NONLINEAR ASYMPTOTIC STABILITY OF THE SEMI-STRONG PULSE DYNAMICS IN A REGULARIZED GIERER-MEINHARDT MODEL

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Abstract. We use renormalization group (RG) techniques to prove the nonlinear asymptotic stability for the semi-strong regime of two-pulse interactions in a regularized Gierer-Meinhardt system. In the semi-strong limit the localized activator pulses interact strongly through the slowly varying inhibitor. The interaction is not tail-tail as in the weak interaction limit, and the pulse amplitudes and speeds change as the pulse separation evolves on algebraically slow time scales. In addition the point spectrum of the associated linearized operator evolves with the pulse dynamics. The RG approach employed here validates the interaction laws of quasi-steady two-pulse patterns obtained formally in the literature, and establishes that the pulse dynamics reduce to a closed system of ordinary differential equations for the activator pulse locations. Moreover, we fully justify the reduction to the nonlocal eigenvalue problem (NLEP) showing that the large difference between the quasi-steady NLEP operator and the operator arising from linearization about the pulse is controlled by the resolvent.

Key words. Gierer-Meinhardt, semi-strong interaction, renormalization group, geometric singular perturbation, nonlocal eigenvalue problem (NLEP)

1. Introduction. Pulse solutions are the building blocks for the analysis of complex patterns in reaction-diffusion equations. Within the proper scaling limit, the dynamics exhibited by many reaction-diffusion systems are governed by the interactions of localized solutions of pulse type. A prototypical example is given by the spatio-temporal chaotic dynamics of the one-dimensional Gray-Scott system for which numerical simulations indicate that the chaotic dynamics originate from the interactions and bifurcations of pulse solutions [13].

In the context of singularly perturbed equations in one spatial dimension, there is a well-developed literature addressing the existence and stability of stationary pulse solutions based on the geometric singular perturbation theory and the Evans function methods (see [15, 5] and the references therein). There is no such general theory for pulse interactions. In fact, strong pulse interactions, and especially the phenomena of pulse-replication and annihilation, have been studied computationally, but are not yet understood mathematically. On the other hand, there are methods to study the behavior of pulses in the weak interaction limit where the pulses are so greatly separated that they can be considered at leading order as copies of a solitary pulse. In this regime the exponentially weak interactions only affect the position of the pulses and has no leading order influence on their shape or stability (see [8, 9, 14, 15] and the references therein).

Recently, an intermediate concept has been introduced in the context of singularly perturbed equations, the semi-strong interaction case (see [6, 16] and the references therein). The semi-strong regime exists in systems whose components decay at asymptotically distinct rates, so that some of the components of the system approach the trivial background state between pulses, while others do not. Moreover, the pulse positions, amplitudes, and shapes change at rates that are algebraically small in the perturbation parameter, see Figure 1.1, and may bifurcate due to the interactions [6, 16].

Up to now the semi-strong pulse interaction has only been studied formally (see Remark 1.3). In this paper we show that the semi-strong interaction fits naturally into the framework of the renormalization group (RG) methods developed to study the stability of slowly evolving patterns [14, 12]. For the Gierer-Meinhardt equations, the geometric singular perturbation theory shows that the activator-inhibitor interaction reduces the highly diffusive inhibitor to a local constant within each activator pulse. The value of this constant determines the activator pulse interactions, in particular causing the pulse amplitudes to depend upon the pulse positions, which evolve on an $\mathcal{O}(\frac{1}{\varepsilon^4})$ time scale. The RG analysis makes these statements rigorous, in particular fully justifying the reductions made in the non-local eigenvalue problem (NLEP) analysis which arises in the linear stability analysis of the pulses.

The singular perturbation theory typically constructs a family of pulse type patterns which are approximate solutions of a given system of equations [2, 3, 6, 11, 16]. The solutions are characterized by parameters $\vec{p} \in \mathbb{R}^k$, which are often – but not exclusively – pulse locations. The linearizing about the global manifold of slowly evolving pulse patterns for a particular choice of parameters $\vec{p}$, allows one to decompose the phase space into tangential (or active) and normal (or decaying) modes. In the RG approach, rather than using the exact linearization, reduced linearized operators are identified at a discrete family of base points on the manifold. These form a loose covering of approximate tangent planes, much like the scales of a fish form a piece-wise linear envelope of the underlying body. In the current setting this gives two specific advantages: first, in a neighborhood of each base point we identify a temporally constant linearized operator and associated phase space decomposition, and second we are free to modify the governing linear operator in ways that simplify the analysis. For the Gierer-Meinhardt equations, the singularly perturbed structure of the linearized operators makes them strongly contractive on certain regions of the phase space, this permits a nontrivial replacement of spatially varying potentials with delta functions, affording dramatic simplification to the analysis of the principle linear operator. Indeed we replace the exact linearization with a putatively $\mathcal{O}(\varepsilon^{-2})$ "perturbation". This reduced linear operator gives rise to exactly the NLEP operators introduced previously in the formal linear stability analysis, [2, 6, 16], and justifying the observation that, at the linear level, the inhibitor equation averages perturbations into a mean-field.

The RG method shows that the NLEP operators control the flow in a neighborhood of the pulse configurations, generating a thin absorbing set in the phase space. Moreover we recover the leading order pulse evolution by projecting the flow onto the tangent plane of the manifold of two-pulse solutions. In this paper we only consider two-pulse solutions. Although there are no new conceptual features, the generalization to N-pulses is technical. In particular determining the amplitudes of each pulse within an N-pulse configuration requires a nontrivial nonlinear computation, and the stability of the underlying pattern will depend sensitively upon the pulse amplitudes and separations. These issues have been studied numerically in [11] and the construction and interaction of N-pulses on bounded domains has been considered in [16]. However, the nonlinear aspects of the stability approach we developed here generalizes directly from the two-pulse to the N-pulse case. Our methods can also be applied to the weak and semi-strong N-pulse interactions in classes of singularly perturbed reaction-diffusion equations as considered in [6] and in [17] on bounded domains. Nevertheless, interesting additional issues may emerge in specific settings, such as the semi-strong evolution of pulses in the Gray-Scott equation, see [4], in which the essential spectrum is asymptotically close to the origin.

