision of two pure imaginary eigenvalues of the associated linearized operator, one detaching from the essential spectrum and the other originating from the broken $U(1)$ gauge invariance.

In the context of reaction-diffusion equations and other parabolic PDEs, the center manifold approach has been applied to address the stability of oscillatory solutions generated in Poincaré-Hopf bifurcations. For the bifurcation of planar solutions, this center-manifold technique is outlined in [15]. More recently the center-manifold method was extended to any PDE which generates a C^1 semigroup with appropriate structure, [8]. This has been applied to investigate stability of so-called galloping waves in reaction diffusion systems was demonstrated in exponentially weighted norms which remove the essential spectrum from the origin, [18]. Applications to the meander of spiral waves in higher dimension have also been investigated, [9]-[10]. The novelty in the application of the renormalization group approach lies in its simplicity. The manifold of oscillatory pulses is constructed explicitly, at the order of accuracy needed, without the technical contraction mapping arguments implicit in the center manifold construction. Moreover the extension of these results to interaction of families of oscillatory N-pulses is simpler and more natural via the RG approach as it does not require an explicit construction of N-pulse solutions before considering their possible bifurcations and subsequent interaction.

We state our main result below.

THEOREM 1.1. *The Poincaré-Hopf bifurcation of the PNLS equation (2.4) is super-critical. Specifically, let $\delta > 0$ be sufficiently small and $M > 0$ sufficiently large, in terms of the parameter $a > a_c$, see Numerical Result 2.1, of the PNLS equation. For $\gamma > \gamma_c$ but sufficiently close that $\epsilon \equiv \Re\lambda_H > 0$, the real part of the Hopf eigenvalue, satisfies $\delta > \sqrt{\epsilon}$, the oscillatory ansatz $\Phi_{\mathbf{p}}$, see (2.11) and section 2.2, of (2.4) are asymptotically exponentially stable up to $\mathcal{O}(\epsilon^2)$. That is, for any initial value $\vec{U}_0$ satisfying*

$$\min_{q\in\mathbb{R}} \|\Phi_0(\cdot - q) - \vec{U}_0(\cdot)\|_{H^1} \leq \delta, \tag{1.1}$$

where the base pulse Φ_0 is given by (2.7), the corresponding solution $\vec{U}$ of the PNLS equations satisfies

$$\vec{U}(x,t) = \Phi_{\mathbf{p}} + W(x,t), \tag{1.2}$$

where $\mathbf{p}(t)$ evolves at leading order according to (3.30). In particular the oscillation amplitude $\mathbf{p}_1$ and remainder W satisfy

$$|\mathbf{p}_1(t)| \leq M\left(e^{-\epsilon(t-t_0)}\delta + \sqrt{\epsilon}\right), \tag{1.3}$$

and

$$\|W\|_{H^1} \leq M\left(e^{-\nu t}\delta + |\mathbf{p}_1(t)|^4\right). \tag{1.4}$$

The paper is organized as follows, in section 2 we present the PNLS equations and the steady pulse solutions, describe the linearized operator and construct the Hopf-manifold which describes the oscillatory solutions at the requisite order of accuracy. In section 3 we present the RG method, in particular obtaining the projected equations, including the Poincaré-Hopf normal form, decay estimates on the remainder, and prove the main result.

1.1. Notation. The following notation is employed. Transposition is denoted by the superscript ^{t}. The components of a vector quantity A are denoted by $A = (a_1, \dots, a_n)^t$. The L^2 inner product $\langle \cdot\,|\cdot\rangle$ of two complex vector functions A and B is defined as

$$\langle A\,|B\,\rangle \equiv \int_{-\infty}^{\infty} A(x)\cdot \overline{B}(x)dx. \tag{1.5}$$

This inner product induces the L^2 norm $\|\cdot\|$. The H^1 norm is denoted $\|A\|_{H^1} = \sqrt{\|A\|_{L^2}^2 + \|\partial_x A\|_{L^2}^2}$. The orthogonal complement in L^2 is denoted by $^\perp$. Given an operator L, its adjoint with respect to $\langle\cdot\,|\cdot\rangle$ is denoted by $L^\dagger$, and its spectrum and resolvent sets are denoted by $\sigma(L)$ and $\rho(L)$. Given $z \in \mathbb{C}$ we denote its complex conjugate by $\overline{z}$, its argument by $\arg z$, and its real and imaginary parts by $\Re z$ and $\Im z$. Differentiation with respect to the variable x is denoted by ∂_x, or ∇_x.

2. The PNLS Equations. The PNLS equation for the signal field u in a cavity can be written as

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + (i-a)u - \gamma\overline{u} = 0, \tag{2.1}$$

where γ and a are the forcing and detuning parameters respectively, [19, 20, 17]. For sufficiently strong parametric excitation, $\gamma > 1$, the system sustains standing waves

$$\phi_\pm = \left(\sqrt{2\nu_\pm^{-1}}\text{sech}\sqrt{2\nu_\pm^{-1}}x\right)e^{i\theta_\pm}, \tag{2.2}$$

where $\nu_\pm = 1/\left(a \pm \sqrt{\gamma^2-1}\right)$ and $e^{-2i\theta_\pm} = \left(i \pm \sqrt{\gamma^2-1}\right)/\gamma$.

We consider the upper branch, ϕ_+ of pulse solutions. Denoting $\nu \equiv \nu_+$, we rescale the independent variables of the PNLS (2.1) as

$$\widetilde{x} \equiv \sqrt{2\nu^{-1}}x \qquad \text{and} \qquad \widetilde{t} = \nu^{-1}t, \tag{2.3}$$

and recast the PNLS as a scalar system, introducing the new dependent variables

$$\vec{U} = (U_1, U_2)^t = \left(\Re(\sqrt{\nu}e^{-i\theta}\phi), \Im(\sqrt{\nu}e^{-i\theta}\phi)\right)^t. \tag{2.4}$$

to obtain

$$\vec{U}_t = F\left(\vec{U}\right), \tag{2.5}$$

where

$$F\left(\vec{U}\right) \equiv \begin{pmatrix} 0 & -\left(\partial_x^2 - \mu + \left|\vec{U}\right|^2\right) \\ \partial_x^2 - 1 + \left|\vec{U}\right|^2 & -2\nu \end{pmatrix}\vec{U}, \tag{2.6}$$

and $\mu \equiv \nu_+/\nu_-$. For $a \in (0, \infty)$ and $\gamma \in \left(1, \sqrt{1+a^2}\right)$, equation (2.5) possesses the family of pulse solutions

$$\Phi_0(x;q) = \begin{pmatrix} \phi_q \\ 0 \end{pmatrix} \tag{2.7}$$

where for $q \in \mathbb{R}$

$$\phi_q(x) = \sqrt{2}\,\text{sech}(x-q). \tag{2.8}$$

To describe the dynamics in the super-critical regime we decompose the solutions of the PNLS equation in a neighborhood of the pulse profile into a pulse and an remainder term

$$\vec{U}(x,t) = \Phi_{\mathbf{p}(t)} + W(x,t), \tag{2.9}$$

where the parameters $\mathbf{p} = (p_0, p_1, p_2) = (q, r_1, r_2)$ describe the position of the pulse and the magnitude of the oscillation. The graph of $\Phi_{\mathbf{p}}$ forms a three dimensional manifold within the phase space

$$\mathcal{M} = \left\{\Phi_{\mathbf{p}}\middle|\mathbf{p} \in \mathbb{R}\times\mathbb{C}\right\}. \tag{2.10}$$

As outlined in section 2.2. The manifold is further resolved into the pulse and higher-order Hopf modes which capture the interaction of the pulse with the unstable Hopf eigenmodes,

$$\Phi_{\mathbf{p}} = \Phi_0(x,q) + \mathcal{H}(x;\mathbf{p}). \tag{2.11}$$

The evolution for the solution $\vec{U}$ of (2.5) in a neighborhood of the manifold $\mathcal{M}$ is described by the Taylor expansion of the vector field F about the unperturbed pulse Φ_0,

$$W_t + \nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}} = F(\Phi_0) + L_q(\mathcal{H}+W) + \mathcal{N}(\mathcal{H}+W). \tag{2.12}$$

Here L_q is the linearization of F about Φ_0 and $\mathcal{N}$ represents terms nonlinear in $\mathcal{H}$ and W. Since Φ_0 is an exact solution of (2.5), $F(\Phi_0) = 0$. The nonlinearity can be written as quadratic and cubic terms

$$\mathcal{N}(Y) = \phi\mathcal{N}_2(Y,Y) + \mathcal{N}_3(Y,Y,Y), \tag{2.13}$$

where

$$\mathcal{N}_2(A,B) \equiv \begin{pmatrix} -2a_1b_2 \\ 3a_1b_1 + a_2b_2 \end{pmatrix} \quad \text{and} \quad \mathcal{N}_3(A,B,C) \equiv \begin{pmatrix} -a_1b_1c_2 - a_2b_2c_2 \\ a_1b_1c_1 + a_1b_2c_2 \end{pmatrix}. \tag{2.14}$$

Each of $\mathcal{N}_2$ and $\mathcal{N}_3$ are linear in their arguments. We also denote $\mathcal{N}_2(Y,Y)$ and $\mathcal{N}_3(Y,Y,Y)$ by $\mathcal{N}_2(Y)$ and $\mathcal{N}_3(Y)$ respectively, while $\mathcal{N}_2^s(A,B) \equiv \mathcal{N}_2(A,B) + \mathcal{N}_2(B,A)$.