The main result of this paper addresses the semi-strong evolution of two-pulse solutions Φ, given in (2.5), of the regularized Geier-Meinhardt equation (2.3). The solutions are parameterized by pulse positions $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)^t$, with pulse separation $\Delta\Gamma = |\Gamma_1 - \Gamma_2|$. In the application at hand the RG approach models the the impact of transient initial perturbations on the semi-strong pulse evolution, and after the decay of the transients, recovers the formal pattern evolution at leading order. For the Gierer-Meinhardt equation the semi-strong regime is comprised of pulses whose separations satisfy $\Delta\Gamma > \Delta\Gamma^*(\mu)$, where $\Delta\Gamma^* = \mathcal{O}(\varepsilon^{-2})$ is defined in (3.37) of Proposition 3.4. For $\Delta\Gamma < \Delta\Gamma^*$ one enters the strong interaction regime, the two-pulse solution has quasi-stationary eigenvalues which are incompatible with the two-pulse manifold and pulse splitting bifurcations are observed. Our approach works uniformly for the weak, $\Delta\Gamma \gg \varepsilon^{-2}$, and semi-strong interactions, recovering prior results, [8, 9, 14, 15], for the weak interaction limit.

We introduce the norm $\|\cdot\|_X$ on $H^1 \times H^1$ defined by

$$\|G\|_X = \varepsilon\|G_1\|_{L^2} + \varepsilon^{-1}\|\partial_\xi G_1\|_{L^2} + \|G_2\|_{H^1}, \tag{1.1}$$

and remark that it controls the L^∞ norm uniformly

$$\|G_1\|_{L^\infty} \leq (2\|G_1\|_{L^2}\|\partial_\xi G_1\|_{L^2})^{\frac{1}{2}} \leq \varepsilon\|G_1\|_{L^2} + \varepsilon^{-1}\|\partial_\xi G_1\|_{L^2} \leq \|G\|_X. \tag{1.2}$$

We state below our main theorem for the pulses in the semi-strong interaction regime.

Theorem 1.1. *Let ε be sufficiently small, let $\mu > \mu_{\mathrm{Hopf}}$, and let the pulse separation satisfy $\Delta\Gamma > \Delta\Gamma^*(\mu)$, where μ_{Hopf} and $\Delta\Gamma^*(\mu)$ are given in Proposition 3.4. The manifold $\mathcal{M}$ of two-pulse solutions (2.6), of the regularized Gierer-Meinhardt equation (2.3) is asymptotically exponentially stable up to $\mathcal{O}(\varepsilon^3)$. That is, there exists M and $\nu > 0$, independent of ε, such that for all initial data $\vec{U}_0$ sufficiently close to the $\mathcal{M}$, the corresponding solution $\vec{U} = (U, V)^t$ of the regularized Gierer-Meinhardt equations can be decomposed as*

$$\vec{U}(\xi, t) = \Phi_{\mathbf{\Gamma}} + W(\xi, t), \tag{1.3}$$

where the parameters $\mathbf{\Gamma}(t)$ of the two pulse solution Φ evolve at leading order according to (4.17). Moreover the remainder W satisfies

$$\|W\|_X \leq M(e^{-\nu t}\|W_0\|_X + \varepsilon^3). \tag{1.4}$$

In particular, after the perturbation W has decayed to $\mathcal{O}(\varepsilon^3)$, the pulse evolution is given by the ordinary differential equations (4.75) which are equivalent at leading order to

$$\frac{d}{dt}\Delta\Gamma = \varepsilon^2\sqrt{\mu}\frac{e^{-\varepsilon^2\Delta\Gamma\sqrt{\mu}}}{1 + e^{-\varepsilon^2\Delta\Gamma\sqrt{\mu}}}. \tag{1.5}$$

Since the pulses are repelling, Theorem 1.1 governs the evolution of all two pulses solutions in the semi-strong or the weak interaction regime. That is, any two-pulse solutions with $\Delta\Gamma(0) > \Delta\Gamma^*(\mu)$, will evolve according to (1.5) for all subsequent time.

Remark 1.2. The pulse dynamics (1.5) were obtained formally in [6].

Remark 1.3. In [2, 3], slowly-modulated two-pulse solutions were constructed for the Gray-Scott model on the infinite line, along with ODEs for the pulse positions, using the method of multiple scales. Various bifurcations, including the bifurcation to self-replicating two-pulse solutions were identified. Moreover, the NLEP method, which was initially developed for studying the stability of stationary, one-pulse solutions, see [5] and the references therein, was (formally) extended in [2] to the stability analysis of two-pulse solutions. For the generalized Gierer-Meinhardt equations on bounded domains, [11] presents ODEs for $N-$spike quasi-equilibrium solutions in the semi-strong and weak interaction regimes. Also, the NLEP method is employed to formally derive explicitly computable stability criterions. Further formal study of the instabilities (competition and oscillatory) for two-pulse solutions of the Gierer-Meinhardt equations has been reported in [16]. The analysis is primarily on a bounded domain, and results for the infinite line – some of which extend those reported in [2, 3] – are obtained by taking the domain to be large. Semi-strong pulse interactions have also been studied in [6] for a large class of reaction-diffusion equations, including for the generalized Gierer-Meinhardt equations, the Gray-Scott model, the Thomas equations, the Schnakenberg model, and others. Conditions were derived to determine formally whether adjacent pulses attract or repel, and the interactions between stationary and dynamically-evolving $N-$pulse solutions were studied.

2. The Two-Pulse Solutions of the Gierer-Meinhardt equations. As proposed the Gierer-Meinhardt model, [10], has an artificial singularity in its nonlinear term, which suggests infinite production of the activator, V, in the absence of the inhibitor, U. While the singular model can be studied by working with exponentially weighted norms which preserve positivity of the inhibitor, the behavior of the model for small concentrations bears little resemblance to chemical reality. Moreover the singularity has vanishingly small impact on both the two-pulse construction and their evolution. To avoid clouding the analysis, we truncate the superfluous singularity of the Gierer-Meinhardt reaction term, replacing it with a variation of the classic

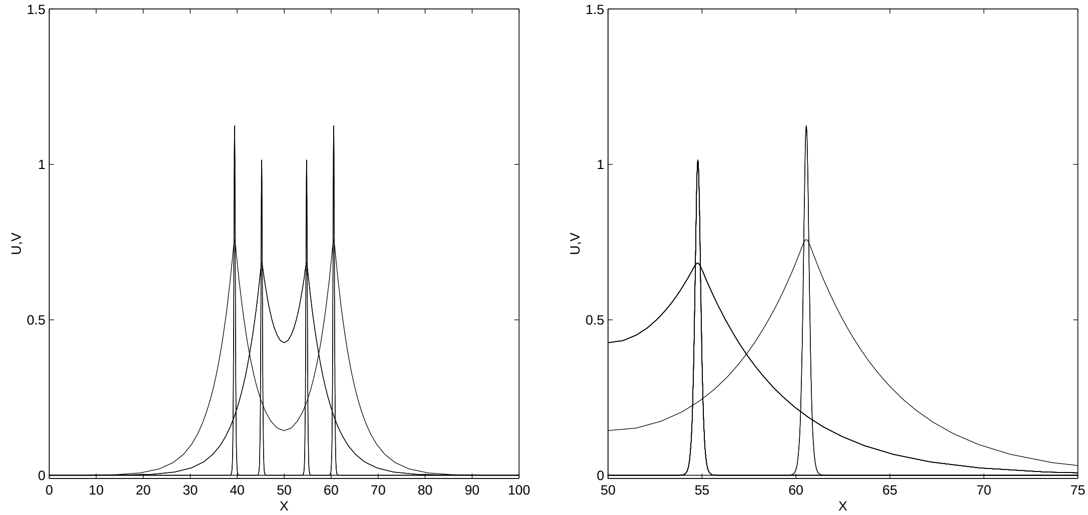

FIG. 1.1. *The two-pulse solution of the Gierer-Meinhardt equation, shown at $t = 500$ and $t = 5000$, in the slow spatial variable. The figures are obtained from numerical simulation of (2.1) with $\varepsilon^2 = 0.01$ and $\mu = 5$.*

Rice-Hertzfeld mechanism typical of complex reactions with inhibition steps, see chapter 26 of [1]. In the slow spatial variable x, the regularized Gierer-Meinhardt equation is given by,

$$\left\{ \begin{array}{lllll} U_t & = & \frac{1}{\varepsilon^2}U_{xx} & -\mu U & +\frac{1}{\varepsilon^2}V^2 \\ V_t & = & \varepsilon^2 V_{xx} & -V & +\frac{V^2}{\kappa(U)}, \end{array} \right. \tag{2.1}$$

where $U(x,t), V(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $\mu > 0$ is the main (bifurcation) parameter, and $\varepsilon > 0$ is asymptotically small, $0 < \varepsilon \ll 1$. The regularizing function κ takes the form

$$\kappa(s) = \left\{ \begin{array}{ll} s & \text{if } s > 2\delta, \\ \delta & \text{if } 0 < s < \delta, \end{array} \right. \tag{2.2}$$

and is smooth for $s \in (\delta, 2\delta)$, with derivative less than two. In the absence of the inhibitor, U, the production rate of the activator V reduces to V^2/δ, where δ is a small parameter. The regularization introduces an $\mathcal{O}(e^{-\varepsilon^{-2}|\ln\delta|})$ perturbation to the pulse dynamics. The fast spatial scale is defined by $\xi = \frac{x}{\varepsilon}$, so that (2.1) transforms into

$$\left\{ \begin{array}{lllll} U_t & = & \frac{1}{\varepsilon^4}U_{\xi\xi} & -\mu U & +\frac{1}{\varepsilon^2}V^2, \\ V_t & = & V_{\xi\xi} & -V & +\frac{V^2}{\kappa(U)}. \end{array} \right. \tag{2.3}$$

We denote the right-hand side of (2.3) by $F(U,V)$. Since the regularizing term has only an exponentially small impact on the pulse construction we carry over the asymptotic results for the singular Gierer-Meinhardt equation without modification.

PROPOSITION 2.1. *The construction and spectral analysis of pulse solutions for the classical GM model given in [5] and the construction and formal dynamics of semi-strong two-pulses given in [6] hold up to exponentially small terms for the regularized models (2.1)/(2.3).*

2.1. Notation. We write $f = g + \mathcal{O}(\varepsilon)$ in norm $\|\cdot\|$ if

$$\|f - g\| \leq c\varepsilon, \tag{2.4}$$

and assume the $\|\cdot\|_X$ norm if no norm is specified. The solution (U,V) of the Gierer-Meinhardt equation is denoted $\vec{U}$. The two-pulse solutions are denoted by $\Phi_{\mathbf{\Gamma}} = (U_0+\varepsilon^2 U_2+\ldots, V_0+\varepsilon^2 V_2+\ldots)^t$, while the initial data of the Gierer-Meinhardt equation is given by $\vec{U}_0$. We denote by $\|f\|_{\widehat{L}^p}$ and $\|f\|_{\widehat{H}^2}$ the L^p and H^2 norms of the Fourier transform of f. We remark that $\|f\|_{L^\infty} \leq c\|f\|_{\widehat{L}^1}$ and conversely $\|f\|_{\widehat{L}^\infty} \leq c\|f\|_{L^1}$, and in particular that the delta function resides in $\widehat{L}^\infty$ but is not in L^1. Also the norm $\| <x>f\|_{L^1}$ with $<x> \equiv 1+|x|$ controls the L^∞ norm of the derivative of the fourier transform of f. We denote the mass of a function f by $\overline{f} = \int_{-\infty}^{\infty} f\,d\xi$. The quantity $[\vec{F}]_k$ will denote the k'th component of the vector $\vec{F}$ when less cumbersome notation is not available.

2.2. Asymptotic Pulse Solutions. Within the semi-strong pulse regime the two pulses interact strongly through the inhibitor component, U, and weakly through the activator, V. The asymptotic family of semi-strong two-pulse solutions is parameterized by the pulse location $\mathbf{\Gamma} \in \mathcal{K} = \{(\Gamma_1,\Gamma_2)\big|\Gamma_1 < \Gamma_2, |\Gamma_1-\Gamma_2| \geq \Delta\Gamma^*(\mu)\}$, where $\Delta\Gamma^*(\mu)$ is defined in Proposition 3.4. We denote the two-pulse solution by $\Phi_{\mathbf{\Gamma}}(\xi)$ which we expand as

$$\Phi_{\mathbf{\Gamma}}(\xi) = \begin{pmatrix}\Phi_{\mathbf{\Gamma},1}\\ \Phi_{\mathbf{\Gamma},2}\end{pmatrix} = \begin{pmatrix}U_0(\xi;\mathbf{\Gamma})+\varepsilon^2 U_2(\xi;\mathbf{\Gamma})+\varepsilon^4 U_4(\xi;\mathbf{\Gamma})\\ V_0(\xi;\mathbf{\Gamma})+\varepsilon^2 V_2(\xi;\mathbf{\Gamma})\end{pmatrix}, \tag{2.5}$$

and define the manifold $\mathcal{M} \subset H^1 \times H^1$ of two-pulse solutions by

$$\mathcal{M} = \{\Phi_{\mathbf{\Gamma}}\big|\mathbf{\Gamma} \in \mathcal{K}\}. \tag{2.6}$$