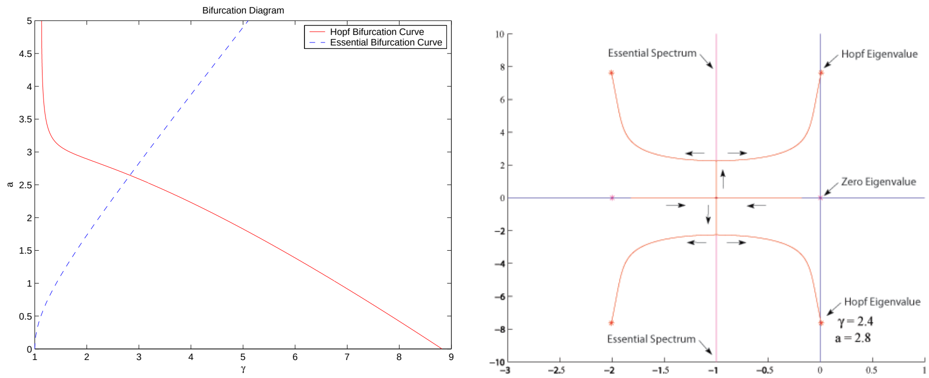

FIG. 2.1. *(left) Bifurcation Diagram for the PNLS. For values of (γ, a) to the left of the solid Hopf bifurcation curve, $\gamma = \gamma_c(a)$, the Hopf eigenvalues have negative real part, with positive real part to the right. Above the dotted essential bifurcation curve $a = \sqrt{1+\gamma^2}$, the essential spectrum resides in the left-half plane, below it the essential spectrum crosses into the right-half plane. (right) The trajectory of the Hopf eigenvalues for a fixed at $a = 2.8$ and γ increasing from slightly larger than 1 to just crossing the Hopf curve. The Hopf eigenvalues start near the origin and -2, move towards each other along the real axis, collide at -1, move vertically up the line $-1 + i\mathbb{R}$, and collide with eigenvalues created in edge bifurcations from the essential spectrum, forming a quartet of complex eigenvalues symmetric about the point -1. The right-most two of these eigenvalues cross the imaginary axis as γ crosses $\gamma_c(a = 2.8)$.*

2.1. The Linearized Operator. The linearized operator L_q takes the form

$$L_q = \begin{pmatrix} 0 & D_q \\ -C_q & -2\nu \end{pmatrix}, \tag{2.15}$$

where C_q and D_q are the self-adjoint operators

$$C_q = -\left(\partial_x^2 + 3\phi_q^2 - 1\right), \tag{2.16}$$

$$D_q = -\left(\partial_x^2 + \phi_q^2 - \mu\right). \tag{2.17}$$

The operator C_q has the eigenvalue-eigenfunction pairs $\{-3, \phi_q^2\}$, $\{0, \phi_q'\}$ and essential spectra $[1, \infty)$, while D_q has the eigenvalue-eigenfunction pair $\{\mu - 1, \phi_q\}$ and essential spectra $[\mu, \infty)$. In particular D_q has a bounded self-adjoint inverse for $\gamma \in \left(1, \sqrt{1+a^2}\right)$. See [17] for details.

NUMERICAL RESULT 2.1. *There exists a curve $\gamma = \gamma_c(a)$ on which the point spectrum of $\sigma(L)$ possesses two complex conjugate eigenvalues λ_1 and $\lambda_2 = \overline{\lambda}_1$ which lie on the imaginary axis. For $\gamma - \gamma_c(a) > 0$ the point spectrum takes the form*

$$\sigma_p(L) = \{\lambda_0, \lambda_1, \lambda_2\} \cup \sigma_p^s(L), \tag{2.18}$$

where $\sigma_p^s(L)$ is comprised of a finite number of eigenvalues contained uniformly within the left-half complex plane. For $a > a_c \approx 2.645$, the essential spectrum of L is strictly in the left-half plane for $|\gamma - \gamma_c(a)|$ sufficiently small.

The eigenvalue problem

$$L_q \Psi = \lambda \Psi, \tag{2.19}$$

is re-written as a first order evolution equation for the system $Y = (\Psi_1, \Psi_2, \Psi_1', \Psi_2')^t$,

$$Y' = A(x, \lambda) Y. \tag{2.20}$$

In [6], using the technique from [14], the Hopf bifurcation curve $\gamma = \gamma_c(a)$ was computed by constructing the Dirichlet expansions for the stable and unstable manifolds of zero for (2.20). This expansion yields analytic expressions, valid in $x > x_0 > 0$, for the solutions Y^1 and Y^2 which decay as $x \to \infty$ and expression, valid in $x < -x_0 < 0$, for Y^3, Y^4 which decay as $x \to -\infty$. These expressions are numerically continued to $x = 0$ where the Evans function, $E(\lambda)$ defined by

$$E(\lambda) = \det \begin{pmatrix} Y^1 & Y^2 & Y^3 & Y^4 \end{pmatrix}\Big|_{x=0}, \tag{2.21}$$

is evaluated. From the nature of the Dirichlet expansion the function Y^k for $k = 1, 2$ are linearly independent, as are Y^k for $k = 3, 4$. It is well known that the Evans function is analytic in λ away from the branches of the essential spectrum and that the zeros of E correspond up to multiplicity with the eigenvalues of L_q, see, [1]. In this way the stability diagram depicted in Figure 2.1 was obtained.

The point spectrum of L_1 enjoys a four-fold symmetry. For each $\lambda \in \sigma_p(L_q)$ the points $\overline{\lambda}$, $-2 - \lambda/\nu$, and $\overline{-2 - \lambda/\nu}$ also lie in $\sigma_p(L_q)$. The linearized operator possesses always possesses eigenvalues at $\lambda = 0, -2/\nu$, corresponding to the translational eigenvalue of the pulse and its symmetry. The essential spectrum takes the form

$$\sigma_{\text{ess}}(L) = \left\{ \lambda = \lambda_1 + i\lambda_2 \middle| \ \lambda_1 \in \left(-\nu - \nu\sqrt{\gamma^2 - a^2}, -\nu + \nu\sqrt{\gamma^2 - a^2}\right), \lambda_2 = 0 \text{ or} \right.$$
$$\left. \lambda_1 = -\nu, \lambda_2 \in \left(-\infty, -\nu\sqrt{a^2 - \gamma^2}\right) \cup \left(\nu\sqrt{a^2 - \gamma^2}, \infty\right) \right\}. \tag{2.22}$$

In Figure 2.1 (right) we plot the Hopf eigenvalues for $a \approx 2.645$. As γ increases from 1, the eigenvalue λ_H leaves the origin along the real axis and moves towards -1. A pair of eigenvalues $\lambda_E^{\pm}$ are created in an edge bifurcation from the two branch points of the essential spectrum located at $-1 \pm i\sqrt{a^2 - 1}$ and move along line $-1 + i\mathbb{R}$ towards -1. Although the branch points of the essential spectrum also move on the line $-1 + i\mathbb{R}$ towards -1 from above and below, they always trail behind $\lambda_E^{\pm}$. As γ further increases, λ_H collides with its symmetry $-2 - \lambda_H$ at $\lambda = -1$. The pair move vertically, one up and one down, along the line $-1 + i\mathbb{R}$ towards $\lambda_E^{\pm}$. The Hopf eigenvalue λ_H and its symmetry collide with $\lambda_E^{\pm}$ at a critical value of γ, leaving the line $-1 + i\mathbb{R}$ as a symmetric quartet of eigenvalues with the two right most, now denoted λ_H and $\overline{\lambda}_H$, crossing the imaginary axis as γ increases through $\gamma_c(a)$.

The translational eigenfunction is given by

$$\Psi_0(x; q) = \begin{pmatrix} \phi_q' \\ 0 \end{pmatrix} \tag{2.23}$$

while the complex conjugate Hopf eigenfunctions are denoted by $\Psi_1(x; q) = (\Psi_{1,1}, \Psi_{1,2})^t$ and $\Psi_2(x; q) = \overline{\Psi}_1$. They are depicted in Figure 2.2 and were numerically computed from the Evans function in [6]. They are oscillatory with a much slower spatial decay rate than the base pulse Ψ_0.