We first describe the leading order terms $(U_0,V_0)^t$ which were derived in [6]. Full resolution of the pulse dynamics the renormalization procedure of section 4 requires a more accurate description of the two-pulse solution which requires the construction of the higher order corrections, outlined in Lemma 2.1

The V-components of the two-pulse solutions are centered around the pulse-positions $\xi = \Gamma_k(t)$, where

$$\Gamma_1(t) = \Gamma_0 - \varepsilon^2\int_0^t \hat{c}(s)ds, \quad \Gamma_2(t) = \Gamma_0 + \varepsilon^2\int_0^t \hat{c}(s)ds. \tag{2.7}$$

In the two-pulse configuration each pulse moves away from their mutual center Γ_0 with equal and opposite speed given by

$$\hat{c} = \frac{1}{2}\sqrt{\mu}\frac{e^{-\varepsilon^2\Delta\Gamma\sqrt{\mu}}}{1+e^{-\varepsilon^2\Delta\Gamma\sqrt{\mu}}}, \tag{2.8}$$

where $\Delta\Gamma = \Delta\Gamma(t) = |\Gamma_1-\Gamma_2|$, see (1.5).

The leader order term, V_0, of the V component of Φ_Γ is given by the sum of two one-pulses

$$V_0(\xi;\mathbf{\Gamma}(t)) = \phi_1+\phi_2, \tag{2.9}$$

where for $k=1,2$ the one pulse solution is

$$\phi_k(\xi) = \frac{3}{2}A(\mathbf{\Gamma})\,\mathrm{sech}^2\,\frac{1}{2}(\xi-\Gamma_k(t)). \tag{2.10}$$

A key distinction between the semi-strong interaction depicted here and the weak pulse interaction is that the pulse amplitude, $A(\mathbf{\Gamma})$, depends nontrivially upon the pulse separation, $\Delta\Gamma = |\Gamma_1-\Gamma_2|$, via

$$A(\mathbf{\Gamma}) = \frac{\sqrt{\mu}}{3}\frac{1}{1+e^{-2\varepsilon^2\Delta\Gamma\sqrt{\mu}}}. \tag{2.11}$$

The pulse regions $\mathcal{I}_k = \mathcal{I}_k(t)$, $k = 1, 2$, are defined as regions outside which V_0 is exponentially small, and such that U_0 remains constant at leading order over a pulse region. We set the width of the pulse regions to be $\mathcal{O}(1/\sqrt{\varepsilon})$, i.e. we define

$$\mathcal{I}_k = \left(\Gamma_k(t) - \frac{1}{\sqrt{\varepsilon}}, \Gamma_k(t) + \frac{1}{\sqrt{\varepsilon}}\right), \quad k = 1, 2. \tag{2.12}$$

The choice of pulse region width is somewhat arbitrary but standard. Another distinguishing feature of the semi-strong pulse interaction is that the slowly varying U-component of Φ_Γ is not the sum of two one-pulses. To the left of $\mathcal{I}_1$ and to the right of $\mathcal{I}_2$, $U_0(x,t)$ decays slowly, while in the region between $\mathcal{I}_1$ and $\mathcal{I}_2$ it is cosh-like, but again on the slow spatial scale,

$$U_0(\xi; \mathbf{\Gamma}) = \begin{cases} A e^{\varepsilon^2\sqrt{\mu}(\xi-\Gamma_1)} & \text{for} \qquad\qquad \xi < \Gamma_1 - \varepsilon^{-\frac{1}{2}}, \\ A\dfrac{\cosh \varepsilon^2\sqrt{\mu}\,(\xi - (\Gamma_1+\Gamma_2)/2)}{\cosh \varepsilon^2\sqrt{\mu}\Delta\Gamma/2} & \text{for} \quad \Gamma_1 + \varepsilon^{-\frac{1}{2}} < \xi < \Gamma_2 - \varepsilon^{-\frac{1}{2}}, \\ A e^{-\varepsilon^2\sqrt{\mu}(\xi-\Gamma_2)} & \text{for} \quad \Gamma_2 + \varepsilon^{-\frac{1}{2}} < \xi. \end{cases} \tag{2.13}$$

As defined above, U_0 would be non-smooth if extended into the pulse regions $\mathcal{I}_k$. Rather we define the U-component of the two-pulse solution inside $\mathcal{I}_k$ as $U_0 + \varepsilon^2 U_2(\xi)$, where $U_0 \equiv A$, and $U_2(\xi)$ is a solution of $U_{\xi\xi} + \phi_k^2 = 0$ – see (2.3). Using (2.13) as boundary or matching conditions and the pulse amplitude (2.11), we find

$$U_0 + \varepsilon^2 U_2(\xi; \mathbf{\Gamma}) = A + \varepsilon^2 \begin{cases} A[\sqrt{\mu} - 3A](\xi - \Gamma_1) - \int_{\Gamma_1}^{\xi}\int_{\Gamma_1}^{\xi_1} \phi_1^2(\xi_2)\, d\xi_2\, d\xi_1 & \text{for} \quad \xi \in \mathcal{I}_1, \\ A[\sqrt{\mu} - 3A](\xi - \Gamma_2) - \int_{\Gamma_2}^{\xi}\int_{\Gamma_2}^{\xi_1} \phi_2^2(\xi_2)\, d\xi_2\, d\xi_1 & \text{for} \quad \xi \in \mathcal{I}_2, \end{cases} \tag{2.14}$$

which gives that $U_0 + \varepsilon^2 U_2(\xi) \in C^1 \cap H^2$. The C^1-smoothness of $U_0 + \varepsilon^2 U_2(\xi)$ is equivalent to the amplitude-pulse separation relation (2.11), i.e. $U_0 + \varepsilon^2 U_2(\xi)$ can only be smooth for $A(\mathbf{\Gamma})$ given by (2.11).