The adjoint of L_q is given by

$$L_q^\dagger = \begin{pmatrix} 0 & -C_q \\ D_q & -2\nu \end{pmatrix} \tag{2.24}$$

Since ϕ_q' spans the kernel of C_q it follows that

$$\Psi_0^\dagger(x; q) = \frac{1}{\Theta_0} \begin{pmatrix} 2\nu D_q^{-1} \phi_q' \\ \phi_q' \end{pmatrix} \tag{2.25}$$

is the adjoint eigenfunction which corresponds to the zero eigenvalue of L_q and Θ_0 is a normalization parameter. The adjoint Hopf-eigenfunctions $\Psi_k^\dagger$ for $k = 1, 2$ are also complex conjugates and satisfy

$$\Psi_1^\dagger(x; q) = \frac{1}{\Theta_1} \begin{pmatrix} (\lambda_1 + 2\nu)\overline{\Psi}_{1,2} \\ \lambda_1 \overline{\Psi}_{1,1} \end{pmatrix} \tag{2.26}$$

The normalization constants Θ_0 and Θ_1 are independent of q and chosen as

$$\Theta_0 \equiv -2\nu \left\langle D^{-1}\phi' \,\middle|\, \phi' \right\rangle, \tag{2.27}$$
$$\Theta_1 \equiv 2(\lambda_1 + \nu) \left\langle \Psi_{1,1} \,\middle|\, \overline{\Psi}_{1,2} \right\rangle, \tag{2.28}$$

so that the Hopf and translational eigen- and adjoint eigenfunctions form a bi-orthonormal set

$$\left\langle \Psi_j \,\middle|\, \Psi_k^\dagger \right\rangle = \delta_{jk}, \tag{2.29}$$

where δ_{jk} is the Kronecker delta. The eigenfunctions Ψ_j are simple exactly when the normalization constants Θ_j are nonzero, see Theorem 1.1 of [11]. The following Lemma is from [17],

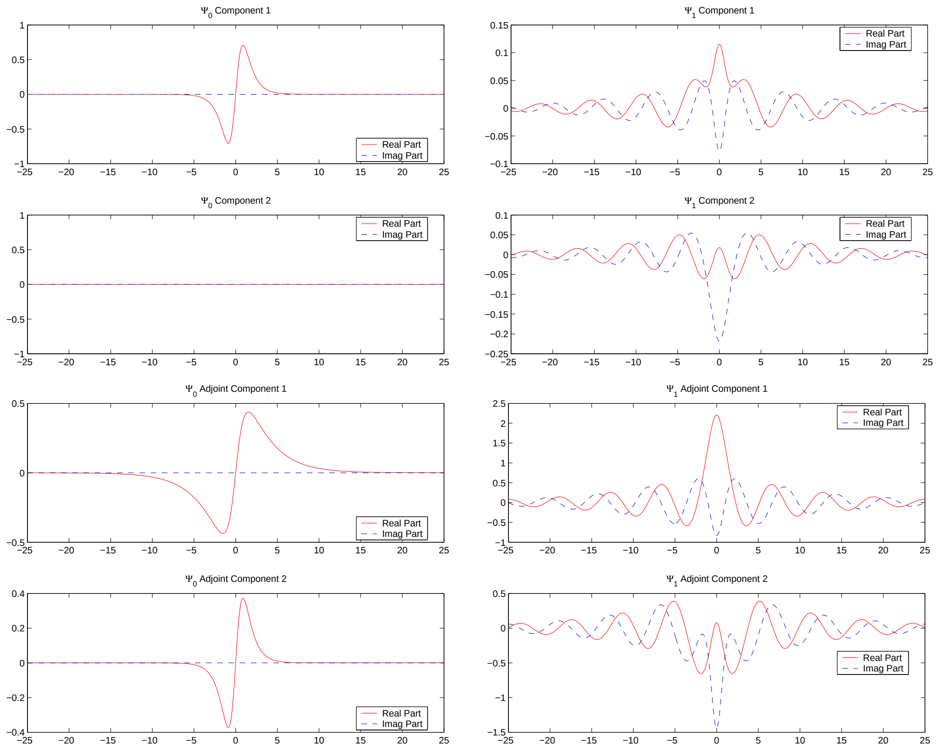

FIG. 2.2. *The eigenfunctions and adjoint eigenfunctions of L_q. The Hopf eigenfunctions decay more slowly as $|x| \to \infty$ then the pulse and the translational eigenfunctions.*

LEMMA 2.2. *If $\gamma \in \left(1, \sqrt{1+a^2}\right)$, then the translational eigenvalue $\lambda_0 = 0$ of L_q has algebraic multiplicity 1. Furthermore, $\Theta_0 < 0$.*

NUMERICAL RESULT 2.3. *For $|\gamma - \gamma_c(a)|$ sufficiently small and $a > a_c$, the Hopf eigenvalues λ_1 and λ_2 of L_q have algebraic multiplicity 1. Moreover $|\Theta_1|$ is uniformly bounded away from zero in this region.*

We define the spectral projections

$$\pi_q(\cdot) = \sum_{j=0}^{2} \left\langle \cdot \middle| \Psi_j^\dagger \right\rangle \Psi_j, \tag{2.30}$$

$$\pi_q^-(\cdot) = (I - \pi_q)(\cdot). \tag{2.31}$$

The associated eigenspaces are $X_q = \{\vec{U} \in H^1 \times H^1 | \pi_q \vec{U} = 0\}$, and its compliment $X_q^- = \{\vec{U} \in H^1 \times H^1 | \pi_q^- \vec{U} = 0\}$. The spectrum of L_q is not contained in any sector of the complex plane, and L_q generates only a C_0 semigroup $S_q(t)$. However, because $\sigma(L_q) \setminus \{\lambda_0, \lambda_1, \lambda_2\}$ is a strict subset of the left-half complex plane, the restriction of L_q to X_q^- enjoys the following estimate. See Proposition 4.1 of [17] for details.

LEMMA 2.4. *Each operator L_q generates a C_0 semigroup S_q which satisfies*

$$\|S_q(t) U\|_{H^1} \le M e^{-\nu t} \|Z\|_{H^1} \tag{2.32}$$

for some constant $M \ge 1$, for all $U \in X_q^-$, and for all $t \ge 0$. The constants M and ν are independent of $q \in \mathbb{R}$.

The following technical lemma is used in the sequel.

LEMMA 2.5. *If $\gamma \in \left(1, \sqrt{1+a^2}\right)$ then*

$$\left\langle \Psi_j' \middle| \Psi_k^\dagger \right\rangle = 0 \tag{2.33}$$

for all $j, k = 1, 2$.

Proof. The operator L_q is invariant under the reflection $x \mapsto -x$. It follows that $\Psi_j(-x)$ is also an eigenfunction of L of eigenvalue λ_j, and hence by simplicity of the Hopf eigenfunctions, $\Psi_j(-x) = \pm\Psi_j(x)$. Since the Hopf eigenfunctions are complex conjugates of each other, and the adjoint eigenfunctions are related to the eigenfunctions through (2.26) it follows that the Hopf eigenfunctions and adjoint eigenfunctions are either all even, or all odd. Since differentiation changes parity, the relations (2.33) follow. ☐

Remark: The numerical computations shown in Figure 2.2 show that the Hopf eigenfunctions are even. The translational eigenfunctions and their adjoints are clearly odd by their construction.

2.2. The Hopf Manifold . To describe the oscillations of the supercritical pulse we form a manifold parameterized by the pulse position q and the complex amplitude r of the Hopf eigenfunctions. We anticipate that the amplitude r evolves on a $\mathcal{O}(1)$ time scale, while the pulse position q evolves far more slowly. As part of the RG approach we keep track of both a fixed, base-point, value of the pulse position, $\hat{q}$, and an evolving pulse position q. The base-point value is used to define the spectral decomposition, and the higher-order elements of the manifold, while the sliding point q defines the current pulse location. When the quantity $q - \hat{q}$ becomes sufficiently large, we will renormalize, up-dating the fixed point $\hat{q}$, and the associated spectral decomposition and manifold. The evolving parameters are

$$\mathbf{p} = (p_0, p_1, p_2) = (q, r_1, r_2) \in \mathbb{R} \times \mathbb{C}^2, \tag{2.34}$$

where for convenience we have introduced $r_1 = r$ and $r_2 = \overline{r}$. We recall the Hopf manifold

$$\Phi_{\mathbf{p}}(x) = \Phi_0(x, q) + \mathcal{H}(x; \mathbf{p}), \tag{2.35}$$

and expand $\mathcal{H} = \mathcal{H}^{(1)}(x; \mathbf{p}) + \mathcal{H}^{(2)}(x; \mathbf{p}) + \mathcal{H}^{(3)}(x, \mathbf{p})$. The $\mathcal{H}^{(1)}$ terms are linear in r and their tangent plane spans the Hopf eigenspace, given explicitly as

$$\mathcal{H}^{(1)}(x; \mathbf{p}) = r_1\Psi_1(x - q) + r_2\Psi_2(x - q). \tag{2.36}$$

The higher order terms take the form

$$\mathcal{H}^{(k)}(x, \mathbf{p}) = \sum_{j_1, \ldots, j_k = 1}^{2} r_{j_1} \cdots r_{j_k} \mathcal{H}_{j_1 \cdots j_k}(x; \mathbf{p}). \tag{2.37}$$

To make $\Phi_{\mathbf{p}}$ purely real we take $\mathcal{H}_{jk}$ and $\mathcal{H}_{jkl}$ to be symmetric with respect to the interchange of any two indices, and in addition, denoting $\overline{1} = 2$ and $\overline{2} = 1$, we demand that

$$\mathcal{H}_{\overline{jk}} = \overline{\mathcal{H}_{jk}} \quad \text{and} \quad \mathcal{H}_{\overline{jkl}} = \overline{\mathcal{H}_{jkl}}. \tag{2.38}$$

Under these conditions, $\mathcal{H}_{12}$ is real since $\mathcal{H}_{12} = \mathcal{H}_{21} = \overline{\mathcal{H}_{12}}$.

The local tangent plane of the manifold $\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}$ closely resembles the Hopf+translational eigenspace X_q. Demanding that the remainder W lie in the complimentary space X_q^- we project the dynamics (2.12) onto this space and find the the remainder satisfies

$$W_t = L_q W + \pi_q^- \left(L_q\mathcal{H} + \mathcal{N}(\mathcal{H} + W) - \nabla_{\mathbf{p}}\Phi_p\dot{\mathbf{p}} \right). \tag{2.39}$$

This motivates the following definition of the residual $R = R(\mathbf{p})$

$$R \equiv L_q\mathcal{H} + \mathcal{N}(\mathcal{H}) - \nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\mathcal{F}(\mathbf{p}), \tag{2.40}$$

since if $\pi_q^- R = 0$ then $W \equiv 0$ is a solution of (2.39). The construct the higher order elements of the Hopf manifold seeks to minimize the projection of the residual. We impose the condition $\pi_q\mathcal{H}^{(2)} = \pi_q\mathcal{H}^{(3)} = 0$, and solve for the $\mathcal{H}^{(k)}$ corrections, $k = 2, 3$, so as to render $\|\pi_q^- R\|_{H^1} = \mathcal{O}(\epsilon^2)$. It is these damped components of the residual which, if unaccounted for in the manifold, would force the remainder W to grow. Eliminating these terms to $\mathcal{O}(\epsilon^2)$ affords a normal form calculation on the manifold $\mathcal{M}$ up to $\mathcal{O}(r^4)$, permitting the determination of the stability of the oscillatory modes.

The projected dynamics are expanded as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3$ where $\mathcal{F}_k = \mathcal{O}(|r|^k)$ for $k = 1, 2, 3$. The leading order term $\mathcal{F}_1$ substitutes 0 for $\dot{q}$ and $\lambda_j r_j$ for $\dot{r}_j$. This yields

$$\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_1 = \sum_{j=1}^{2} \lambda_j r_j \Psi_j + \sum_{j,k=1}^{2} \mu_{jk} r_j r_k \mathcal{H}_{jk} + \sum_{j,k,l=1}^{2} \mu_{jkl} r_j r_k r_l \mathcal{H}_{jkl}, \tag{2.41}$$

where we have introduced the resonances

$$\mu_{jk} = \lambda_j + \lambda_k \quad \text{and} \quad \mu_{jkl} = \lambda_j + \lambda_k + \lambda_l. \tag{2.42}$$

The quadratic term, given by (3.32) takes the form

$$\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_2 = \sum_{j=1}^{2} \mathcal{F}_{2j} \Psi_j + 2 \sum_{j,k=1}^{2} r_j \mathcal{F}_{2k} \mathcal{H}_{jk} + \mathcal{O}(|r|^4), \tag{2.43}$$

where $\mathcal{F}_{2j}$ is a **known** quadratic polynomial in r_1 and r_2 whose coefficients may depend upon q. The cubic term takes the form,

$$\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_3 = \sum_{j=1}^{2} \mathcal{F}_{3j} \Psi_j + \mathcal{O}(|r|^4), \tag{2.44}$$

where $\mathcal{F}_{3j}$ is a cubic polynomial in r_1 and r_2 whose coefficients depend upon $\mathcal{H}^{(2)}$ and q. Examining the linear term in (2.40) we find

$$L_q \mathcal{H} = \lambda_1 r_1 \Psi_1 + \lambda_2 r_2 \Psi_2 + \sum_{j,k=1}^{2} r_j r_k L_q \mathcal{H}_{jk} + \sum_{j,k,l=1}^{2} r_j r_k r_l L_q \mathcal{H}_{jkl}, \tag{2.45}$$

while substituting the form (2.37) of $\mathcal{H}^{(k)}$ into the nonlinearity and using (2.14) we find

$$\begin{aligned} \mathcal{N}(\mathcal{H}) = \sum_{j,k=1}^{2} r_j r_k \phi_q \mathcal{N}_2(\Psi_j, \Psi_k) + \sum_{j,k,l=1}^{2} r_j r_k r_l \Big(& \phi_q \mathcal{N}_2(\Psi_j, \mathcal{H}_{kl}) + \phi_q \mathcal{N}_2(\mathcal{H}_{jk}, \Psi_l) + \\ & + \mathcal{N}_3(\Psi_j, \Psi_k, \Psi_l) \Big) + \mathcal{O}(r^4). \end{aligned} \tag{2.46}$$

Substituting the time derivative (2.41-2.44), linear (2.45), and nonlinear (2.46) terms into the residual (2.40) and projecting with π_q^- we set the quadratic coefficients in r to zero, solving for the second order Hopf corrections

$$\mathcal{H}_{jk} = \left(\pi_q^- L_q - \mu_{jk}\right)^{-1} \Big(\left[\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_2\right]_{jk} - \phi_q \mathcal{N}_2(\Psi_j, \Psi_k) \Big), \tag{2.47}$$

where $[F]_{jk}$ denotes the coefficient of $r_j r_k$ in the polynomial F. With $\mathcal{H}_{jk}$ computed the cubic coefficients of $\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_2$ and $\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \mathcal{F}_3$ are known. The third order Hopf correction is then given by

$$\mathcal{H}_{jkl} = \left(\pi_q^- L_q - \mu_{jkl}\right)^{-1} \Big(\left[\nabla_{\mathbf{p}} \Phi_{\mathbf{p}} (\mathcal{F}_2 + \mathcal{F}_3)\right]_{jkl} - \phi_q (\mathcal{N}_2(\Psi_j, \mathcal{H}_{kl}) + \mathcal{N}_2(\mathcal{H}_{jk}, \Psi_l)) - \mathcal{N}_3(\Psi_j, \Psi_k, \Psi_l) \Big). \tag{2.48}$$

The operator $\pi_{\hat{q}}^- L_{\hat{q}} - \mu$ is boundedly invertible, uniformly in q, for $\mu \notin \big(\sigma_p^s(L) \cup \sigma_{\rm ess}(L)\big)$. Since each λ_j has positive real part, so do the resonances. We deduce that the functions $\mathcal{H}^{(k)}$ are uniformly bounded in $H^1 \times H^1$ and depend smoothly on q. Finally L_q, π_q, and π_q^- preserve evenness and oddness with respect to $x = q$. Since each Ψ_k is even with respect to $x = q$ we see that $\mathcal{H}^{(2)}$ and $\mathcal{H}^{(3)}$ are even with respect to $x = q$ as well. We have proved the following lemma,

Lemma 2.6. *With the Hopf corrections $\mathcal{H}^{(2)}$ and $\mathcal{H}^{(3)}$ given by (2.47-2.48) the residual defined by (2.40) satisfies*

$$\|\pi_q^- R\|_{H^1} \leq M |r|^4, \tag{2.49}$$

for some $M > 0$ independent of $q \in \mathbb{R}$. Moreover the corrections depend smoothly upon q, are uniformly bounded in H^1, reside in X_q^-, satisfy the symmetry conditions (2.38), and are even about the point $x = q$.

3. The Renormalization Group Method. We restrict the manifolds $\mathcal{M}$ constructed in section 2.2 to the tube $\mathcal{K}_\delta = \{\mathbf{p} \in \mathbb{R}\times\mathbb{C}^2 \big| \mathbf{p}_1 = \overline{\mathbf{p}}_2 \text{ and } |\mathbf{p}_1| \le \delta\}$. We assume at time t_0 that the initial data $\vec{U}_0$ satisfies

$$\|\Phi_{\mathbf{p}_*} - \vec{U}_0\| \le \delta, \tag{3.1}$$

for some $\mathbf{p}_* \in \mathcal{K}$. The following proposition establishes the existence of a unique nearby base point $\mathbf{p}$ satisfying the nonlinear condition (3.2) below. It is about this base point $\mathbf{p}$ that we develop our local coordinate system.

PROPOSITION 3.1. *Let $\delta > 0$ be sufficiently small but independent of ϵ. For all $\vec{U}_0$ and $\mathbf{p}_* \in \mathcal{K}$ satisfying $\|W_*\|_{H^1} \le \delta$, where $W_* \equiv \Phi_{\mathbf{p}_*} - \vec{U}_0$, there exists $M > 0$, independent of $\vec{U}_0$ and $\mathbf{p} \in \mathcal{K}$, and a smooth base-point map $\mathcal{B} : H^1 \mapsto \mathcal{K}$ such that $\mathbf{p} = \mathbf{p}_* + \mathcal{B}(W_*)$ satisfies*

$$W_0 \equiv \vec{U}_0 - \Phi_{\mathbf{p}} \in X^-_{\mathbf{p}}. \tag{3.2}$$

Moreover, if $W_* \in X_{\tilde{\mathbf{p}}}$ for some $\tilde{\mathbf{p}} \in \mathcal{K}$ then

$$|\mathbf{p} - \mathbf{p}_*| \le M_0\|W_*\|_{H^1}|q_* - \tilde{q}|. \tag{3.3}$$

Proof. Since

$$W_0 = W_* + \Phi_{\mathbf{p}} - \Phi_{\mathbf{p}_*}, \tag{3.4}$$

the condition (3.2) is equivalent to

$$0 = \pi_{\mathbf{p}} W_0 = \pi_{\mathbf{p}}\left(W_* + \Phi_{\mathbf{p}} - \Phi_{\mathbf{p}_*}\right). \tag{3.5}$$