Relations (2.7), (2.8), (2.9), (2.10), (2.11), (2.13), (2.14) give a leading order description of the two-pulse solution $\Phi_{\mathbf{\Gamma}}(\xi)$. The corrections $U_4(\xi)$ and $V_2(\xi)$ can be obtained by a straightforward regular asymptotic expansion and are both only defined in the pulse regions $\mathcal{I}_{1,2}$ (see (2.23) in the proof of Lemma 2.1 below). The residual of $\Phi_{\mathbf{\Gamma}}$,

$$\mathbf{R} = \mathbf{F}(\Phi_{\mathbf{\Gamma}}) = \begin{pmatrix} F_1(\Phi_{\mathbf{\Gamma}}) \\ F_2(\Phi_{\mathbf{\Gamma}}) \end{pmatrix}, \tag{2.15}$$

is determined by the right-hand side of (2.3), denoted by $(F_1, F_2)^t$, evaluated at $\Phi_{\mathbf{\Gamma}}$. Obtaining L^1- and L^2-estimates on the residual is a key step to controlling the remainder in the renormalization process,

Lemma 2.1. *For the residual $\mathbf{R} = \mathbf{F}(\Phi_{\mathbf{\Gamma}})$ defined in (2.15) we have,*

$$\sup_{\mathbb{R}} |F_2(\Phi_{\mathbf{\Gamma}})| = \mathcal{O}(\varepsilon^2), \quad \sup_{\mathbb{R}\setminus\mathcal{I}_1\cup\mathcal{I}_2} |F_1(\Phi_{\mathbf{\Gamma}})| = \mathcal{O}(\varepsilon^4), \quad \sup_{\mathcal{I}_1\cup\mathcal{I}_2} |F_1(\Phi_{\mathbf{\Gamma}})| = \mathcal{O}(\varepsilon\sqrt{\varepsilon}). \tag{2.16}$$

More specifically

$$R_2(\mathbf{\Gamma}) = \varepsilon^2 \hat{c}\,(\phi_1' - \phi_2') + \mathcal{O}(\varepsilon^4) \;\text{ in }\; L^2(\mathbb{R}), \tag{2.17}$$

while,

$$\|R_1(\mathbf{\Gamma})\|_{L^1} = \mathcal{O}(\varepsilon). \tag{2.18}$$

The $\mathcal{O}(\varepsilon\sqrt{\varepsilon})$ bound on F_1 in (2.16) and the $\mathcal{O}(\varepsilon)$ bound on R_1 in (2.18) deteriorate to $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ bounds, if we do not introduce the leading order corrections $\varepsilon^4 U_4$ and $\varepsilon^2 V_2$ in (2.5). Moreover, (2.17) no longer holds in that case. On the other hand, the bounds on $R_2(\mathbf{\Gamma})$ given in the Lemma are sharp. The bounds on $R_1(\mathbf{\Gamma})$ may be sharpened, but this does not lead to any improvements in the renormalization analysis of section 4.

Proof: In [6], $\Phi_{\mathbf{\Gamma}}$ is constructed as the solution of the classical Gierer-Meinhardt system

$$\left\{ \begin{array}{rclll} -\varepsilon^6 \hat{c}_k U_\xi & = & U_{\xi\xi} & -\varepsilon^4 \mu U & +\varepsilon^2 V^2 \\ -\varepsilon^2 \hat{c}_k V_\xi & = & V_{\xi\xi} & -V & +\frac{V^2}{U} \end{array} \right. \tag{2.19}$$

with $\hat{c}_1 = -\hat{c} < 0$ for $\xi < \Gamma_0$ and $\hat{c}_2 = \hat{c} > 0$ for $\xi < \Gamma_0$, and $\hat{c} = \hat{c}(t)$ as in (2.8). Note the factor ε^4 difference between the right hand sides of the U-equation here and in (2.3). We may employ a regular perturbation expansion, writing

$$\begin{array}{rcl} U(\xi,t) & = & U_0(\xi;\mathbf{\Gamma}) + \varepsilon^2 U_2(\xi;\mathbf{\Gamma}) + \varepsilon^4 U_4(\xi;\mathbf{\Gamma}) + \varepsilon^6 U_r(\xi,t;\varepsilon^2), \\ V(\xi,t) & = & V_0(\xi;\mathbf{\Gamma}) + \varepsilon^2 V_2(\xi;\mathbf{\Gamma}) + \varepsilon^4 V_r(\xi,t;\varepsilon^2), \end{array} \tag{2.20}$$

where A, U_2 and V_0 are given in (2.9), (2.10), (2.11), (2.13), (2.14), and U_4 and V_2 have already been introduced in (2.5). Likewise, we expand F_1 and F_2 (Proposition 2.1),

$$\begin{array}{l} F_1(\Phi_{\mathbf{\Gamma}}) = \frac{1}{\varepsilon^2}\left[U_{2,\xi\xi} + V_0^2\right] + \left[U_{4,\xi\xi} - \mu U_0 + 2V_0V_2\right] + \varepsilon^2\left[U_{r,\xi\xi} - \varepsilon^4 \mu U_r - F_{1,r}^{\text{inh}}(U_2, V_{0,2,r};\varepsilon^2)\right], \\ F_2(\Phi_{\mathbf{\Gamma}}) = \left[V_{0,\xi\xi} - V_0 + \frac{V_0^2}{U_0}\right] + \varepsilon^2\left[L_{22}V_2 - \frac{V_0^2 U_2}{U_0^2}\right] + \varepsilon^4\left[L_{22}V_r - F_{2,r}^{\text{inh}}(U_{0,2,4}, \varepsilon^2 U_r, V_{0,2}, \varepsilon^2 V_r;\varepsilon^2)\right], \end{array} \tag{2.21}$$

where

$$L_{22}V = V_{\xi\xi} - V + \frac{V_0^2}{U_0}V, \tag{2.22}$$

and the expressions for $F_{1,r}^{\text{inh}}(U_2, V_{0,2,r};\varepsilon^2)$ and $F_{2,r}^{\text{inh}}(U_{0,2,4}, \varepsilon^2 U_r, V_{0,2}, \varepsilon^2 V_r;\varepsilon^2)$ follow directly by substitution of (2.20) in (2.3). We obtain by (2.19) the following equations for U_4 and V_2,

$$U_{4,\xi\xi} = \mu U_0 - 2V_0V_2, \;\; L_{22}V_2 = \frac{V_0^2 U_2}{U_0^2} - \hat{c}_k V_{0,\xi}, \tag{2.23}$$