The range of $\pi_{\mathbf{p}}$ is spanned by Ψ_0, Ψ_1, Ψ_2, the equation (3.5) is equivalent to

$$\Upsilon_k(\mathbf{p}, W_*) \equiv \left\langle W_* + \Phi_{\mathbf{p}} - \Phi_{\mathbf{p}_*} \middle| \Psi_k(\cdot,\mathbf{p})^\dagger \right\rangle = 0, \tag{3.6}$$

for $k = 0, 1, 2$. These equations have the trivial solution $\Upsilon_k(\mathbf{p}_*, 0) = 0$. From Lemma 2.6 and (2.35-2.36) we calculate that

$$\partial_q \Phi_{\mathbf{p}} = \Psi_0 + r_1\Psi_1' + r_2\Psi_2' + \mathcal{O}(r^2), \tag{3.7}$$

where we used $\Phi_0' = \Psi_0$. Imposing the symmetry conditions (2.38), the r_j derivative takes the form

$$\partial_{r_j}\Phi_{\mathbf{p}} = \Psi_j + 2\sum_{k=1}^{2} r_k\mathcal{H}_{jk} + 3\sum_{k,l=1}^{2} r_k r_l \mathcal{H}_{jkl}, \tag{3.8}$$

We define the matrix Π by

$$[\Pi]_{jk} \equiv \left\langle \partial_{p_k}\Phi_{\mathbf{p}} \middle| \Psi_j^\dagger \right\rangle, \tag{3.9}$$

and we see from (3.7-3.8) that

$$\Pi = \begin{pmatrix} 1 + \sum_{j=1}^2 r_j \left\langle \Psi_j' \middle| \Psi_0^\dagger \right\rangle & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

where we have used the orthonormality condition (2.29), even-odd parity of Lemma 2.5, and that $\mathcal{H}_{jk}$ and $\mathcal{H}_{jkl}$ lie in X_q^- and are orthogonal to $\Psi_k^\dagger$ for $k = 0, 1, 2$. Forming the vector function $\vec{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3)^t$, the $\mathbf{p}$ gradient of $\vec{\Upsilon}$ can be written as

$$\nabla_{\mathbf{p}}\vec{\Upsilon}\Big|_{(\mathbf{p}=\mathbf{p}_*, W_*=0)} = -\Pi. \tag{3.11}$$

Clearly, $\nabla_{\mathbf{p}}\vec{\Upsilon}\Big|_{(\mathbf{p}=\mathbf{p}_*,W_*=0)}$ is uniformly invertible for $\mathbf{p}\in\mathcal{K}$, for δ sufficiently small. The implicit function theorem guarantees the existence of a smooth function $\mathcal{B}$ which provides the solution of (3.2) in a neighborhood about the manifold $\mathcal{M}$ defined in (2.10). The interval of existence of $\mathcal{B}$ may be chosen uniformly in $\mathbf{p}$ since the solution of (3.2) is translationally independent.

If in addition we have $W_*\in X^-_{\tilde{\mathbf{p}}}$, then $\left\langle W_*(\cdot)\,\middle|\,\Psi_k^\dagger(\cdot,\tilde{\mathbf{p}})\right\rangle=0$ for $k=0,1,2$. We see that

$$\left|\left\langle W_*\,\middle|\,\Psi_k^\dagger(\mathbf{p}_*)\right\rangle\right|\le\left|\left\langle W_*\,\middle|\,\Psi_k^\dagger(\tilde{\mathbf{p}})-\Psi_k^\dagger(\mathbf{p}_*)\right\rangle\right|\le M_0\|W_*\|_{L^2}|q_*-\tilde{q}|, \tag{3.12}$$

since $\Psi_k^\dagger$ depends upon $\mathbf{p}$ only through $p_1=q$. This shows that $|\partial_W\vec{\Upsilon}|\le M_0\|W_*\|_{L^2}|q_*-\tilde{q}|$, so the implicit function theorem, together with the $\mathcal{O}(1)$ bound on Π^{-1}, yields (3.3). □

3.1. The Projected Equations. We freeze the slow parameter $q=\hat{q}$ in the definition of the damped eigenspace, $X_{\hat{q}}$, where $\hat{q}$ is the pulse position corresponding to the base point $\hat{\mathbf{p}}=(\hat{q},\overline{r}_1,\overline{r}_2)$ constructed in Proposition 3.1. To identify the duration of each renormalization interval, quantify the decay of the remainder W over this interval, and control the growth of the oscillation amplitudes, we introduce the quantities

$$T_W(t)=\sup_{t_0<s<t}e^{\nu(s-t_0)}\|W(s)\|_{H^1}, \tag{3.13}$$

$$T_q(t)=\sup_{t_0<s<t}|q(s)-\hat{q}|,, \tag{3.14}$$

$$T_r(t)=\sup_{t_0<s<t}|r(s)|. \tag{3.15}$$

We recall the notation $\mathbf{p}=(q(t),r_1(t),r_2(t))^t$ for the evolving parameters and change variables as

$$\vec{U}(x,t)=\Phi_{\mathbf{p}}(x)+W(x,\mathbf{p}), \tag{3.16}$$

where $W\in X^-_{\hat{q}}$. Returning to (2.12) we write the equation for the evolution of the remainder W as

$$W_t+\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}=L_{\hat{q}}W+\Delta L\;W+L_q\mathcal{H}+\mathcal{N}(\mathcal{H}+W), \tag{3.17}$$

$$W(x,0)=W_0, \tag{3.18}$$

$$\mathbf{p}(0)=\hat{\mathbf{p}}, \tag{3.19}$$

where $W_0=W_*+\Phi_{\mathbf{p}_0}-\Phi_{\mathbf{p}_*}\in X^-_{\hat{q}}$, and $\Delta L\;\equiv L_q-L_{\hat{q}}$ is the secularity implicit in the sliding of $\mathbf{p}$ away from $\hat{\mathbf{p}}$.

To enforce $W\in X^-_{\hat{q}}$ we impose the non-degeneracy condition $\pi_{\hat{q}}W_t=0$. Since $\pi_{\hat{q}}$ commutes with $L_{\hat{q}}$ it follows that $\pi^-_{\hat{q}}L_{\hat{q}}W=L_{\hat{q}}\pi^-_{\hat{q}}W=0$, while $\pi_qL_q\mathcal{H}=L_q\mathcal{H}^{(1)}$. The non-degeneracy condition is equivalent to the equations

$$\pi_{\hat{q}}\Big(\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}-L_qH-\Delta L\;W+\mathcal{N}(H+W)\Big)=0, \tag{3.20}$$

where by slight abuse of notation we view π as mapping into $\mathbb{R}^3$ by reading off the coefficients of Ψ_k for $k=0,1,2$. Taking the equation term by term, we address first the dynamic term which we rewrite as

$$\Big[\pi_{\hat{q}}\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}\Big]_k=\left\langle\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}\,\middle|\,\Psi_k^\dagger(q)\right\rangle+\left\langle\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}\,\middle|\,\Psi_k^\dagger(\hat{q})-\Psi_k^\dagger(q)\right\rangle, \tag{3.21}$$

so that

$$\pi_{\hat{q}}\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\dot{\mathbf{p}}=(\Pi+(\hat{q}-q)\Pi')\,\dot{\mathbf{p}}+\mathcal{O}(T_q^2|\dot{\mathbf{p}}|), \tag{3.22}$$

where Π given by (3.10) is independent of base-point and

$$[\Pi']_{jk}=\left\langle\partial_{p_k}\Phi_{\mathbf{p}}\,\middle|\,\left(\Psi_j^\dagger\right)'(q)\right\rangle. \tag{3.23}$$

Using the parity of the eigenfunctions, and the expressions (3.7-3.8) we see that

$$\Pi' = \begin{pmatrix} \mathcal{O}(r^2) & \left\langle \Psi_1 \middle| \Psi_0^{\dagger\prime} \right\rangle + \mathcal{O}(r) & \left\langle \Psi_1 \middle| \Psi_0^{\dagger\prime} \right\rangle + \mathcal{O}(r) \\ \left\langle \Psi_0 \middle| \Psi_1^{\dagger\prime} \right\rangle + \mathcal{O}(r^3) & \mathcal{O}(r) & \mathcal{O}(r) \\ \left\langle \Psi_0 \middle| \Psi_2^{\dagger\prime} \right\rangle + \mathcal{O}(r^3) & \mathcal{O}(r) & \mathcal{O}(r) \end{pmatrix} \tag{3.24}$$

The linear term gives the leading order dynamics on the tangent plane of the manifold,

$$\left[\pi_{\hat{q}} L_q \mathcal{H}\right]_k = \left\langle L_q \mathcal{H} \middle| \Psi_k^\dagger(q) \right\rangle + (\hat{q} - q) \left\langle L_q \mathcal{H} \middle| \left(\Psi_k^\dagger(q)\right)' \right\rangle + \mathcal{O}(T_q^2). \tag{3.25}$$

Writing in the vector notation, and using the identities $\pi_q \mathcal{H}^{(2)} = \pi_q \mathcal{H}^{(3)} = 0$ and the symmetries of Lemma 2.6, we find

$$\pi_{\hat{q}} L_q \mathcal{H} = \begin{pmatrix} 0 \\ \lambda_1 r_1 \\ \lambda_2 r_2 \end{pmatrix} + (\hat{q} - q) \begin{pmatrix} \sum_{k=1}^2 \lambda_k r_k \left\langle \Psi_k \middle| \Psi_0^{\dagger\prime} \right\rangle \\ \mathcal{O}(r^2) \\ \mathcal{O}(r^2) \end{pmatrix} + \mathcal{O}(T_q^2). \tag{3.26}$$

Since ΔL is the difference of two smooth potentials evaluated at $x-q$ and $x-\hat{q}$, the secular term is estimated as

$$|\pi_{\hat{q}} \Delta L \; W| \leq c T_q \|W\|_{H^1}. \tag{3.27}$$

Finally the nonlinearity takes the form

$$\mathcal{N}(\mathcal{H} + W) = \phi_q \mathcal{N}_2(\mathcal{H}^{(1)}) + \mathcal{N}_3(\mathcal{H}^{(1)}) + \phi_q \mathcal{N}_2^s(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}) + \mathcal{O}(r^4, r\|W\|_{H^1}) \tag{3.28}$$

From Lemma 2.6 we know that $\mathcal{N}(\mathcal{H})$ is even about $x = q$ while $\Psi_0(\hat{q})$ is odd, so at leading order the first component vanishes, we find

$$\pi_{\hat{q}} \mathcal{N}(\mathcal{H} + W) = \begin{pmatrix} \mathcal{O}(r\|W\|_{H^1}, T_q r^2) \\ \left\langle \phi_q \mathcal{N}_2(\mathcal{H}^{(1)}) + \mathcal{N}_3(\mathcal{H}^{(1)}) + \phi_q \mathcal{N}_2^s(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}) \middle| \Psi_1^\dagger(q) \right\rangle + \mathcal{O}(r^4, r\|W\|_{H^1}, r^2 T_q) \\ \left\langle \phi_q \mathcal{N}_2(\mathcal{H}^{(1)}) + \mathcal{N}_3(\mathcal{H}^{(1)}) + \phi_q \mathcal{N}_2^s(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}) \middle| \Psi_2^\dagger(q) \right\rangle + \mathcal{O}(r^4, r\|W\|_{H^1}, r^2 T_q) \end{pmatrix}. \tag{3.29}$$

Combining the term-wise reductions of the non-degeneracy equation (3.20), and inverting the matrix Π, we obtain the leading order pulse parameter evolution

$$\dot{\mathbf{p}} = \mathcal{F}(\mathbf{p}) + \begin{pmatrix} \mathcal{O}\big(|r|T_q + (|r| + T_q)\|W\|_{H^1}\big) \\ \mathcal{O}\big(|r|^2 T_q + (T_q + |r|)\|W\|_{H^1} + |r|^4\big) \\ \mathcal{O}\big(|r|^2 T_q + (T_q + |r|)\|W\|_{H^1} + |r|^4\big) \end{pmatrix}, \tag{3.30}$$

$$\mathbf{p}(0) = \hat{\mathbf{p}}, \tag{3.31}$$

where the leading order projected vector field $\mathcal{F}$, broken down into terms that are linear, quadratic, and cubic in r, is given by

$$\mathcal{F} = \begin{pmatrix} 0 \\ \lambda_1 r_1 \\ \lambda_2 r_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \left\langle \phi_q \mathcal{N}_2(\mathcal{H}^{(1)}) \middle| \Psi_1^\dagger(q) \right\rangle \\ \left\langle \phi_q \mathcal{N}_2(\mathcal{H}^{(1)}) \middle| \Psi_2^\dagger(q) \right\rangle \end{pmatrix} + \begin{pmatrix} 0 \\ \left\langle \mathcal{N}_3(\mathcal{H}^{(1)}) + \phi_q \mathcal{N}_2^s(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}) \middle| \Psi_1^\dagger(q) \right\rangle \\ \left\langle \mathcal{N}_3(\mathcal{H}^{(1)}) + \phi_q \mathcal{N}_2^s(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}) \middle| \Psi_2^\dagger(q) \right\rangle \end{pmatrix}. \tag{3.32}$$

We denote the linear, quadratic, and cubic terms by $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ respectively.

3.2. Normal form for the Poincaré-Hopf Bifurcation. Introducing the quantities

$$\gamma_{jkl} \equiv \left\langle \phi \mathcal{N}_2(\Psi_k, \Psi_l) \middle| \Psi_j^\dagger \right\rangle, \tag{3.33}$$

$$\gamma_{jklm} \equiv \left\langle \phi \mathcal{N}_2(\Psi_k, \mathcal{H}_{lm}) + \mathcal{N}_3(\Psi_k, \Psi_l, \Psi_m) \middle| \Psi_j^\dagger \right\rangle, \tag{3.34}$$

we can rewrite the projected evolution (3.30) as

$$\dot{r}_j = \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m + O\left(T_r^4, T_r^2 T_q, \|W\|_{H^1} T_r\right), \tag{3.35}$$

Defining

$$\begin{array}{rclcrcl} h_{20} & = & 2\gamma_{111}, & & h_{11} & = & \gamma_{121} + \gamma_{211}, \\ h_{02} & = & 2\gamma_{221}, & & h_{30} & = & 6\gamma_{1111}, \\ h_{21} & = & 2\left(\gamma_{1121} + \gamma_{1211} + \gamma_{2111}\right), & & h_{12} & = & 2\left(\gamma_{1221} + \gamma_{2121} + \gamma_{2211}\right), \\ h_{03} & = & 6\gamma_{2221}, & & & & \end{array} \tag{3.36}$$

we rewrite (3.30) as

$$\dot{r} = \lambda r + \sum_{2 \le k+l \le 3} \frac{h_{kl}}{k!l!} r^k \overline{r}^l + O\left(r^4\right), \tag{3.37}$$

where $\lambda = \lambda_1$ is the Hopf eigenvalue. We now apply the normal form, see Lemma 3.6 from [13], which we restate below, to transform the evolution equation (3.37) for r into the Poincaré normal form for the Poincaré-Hopf bifurcation.

LEMMA 3.2. *The equation (3.37) with $\lambda = \lambda(\gamma)$, $\Re\lambda(\gamma_c) = 0$, $\Im\lambda(\gamma_c) > 0$, and $h_{kl} = h_{kl}(\gamma)$, can be transformed by the invertible parameter-dependent change of complex coordinate*

$$r = v + \sum_{2 \le k+l \le 3} \frac{j_{kl}}{k!l!} v^k \overline{v}^l, \tag{3.38}$$

with $j_{21} = 0$ into an equation with only the resonant cubic term

$$\dot{v} = \lambda v + \eta |v|^2 v + \mathcal{O}(v^4), \tag{3.39}$$

for all γ such that $|\gamma - \gamma_c|$ is sufficiently small. Furthermore,

$$\eta(\gamma) = \frac{h_{20} h_{11}\left(2\lambda + \overline{\lambda}\right)}{2|\lambda|^2} + \frac{|h_{11}|^2}{\lambda} + \frac{|h_{02}|^2}{2\left(2\lambda - \overline{\lambda}\right)} + \frac{h_{21}}{2}, \tag{3.40}$$

which, at the critical bifurcation value γ_c, reduces to

$$\eta_c(a) = \eta\left(\gamma_c(a)\right) = \frac{i}{2\Im\lambda(\gamma_c)}\left(h_{20}h_{11} - 2|h_{11}|^2 - \frac{1}{3}|h_{02}|^2\right) + \frac{h_{21}}{2}. \tag{3.41}$$

If $\Re\eta < 0$, the Poincaré-Hopf bifurcation is supercritical, if $\Re\eta > 0$ the Poincaré-Hopf bifurcation is subcritical.

NUMERICAL RESULT 3.3. *The formal projected dynamics,*

$$\dot{\mathbf{p}} = \mathcal{F}(p; \gamma), \tag{3.42}$$

experience a super-critical Poincaré-Hopf bifurcation as γ increases through $\gamma_c(a)$ for $a > a_c$, where a_c is introduced in Numerical Result 2.1.

Discussion: The value of the coefficient η_c of the cubic term of (3.39 evaluated along the Hopf bifurcation curve $\gamma = \gamma_c(a)$, is given by (3.41). This was evaluated numerically along the Hopf bifurcation curve for $a \in (2.8, 4.9)$. The most technical aspect of the computation was solving for the Hopf eigenfunction Ψ_1 and the correction terms $\mathcal{H}_{jk}$ and $\mathcal{H}_{jkl}$. As outlined in Numerical Result 2.1 the Hopf eigenfunction was computed using a Dirichlet expansion on the stable manifold of the associated linearized eigenvalue problem. Moreover the eigenfunction Ψ_1 can be recovered from the kernel of the Evans function E at $\lambda = \lambda_1$. Specifically the kernel serves as an initial condition for Ψ_1 at $x = 0$ for the first order evolution equation (2.20), see [6] for details. On the other hand, the computation of the correction terms involved solving ODEs of the type

$$\left(L_q - \mu\right)\mathcal{H} = Z(y), \tag{3.43}$$

where μ is the resonance. This was done using Matlab's two-point boundary value problem solver *bvp4c* with boundary conditions $\mathcal{H} = 0$ and $\mathcal{H}' = 0$ at some sufficiently large y. The results are shown in Figure 3.1, which shows that $\Re\eta\left(\gamma_c(a)\right) < 0$ and the Poincaré-Hopf bifurcation is supercritical. ∎

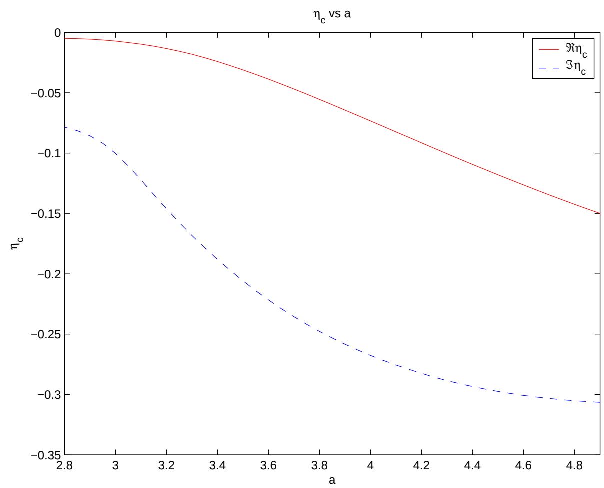

FIG. 3.1. *Hopf bifurcation constant η versus detuning parameter a along Hopf bifurcation curve.*