for $\xi \in \mathcal{I}_{1,2}$. These equations can be solved uniquely by application of the natural boundary/matching conditions. Note that $U_4(\xi)$ grows as $(\xi - \Gamma_{1,2})^2$ for $|\xi - \Gamma_{1,2}| \gg 1$ and that V_2 decays exponentially to 0 as $|\xi - \Gamma_{1,2}| \gg 1$ (see (2.9)). The equations for the remainders $U_r(\xi,t;\varepsilon^2)$ and $V_r(\xi,t;\varepsilon^2)$ are given by,

$$\begin{array}{rcl} U_{r,\xi\xi} - \varepsilon^4 \mu U_r & = & F_{1,r}^{\text{inh}}(U_2, V_{0,2,r};\varepsilon^2) - \hat{c}_k U_{0,\xi} - \varepsilon^2 \hat{c}_k U_{2,\xi} - \varepsilon^4 \hat{c}_k U_{4,\xi} - \varepsilon^6 \hat{c}_k U_{r,\xi}, \\ L_{22}V_r & = & F_{2,r}^{\text{inh}}(U_{0,2,4}, \varepsilon^2 U_r, V_{0,2}, \varepsilon^2 V_r;\varepsilon^2) - \hat{c}_k V_{2,\xi} - \varepsilon^2 \hat{c}_k V_{r,\xi}, \end{array} \tag{2.24}$$

for $\xi \in \mathbb{R}$. It is a straightforward procedure to check that $|V_r|$ and $|F_{2,r}^{\text{inh}}|$ are uniformly bounded for $\xi \in \mathbb{R}$; in fact, both V_r and $F_{2,r}^{\text{inh}}$ decay exponentially to 0 as $|\xi - \Gamma_{1,2}(t)| \gg 1$. Together with the definitions of V_0 and V_2 ((2.9) and (2.23)), substitution of this result in the second equation of (2.21) yields (2.17). This also implies the results on $F_2(\Phi_{\mathbf{\Gamma}})$ in (2.16).

Outside the pulse regions $\mathcal{I}_k$, all $V_{0,2,r}$ components are exponentially small, and U_0 is constructed as a solution of the equation $U_{\xi\xi} - \varepsilon^4 \mu U = 0$, see (2.13). Therefore, the correction U_r to U_0 in the U-component of the two-pulse solution also varies like $\varepsilon^2\xi$, and U_2 and U_4 maybe taken identically zero outside $\mathcal{I}_k$. This implies by (2.21) and (2.24) that outside $\mathcal{I}_k$

$$F_1(\Phi_{\mathbf{\Gamma}}) = -\varepsilon^2 \hat{c}_k U_{0,\xi} = \mathcal{O}(\varepsilon^4).$$

Since U_r decays for $\xi \to \pm\infty$ with the same slow rate as U_0, we find

$$\int_{\mathbb{R}\setminus\mathcal{I}_1\cup\mathcal{I}_2} |F_1(\Phi_{\mathbf{\Gamma}})|\, d\xi = \frac{1}{\varepsilon^2} \times \mathcal{O}(\varepsilon^4) = \mathcal{O}(\varepsilon^2).$$

Inside $\mathcal{I}_k$, we conclude from (2.24) and the fact that U_2 grows linearly with $(\xi - \Gamma_i)$, see (2.14), that U_r may grow as $(\xi-\Gamma_i)^3$. Nevertheless, both $U_{r,\xi\xi}$ and $F_{1,r}^{\mathrm{inh}}(U_2, V_{0,2,r}; \varepsilon^2)$ only grow linearly in $(\xi-\Gamma_i)$. Since the width of the $\mathcal{I}_k$ interval is $\mathcal{O}(1/\sqrt{\varepsilon})$, see (2.12), we deduce from (2.21) and (2.24) that $\sup_{\mathcal{I}_1\cup\mathcal{I}_2} |F_1(\Phi_{\mathbf{\Gamma}})| = \mathcal{O}(\varepsilon\sqrt{\varepsilon})$ (2.16). Hence, by (2.12),

$$\int_{\mathcal{I}_k} |F_1(\Phi_{\mathbf{\Gamma}})|\, d\xi = \frac{1}{\sqrt{\varepsilon}} \times \mathcal{O}(\varepsilon\sqrt{\varepsilon}) = \mathcal{O}(\varepsilon),$$

which yields the L^1-bound (2.18). $\square$

3. Linearization and the Reduced Operators. In a neighborhood of the two-pulse manifold $\mathcal{M}$ we decompose the solutions of (2.3) as

$$\begin{pmatrix} U \\ V \end{pmatrix} = \Phi_{\mathbf{\Gamma}} + W(\xi, t), \tag{3.1}$$

where the remainder $W = (W_1, W_2)^t$ and $\mathbf{\Gamma}$ is taken as a function of time. In terms of the remainder introduced in (3.1) the GM equation (2.3) can then be written as

$$W_t + \frac{\partial \Phi}{\partial \mathbf{\Gamma}} \dot{\mathbf{\Gamma}} = \mathbf{R} + L_{\mathbf{\Gamma}} W + \mathcal{N}(W), \tag{3.2}$$

where R is the residual (2.15) and $L_{\mathbf{\Gamma}}$ is the linearization of F about $\Phi_{\mathbf{\Gamma}}$, given by

$$L_{\mathbf{\Gamma}} = \begin{pmatrix} \epsilon^{-4}\partial_\xi^2 - \mu & 2\varepsilon^{-2}\Phi_{\Gamma,2} \\ -\dfrac{\Phi_{\Gamma,2}^2 \kappa'(\Phi_{\Gamma,1})}{\kappa(\Phi_{\Gamma,1})^2} & \partial_\xi^2 - 1 + 2\dfrac{\Phi_{\Gamma,2}}{\kappa(\Phi_{\Gamma,1})} \end{pmatrix}. \tag{3.3}$$

In the linear operator above $\kappa(\Phi_{\Gamma,1}) = \Phi_{\Gamma,1}$ except for those ξ for which $\Phi_{\Gamma,2}(\xi) = \mathcal{O}(e^{-\varepsilon^{-2}|\ln\delta|})$. Thus the perturbation to the linearization introduced by the regularization is compact and exponentially small. The final term, $\mathcal{N}(W)$, representing the nonlinearity is given at leading order by

$$\mathcal{N}(W) = \begin{pmatrix} \epsilon^{-2} W_2^2 \\ \mathcal{O}(W_2^2) + \mathcal{O}(V_0 W_1 W_2) + \mathcal{O}(V_0^2 W_1^2) \end{pmatrix}. \tag{3.4}$$

From the asymptotic form of the pulse solution given in (2.9), (2.13), and (2.11), we calculate that

$$\varepsilon^2 \Big\| \frac{\partial U_0}{\partial \Gamma_k} \Big\|_{L^1} + \varepsilon \Big\| \frac{\partial U_0}{\partial \Gamma_k} \Big\|_{L^2} + \Big\| \frac{\partial U_0}{\partial \Gamma_k} \Big\|_{L^\infty} = \mathcal{O}(\varepsilon^2), \tag{3.5}$$

while

$$\frac{\partial V_0}{\partial \Gamma_k} = -\phi_k' + \mathcal{O}(\varepsilon^2), \tag{3.6}$$

in L^2.