3.3. Evolution of the Remainder. With the parameter evolution given by (3.30), which is equivalent to the nondegeneracy condition (3.20), the evolution for the remainder $W \in X_{\hat{q}}^-$ is given by

$$W_t = L_{\hat{q}} W + \pi_{\hat{q}}^- \left(\Delta L\ W + L_q \mathcal{H} + \mathcal{N}(\mathcal{H} + W) - \nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \dot{\mathbf{p}} \right), \tag{3.44}$$

$$W(x, 0) = W_0, \tag{3.45}$$

where W_0 and $\hat{q}$ are given by (3.2). Motivated by the definition of the residual R, given in (2.40), we write

$$L_q \mathcal{H} + \mathcal{N}(\mathcal{H} + W) - \nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \dot{\mathbf{p}} = R + \mathcal{N}(\mathcal{H} + W) - \mathcal{N}(\mathcal{H}) - \nabla_{\mathbf{p}} \Phi_{\mathbf{p}} \big(\dot{\mathbf{p}} - \mathcal{F}(r) \big), \tag{3.46}$$

where $\mathcal{F}$ is given by (3.32). For the nonlinearity we estimate

$$\| \mathcal{N}(\mathcal{H} + W) - \mathcal{N}(\mathcal{H}) \| \leq c \left(|r| \|W\|_{H^1} + \|W\|_{H^1}^2 \right), \tag{3.47}$$

while from (3.30) we see that the projected vector field, less its leading order terms, satisfies

$$|\dot{\mathbf{p}} - \mathcal{F}(r)| \leq c \left(|r|^4 + |r| \|W\|_{H^1} + T_q (|r| + \|W\|_{H^1}) \right). \tag{3.48}$$

From Lemma 2.6 the residual satisfies

$$\|\pi_q R\|_{H^1} \leq c |r|^4. \tag{3.49}$$

We observe that

$$\|(\pi_q^- - \pi_{\hat{q}}^-) F\|_{H^1} \leq c T_q \|F\|_{H^1}, \tag{3.50}$$

and hence

$$\|\pi_{\hat{q}} R\|_{H^1} \leq \|\pi_q^- R\|_{H^1} + c T_q \|R\|_{H^1} \leq c \left(|r|^4 + |r| T_q \right). \tag{3.51}$$

Since the differential terms cancel in the secular operator ΔL, it acts as a multiplier operator with a smooth potential which is $\mathcal{O}(T_q)$ in H^1. The secular term satisfies the simple estimate

$$\|\pi_{\hat{q}}^- \Delta L\ W\|_{H^1} \leq c T_q \|W\|_{H^1}. \tag{3.52}$$

Returning to (3.44) we write it as

$$W_t = L_{\hat{q}} W + \mathcal{G}, \tag{3.53}$$

where $\mathcal{G} \in X_{\hat{q}}^-$ satisfies

$$\|\mathcal{G}\|_{H^1} \leq c \left(|r|^4 + |r| T_q + (|r| + T_q) \|W\|_{H^1} + \|W\|_{H^1}^2 \right). \tag{3.54}$$

To estimate the distance T_q that the pulse has moved from its base point we examine the projected evolution equation (3.30). The drift of the pulses is controlled by their speed,

$$T_q(t) \leq \int_{t_0}^{t} |\dot{\mathbf{p}}_1(s)| ds, \tag{3.55}$$

$$\leq \int_{t_0}^{t} c\left((T_r(s) + \|W(s)\|_{H^1})T_q(s) + \|W(s)\|_{H^1} T_r(s)\right) ds, \tag{3.56}$$

$$\leq \int_{t_0}^{t} c\left((T_r(t) + e^{-\nu(s-t_0)}T_W(t))T_q(s) + e^{-\nu(s-t_0)}T_W(t)T_r(t)\right) ds, \tag{3.57}$$

where we have introduced $\Delta t = t - t_0$. From the integral Gronwall estimate we see that

$$T_q(t) \leq c e^{\Delta t T_r(t) + T_W(t)} T_W(t) T_r(t). \tag{3.58}$$

For $\Delta t \leq 1/T_r(t)$ and $T_W \leq 1$ we may replace this estimate with

$$T_q(t) \leq c T_W(t) T_r(t). \tag{3.59}$$

The variation of constants formula applied to (3.53) yields the solution

$$W(x,t) = S(\Delta t)W_0 + \int_{t_0}^{t} S(t-s)\mathcal{G}(s)\, ds, \tag{3.60}$$

Taking the H^1-norm of variation of constants solution for W, (3.60), and using the estimates (3.54) and (3.59) we obtain

$$\|W(t)\|_{H^1} \leq M\left(e^{-\nu\Delta t}\|W(t_0)\|_{H^1} + \int_{t_0}^{t} e^{-\nu(t-s)}\left[T_r^4 + T_r^2 T_W + T_r\|W\|_{H^1} + \|W\|_{H^1}^2\right](s)\, ds\right). \tag{3.61}$$

We denote by $\tau \equiv t_0 + \Delta t$ the as yet undetermined time at which we update the base point from $\hat{q}_0$ to $\hat{q}_1$. To estimate the decay of $\|W\|_{H^1}$ over this time period we multiply (3.61) by $e^{\nu(t-t_0)}$. Taking the sup over $t \in (t_0, t_0+\tau)$ yields

$$T_W(\tau) \leq M\left(T_W(t_0) + \int_{t_0}^{\tau}\left[e^{\nu(s-t_0)}\left(T_r(\tau)^4 + T_r^2 T_W(\tau)\right) + T_r T_W(s) + e^{-\nu(s-t_0)}T_W^2(s)\right] ds\right), \tag{3.62}$$

$$\leq M\Big(T_W(t_0) + e^{\nu\Delta t}T_r^4(\tau) + \left(e^{\nu\Delta t}T_r^2(\tau) + \Delta t T_r(\tau)\right)T_W(\tau) + T_W^2(\tau)\Big). \tag{3.63}$$

For $\Delta t < \frac{|\ln 4MT_r^2(\tau)|}{\nu}$ the term $M\left(e^{\nu\Delta t}T_r^2(\tau) + \Delta t T_r(\tau)\right) < \frac{1}{2}$ and we may eliminate the linear term in T_W from the right-hand side. With this reduction (3.63) becomes

$$T_W \leq 2M\Big(T_W(t_0) + e^{\nu\Delta t}T_r^4(\tau) + T_W^2\Big). \tag{3.64}$$

The quadratic equation in T_W

$$0 = T_W(t_0) + e^{\nu\Delta t}T_r^4 - \frac{1}{2M}T_W + T_W^2, \tag{3.65}$$

has two positive real roots so long as $T_W(t_0) + e^{\nu\Delta t}T_r^4 \ll 1$. The smaller of these roots, ξ_0 takes the form

$$\xi_0 = 2M\left(T_W(t_0) + e^{\nu\Delta t}T_r^4)\right) + \mathcal{O}\Big(T_W(t_0) + e^{\nu\Delta t}T_r^4\Big)^2, \tag{3.66}$$

while the larger is

$$\xi_1 = \frac{1}{2M} + \mathcal{O}\Big(T_W(t_0) + e^{\nu\Delta t}T_r^4\Big). \tag{3.67}$$

Thus if $T_W(t_0) \ll 1$ and $e^{\nu\Delta t}T_r^4(\tau) \ll 1$ then there is an **excluded region**, either $0 < T_W < \xi_0$ or $\xi_1 < T_W < \infty$. Since $T_W(t_0) < \xi_0$ and T_W is continuous in t, we see that

$$T_W(\tau) \le \xi_0 \le M(T_W(t_0) + e^{\nu\Delta t}T_r^4(\tau)) \tag{3.68}$$

so long as

$$\Delta t \le \frac{4\beta|\ln T_r|}{\nu} \tag{3.69}$$

for any fixed $\beta < 1$. This condition on Δt prevents the secularity from dominating the linear operator, in particular it is a stronger condition on Δt than that imposed after equation (3.63) so long as $2\beta > 1$ and T_r is small enough. This implies that

$$\|W(\cdot,t)\|_{H^1} \le M\Big(e^{-\nu(t-t_0)}\|W(\cdot,t_0)\|_{H^1} + T_r^4\Big), \qquad \text{for } t \in \left(t_0, t_0 + \frac{4\beta|\ln T_r(\tau)|}{\nu}\right) \tag{3.70}$$

and in particular for $t_1 = t_0 + \Delta t$,

$$\|W(\cdot,t_1)\|_{H^1} \le M\Big(T_r^{4\beta}\|W(\cdot,t_0)\|_{H^1} + T_r^4\Big). \tag{3.71}$$

Turning to the oscillation amplitude r, we see from Lemma 3.2 that $|v| = |r| + \mathcal{O}(|r|^2)$. By abuse of notation we replace v with r which satisfies

$$\dot{r} = \lambda r + \eta|r|^2 r + \mathcal{O}\left(|r|^2 T_q + (|r| + T_q)\|W\|_{H^1} + |r|^4\right). \tag{3.72}$$