3.1. The Reduced Linearization. A key step in the renormalization group treatment is the replacement of the exact linear operator with a reduced operator whose spectral and semi-group properties are easier to analyze, yet such that the difference between the exact and the reduced operator, the secularity, does not lead to growth of the remainder W. Due to the contractivity of the L_{11} component of $L_{\mathbf{\Gamma}}$, the two-pulse potential which comprises the L_{12} component can be replaced with δ functions located at each pulse position.

The mass of the delta function is chosen to equal the mass of the product of the original potential and the function it operates upon. We also replace the exact two-pulse solution Φ_Γ with its leading order approximation $(U_0, V_0)^t$. With these reductions the linearized operator becomes

$$\tilde{L}_{\mathbf{\Gamma}} = \begin{pmatrix} \epsilon^{-4}\partial_\xi^2 - \mu & 2\varepsilon^{-2}\Big(\delta_{\Gamma_1} \otimes \phi_1 + \delta_{\Gamma_2} \otimes \phi_2\Big) \\ -\dfrac{V_0^2}{A^2} & \partial_\xi^2 - 1 + 2\dfrac{V_0}{A} \end{pmatrix}, \tag{3.7}$$

where the tensor product of f_1 and f_2 is defined by

$$(f_1 \otimes f_2)\, W = (f_2, W)_{L^2} f_1. \tag{3.8}$$

In particular $\delta_{\Gamma_k} \otimes \phi_k$ represents the tensor product of the δ function centered at $\xi = \Gamma_k$ with ϕ_k. In the analysis below we use the notation

$$\alpha_k(W) = (\phi_k, W)_{L^2}, \tag{3.9}$$

for $k = 1, 2$. The scalar operators that appear in the upper left entry, respectively lower right, of the matrix $\tilde{L}$ (3.7) will be denoted by L_{11}, resp. L_{22}, see (2.22). The reduced operator is ostensibly an $\mathcal{O}(\varepsilon^{-2})$ perturbation of the original operator. However it is immediately clear that they share the same essential spectrum

$$\sigma_{\text{ess}} = \{\lambda \in \mathbb{R} : \lambda \leq \max\,(-1, -\mu)\} \tag{3.10}$$

3.2. The Point Spectrum. The two-pulse profiles which comprise the manifold $\mathcal{M}$ are not stationary solutions, and as such it is not self-consistent to determine their linear stability in terms of the spectrum of the associated linearized operator. We say that the two-pulse solution $\Phi_{\mathbf{\Gamma}}$ is *spectrally compatible* with the manifold $\mathcal{M}$ if the spectrum of the associated linear operator can be decomposed into a part contained within the left-half complex plane and a finite-dimensional part whose associated eigenspace approximates the tangent plane of $\mathcal{M}$ at $\mathbf{\Gamma}$.

To determine the point spectrum of $\tilde{L}$ we invert the U component of the eigenvalue equation, eliminate the inhibitor from the activator equation, reducing the eigenvalue problem to a scalar equation for the activator component of the eigenfunction. We call this the NLEP equation, see (3.23), and denote the corresponding linear operator by $\mathcal{L}(\lambda, \Delta\Gamma)$. The NLEP operator controls the point spectrum of $\tilde{L}$, to leading order.

PROPOSITION 3.1. *Up to multiplicity we have $\sigma_p(\tilde{L}) = \{\lambda \big| \mathrm{Ker}(\mathcal{L}(\lambda)) \neq 0\}(1 + \mathcal{O}(\varepsilon^2))$. That is, for each eigenvalue, $\lambda \in \sigma_p(\tilde{L})$, with corresponding eigenvector $\Psi = (\Psi_1, \Psi_2)^t$, there is a $\lambda_{\mathcal{L}}$ and corresponding ψ such that $\mathcal{L}(\lambda_{\mathcal{L}})\psi = \lambda_{\mathcal{L}}\psi$, $|\lambda - \lambda_{\mathcal{L}}| = \mathcal{O}(\varepsilon^2)$ with $\Psi_2 = \psi(1 + \mathcal{O}(\varepsilon^2))$ and Ψ_1 given by (3.12) up to $\mathcal{O}(\varepsilon^2)$. Moreover the small eigenvalues of $\tilde{L}$ and $\mathcal{L}$ are both exponentially small.*

Proof: The eigenvalue problem for the reduced operator is written as

$$\tilde{L}\Psi = \lambda\Psi, \tag{3.11}$$

where $\Psi = (\Psi_1, \Psi_2)^t$ is a possibly complex two-vector. Since $L_{11} - \lambda$ is invertible for $\lambda \notin (-\infty, -\mu]$, we may solve for Ψ_1 as

$$\Psi_1 = -2\varepsilon^{-2}\Big(\alpha_1(L_{11} - \lambda)^{-1}\delta_{\Gamma_1} + \alpha_2(L_{11} - \lambda)^{-1}\delta_{\Gamma_2}\Big), \tag{3.12}$$

where the $\alpha_k = (\phi_k, \Psi_2)_{L^2}$ are as yet undetermined. From the Fourier transform we find

$$\widehat{\Psi}_1(k) = \frac{2}{\sqrt{2\pi}} \frac{\varepsilon^2\Big(\alpha_1 e^{ik\Gamma_1} + \alpha_2 e^{ik\Gamma_2}\Big)}{k^2 + \varepsilon^4(\mu + \lambda)}. \tag{3.13}$$