We eliminate T_q using (3.59) and take the real part of the dot product with $\overline{r}$ to find

$$\frac{1}{2}\partial_t|r|^2 = \epsilon|r|^2 + \Re\eta|r|^4 + \mathcal{O}\Big(|r|^3 T_r T_W + |r|(|r| + T_r T_W)\|W\|_{H^1} + |r|^5\Big), \tag{3.73}$$

where $\Re\eta < 0$ is uniformly $o(1)$. Introducing $\rho = |r|^2 + \frac{\epsilon}{\Re\eta}$, we rewrite (3.73) as

$$\frac{1}{2}\dot{\rho} = -\epsilon\rho + \Re\eta\rho^2 + \mathcal{O}\Big(|r|^3 T_r T_W + |r|(|r| + T_r T_W)\|W\|_{H^1} + |r|^5\Big). \tag{3.74}$$

The time scale for the evolution of ρ, $\dot{\rho}/\rho$ is $\mathcal{O}(\epsilon + \Re\eta|r|^2 + \|W\|_{H^1})$ while $\Delta t = \mathcal{O}(|\ln T_r|)$ so that over the length of one renormalization period ρ is constant at leading order, and we replace T_r with r in (3.74), obtaining

$$\frac{1}{2}\dot{\rho} = -\epsilon\rho + \Re\eta\rho^2 + \mathcal{O}\Big(|r|^4(T_W + |r|) + |r|^2\|W\|_{H^1}). \tag{3.75}$$

We observe that if $|r(t_0)| > 2\sqrt{-\frac{\epsilon}{\Re\eta}}$ then $\rho > \frac{1}{2}|r|^2$ and for T_W and $|r|$ are sufficiently small, independent of ϵ, the first term in the error is dominated by $\Re\eta\rho^2 < 0$, yielding

$$\dot{\rho} \le -2\epsilon\rho + \frac{3}{2}\Re\eta\rho^2 + \mathcal{O}\Big(|r|^2\|W\|_{H^1}\Big). \tag{3.76}$$

For $|r|$ and $\|W\|_{H^1}$ small compared to $1/|\ln T_r|$ and 1, but $\|W(t_0)\|_{H^1} > |r|^{3-\beta}$, then the error term on the right-hand side of (3.76) may dominate and we have the estimate

$$|r(\tau)|^2 \le |r(t_0)|^2 + c\int_{t_0}^{\tau} e^{-\nu(s-t_0)}|r(s)|^2 T_W(\tau)ds, \tag{3.77}$$

which simplifies to

$$|r(\tau)| \leq |r(t_0)| + c|r(t_0)|^2. \tag{3.78}$$

However if $|r(t_0)| > 2\sqrt{-\frac{\epsilon}{\Re\eta}}$ but $\|W(t_0)\|_{H^1} < |r|^{3-\beta}$, then the error term on the right-hand side of (3.76) is dominated and we have

$$\dot{|r|} \leq -\left(\epsilon - \frac{\Re\eta}{2}|r|^2\right)|r|, \tag{3.79}$$

and hence

$$|r(\tau)| \leq |r(t_0)| e^{-\frac{4\beta}{\nu}\left|\ln|r(t_0)|\right|\left(\epsilon - \frac{1}{2}\Re\eta|r(t_0)|^2\right)}. \tag{3.80}$$

Finally if $|r(t_0)| < 2\sqrt{-\frac{\epsilon}{\Re\eta}}$ then ρ does not control $|r|$ effectively, and we obtain the result (3.78).

3.4. The RG Map. We break the time evolution into a series of initial value problems, tracking the decay of the remainder and the over the long-time scale of many RG iterations. We fix $\beta < 1$ and define the renormalization times sequentially

$$t_{n+1} = t_n + \frac{4\beta|\log r_n|}{\nu}, \tag{3.81}$$

where $r_n = |r(t_n)|$. We break the evolution of W and r into disjoint intervals $I_n = [t_n, t_{n+1})$. On each interval I_n we solve the initial value problems (3.44-3.45) and (3.30-3.31) with initial data $W(t_n) \in X^-_{q_n}$ and $\mathbf{p}_n$. The quantities $T_W(n), T_r(n)$, and $T_q(n)$ corresponding to (3.13-3.15) over I_n. The renormalization map takes the initial data, $\Big(W(t_{n-1}), \mathbf{p}_{n-1}\Big)^t$ for the initial value problem on interval I_{n-1} and returns the initial data $\Big(W(t_n), \mathbf{p}_n\Big)^t$ for the initial value problem on the interval I_n,

$$\mathcal{R}\begin{pmatrix} W(t_{n-1}) \\ \mathbf{p}_{n-1} \end{pmatrix} = \begin{pmatrix} W(t_n) \\ \mathbf{p}_n \end{pmatrix}. \tag{3.82}$$

Since both W_0 and r are initially small, we may choose β sufficiently close to 1 so that

$$\|W_0\|_{H^1} \leq |r(t_0)|^{4-4\beta}. \tag{3.83}$$

Arguing inductively, we assume that (3.83) holds for all n where the initial data $\mathbf{p}(t_n) = (q(t_n), r_1(t_n), r_2(t_n))$ and the new base point $\hat{q}_n = \mathbf{p}_1(t_n)$ are obtained from $W(t_n^-)$ and $\mathbf{p}(t_n^-)$, the end-values of the evolution of W and $\mathbf{p}$ over I_{n-1}, by applying Proposition 4.1. Indeed we see that $W(t_n^-) \in X^-_{q_{n-1}}$ and so from (3.3) we have

$$\|\mathbf{p}_n - \mathbf{p}(t_n^-)| \leq M_0\|W(t_n^-)\|_{H^1}|\mathbf{p}(t_n^-) - \mathbf{p}(t_{n-1})| \leq M_0\|W(t_n^-)\|_{H^1} T_q(n-1). \tag{3.84}$$

From the inductive assumption (3.83) we see that the bound (3.70) on T_W implies that

$$T_W(n) \leq M(\|W_{n-1}\|_{H^1} + T_r^{4-4\beta}) \leq M T_r^{4-4\beta}(n). \tag{3.85}$$

From the estimate (3.59) on $T_q(n-1)$, (3.71) on W, and (3.85) on T_W, we bound the jump in $\mathbf{p}$ at renormalization by

$$|\mathbf{p}_n - \mathbf{p}(t_n^-)| \leq M T_W(n-1) T_r(n-1)\|W(t_n^-)\|_{H^1}, \tag{3.86}$$

$$\leq M T_r^{4(1-\beta)} T_r\left(T_r^{4\beta}\|W_{n-1}\|_{H^1} + T_r^4\right), \tag{3.87}$$

$$\leq M T_r^5(n-1). \tag{3.88}$$

The solution at time $t = t_n$ is independent of the decomposition,

$$\vec{U}(t_n) = \Phi_{\mathbf{p}(t_n^-)} + W(t_n^-) = \Phi_{\mathbf{p}_n} + W_n, \tag{3.89}$$

and we may bound the jump in W at each renormalization

$$\|W(t_n^-) - W(t_n)\|_{H^1} = \|\Phi_{\mathbf{p}(t_n^-)} - \Phi_{\mathbf{p}_n}\|_{H^1} \leq c|\mathbf{p}_n - \mathbf{p}(t_n^-)| \leq MT_r^5(n-1). \tag{3.90}$$

Combining the estimates (3.88) and (3.70), we obtain a bound on $W_n = W(t_n)$,

$$\|W_n\|_{H^1} \leq \|W(t_n^-) - W_n\|_{H^1} + \|W(t_n^-)\|_{H^1} \leq M\left(T_r^5 + T_r^{4\beta}\|W_{n-1}\|_{H^1} + T_r^4\right) \leq MT_r^4, \tag{3.91}$$

which amply verifies the inductive hypothesis.

Turning again to the oscillation amplitudes $r_n = |r(t_n)|$ we see from (3.90) that for all $n > 0$ we have $\|W_n\|_{H^1} \leq Mr_n^4$ and hence from (3.80), so long as $r_n > 2\sqrt{-\frac{\epsilon}{\Re\eta}}$,

$$r_n \leq r_{n-1}^{1-\frac{4\beta}{\nu}\left(2\epsilon - \frac{\Re\eta}{2}r_{n-1}^2\right)}. \tag{3.92}$$

Introducing $\xi(t)$ a continuous time upper bound for r_n we see that

$$\frac{\xi(t_n) - \xi(t_{n-1})}{t_n - t_{n-1}} = -\frac{\frac{4\beta}{\nu}\left(\epsilon - \frac{\Re\eta}{2}\xi^2(t_{n-1})\right)}{\frac{4\beta}{\nu}|\ln\xi(t_n)|}\xi(t_{n-1}), \tag{3.93}$$

which suggests the RG equation for an upper bound on the amplitude

$$\xi'(t) = -\frac{\left(\epsilon - \frac{\Re\eta}{2}\xi^2\right)}{|\ln\xi|}\xi. \tag{3.94}$$

So that either $r_n \leq \xi(t_n)$ or $r_n \leq 2\sqrt{-\frac{\epsilon}{\Re\eta}}$. In particular there exists an $M > 0$ such that

$$|r(t)| \leq M\left(e^{-\epsilon(t-t_0)}|r_0| + \sqrt{\epsilon}\right), \tag{3.95}$$

which yields the results (1.3-1.4) in Theorem 1.1.

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