From the integral relation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik\xi}\frac{\varepsilon^2 e^{ik\Gamma}}{k^2+\varepsilon^4(\mu+\lambda)} = \sqrt{\frac{\pi}{2(\mu+\lambda)}}e^{-\varepsilon^2\sqrt{\mu+\lambda}|\xi-\Gamma|}, \tag{3.14}$$

we may invert the Fourier transform of Ψ_1 explicitly,

$$\Psi_1(\xi,t) = \alpha_1 H(\lambda,\xi-\Gamma_1) + \alpha_2 H(\lambda,\xi-\Gamma_2), \tag{3.15}$$

where

$$H(\lambda,x) = \frac{1}{\sqrt{\mu+\lambda}}e^{-\varepsilon^2|x|\sqrt{\mu+\lambda}}. \tag{3.16}$$

Eliminating Ψ_1, the equation for Ψ_2 reduces to

$$(L_{22}-\lambda)\Psi_2 = \frac{V_0^2}{A^2}\Psi_1, \tag{3.17}$$

see also (2.22). Since Ψ_1 is a slowly varying function of ξ, while each term in V_0 decays exponentially to zero at an $\mathcal{O}(1)$ rate in ξ, we may reduce the equation for Ψ_2 to

$$\begin{aligned}(L_{22}-\lambda)\Psi_2 &= \frac{V_0^2}{A^2}\left(\alpha_1 H(\lambda,\xi-\Gamma_1)+\alpha_2 H(\lambda,\xi-\Gamma_2)\right), &(3.18)\\ &= \frac{\phi_1^2+\phi_2^2}{A^2}\left(\alpha_1 H(\lambda,\xi-\Gamma_1)+\alpha_2 H(\lambda,\xi-\Gamma_2)\right)+\mathcal{O}(e^{-\Delta\Gamma}), &(3.19)\\ &= \frac{1}{A^2\sqrt{\mu+\lambda}}\left[\phi_1^2\left(\alpha_1+\alpha_2 E\right)+\phi_2^2\left(\alpha_1 E+\alpha_2\right)\right]+\mathcal{O}(\varepsilon^2), &(3.20)\end{aligned}$$

where

$$E = E(\Delta\Gamma;\lambda) = e^{-\varepsilon^2\sqrt{\mu+\lambda}\Delta\Gamma}. \tag{3.21}$$

In the tensor product notation this is written as

$$(L_{22}-\lambda)\Psi_2 = \frac{1}{A^2\sqrt{\mu+\lambda}}\left[\phi_1^2\otimes(\phi_1+E\phi_2)+\phi_2^2\otimes(E\phi_1+\phi_2)\right]\Psi_2. \tag{3.22}$$

We define the NLEP operator

$$\mathcal{L}(\lambda,\Delta\Gamma) = L_{22} - \frac{1}{A^2\sqrt{\mu+\lambda}}\left[\phi_1^2\otimes(\phi_1+E\phi_2)+\phi_2^2\otimes(E\phi_1+\phi_2)\right]. \tag{3.23}$$

This is a compact perturbation of L_{22} and thus is Fredholm, with the same essential spectrum, but is no-longer self adjoint, indeed its adjoint exchanges the roles of the potentials in each tensor product. □

Proposition 3.2. *Except for the exponentially small eigenvalues, the point spectrum of the* NLEP *operator* $\mathcal{L}$ *is given, up to multiplicity, by the zeros of the equation*

$$\mathcal{R}(\lambda) - 3\frac{\sqrt{\mu+\lambda}}{\sqrt{\mu}}\frac{1+e^{-\varepsilon^2\sqrt{\mu}\Delta\Gamma}}{1\pm e^{-\varepsilon^2\sqrt{\mu+\lambda}\Delta\Gamma}} = 0, \tag{3.24}$$

where $\mathcal{R}$ *is an explicitly known meromorphic function on* $\mathbf{C}\backslash(-\infty,-1]$ *given by (3.33).*

Proof: The spectrum of the NLEP operator $\mathcal{L}$ can be determined explicitly as the zeros of an analytic equation using the methods developed in [5], which we outline below. We introduce $w_h(\xi) \geq 0$ as the scaled homoclinic solution of

$$w_{\xi\xi} - w + w^2 = 0 \tag{3.25}$$

with its maximum at $\xi = 0$. For $k = 1, 2$ we introduce the translates $w_{h,k}(\xi) = w_h(\xi - \Gamma_k)$. Since $\phi_k(\xi) = Aw_{h,k}(\xi)$ the equations (2.10) and (3.22) can be written as

$$\frac{d^2\Psi_2}{d\xi^2} - [(1+\lambda) - 2(w_{h,1} + w_{h,2})]\,\Psi_2 = \frac{1}{\sqrt{\mu+\lambda}}\left[w_{h,1}^2\,(\alpha_1 + \alpha_2 E) + w_{h,2}^2\,(\alpha_1 E + \alpha_2)\right], \tag{3.26}$$

where $\alpha_k = \alpha_k(\Psi_2)$ (3.9). Since both the potential of the Schrödinger operator on the left-hand side of the equation, and the inhomogeneous term on the right-hand side consist of disjoint parts localized about Γ_1 and Γ_2, it is natural to decompose Ψ_2 into

$$\Psi_2 = \psi_1(\xi) + \psi_2(\xi), \tag{3.27}$$

where ψ_k is localized about Γ_k and decays exponentially as ξ moves away from Γ_k. The equation (3.26) is equivalent, up to exponentially small terms, to the coupled system,

$$\left\{\begin{array}{lcl} \frac{d^2\psi_1}{d\xi^2} - [(1+\lambda) - 2w_{h,1}]\,\psi_1 & = & \dfrac{w_{h,1}^2}{\sqrt{\mu+\lambda}}\,(\alpha_1 + \alpha_2 E)\,, \\ \frac{d^2\psi_2}{d\xi^2} - [(1+\lambda) - 2w_{h,2}]\,\psi_2 & = & \dfrac{w_{h,2}^2}{\sqrt{\mu+\lambda}}\,(\alpha_1 E + \alpha_2)\,. \end{array}\right. \tag{3.28}$$

We define $\bar{\psi} = \bar{\psi}(\xi;\lambda)$ as the uniquely determined bounded solution of

$$\frac{d^2\psi}{d\xi^2} - [(1+\lambda) - 2w_h]\,\psi = w_h^2; \tag{3.29}$$

and its translates $\bar{\psi}_k(\xi)$ are defined by $\bar{\psi}_k(\xi) = \bar{\psi}(\xi - \Gamma_k)$. The functions $\bar{\psi}$ can be determined explicitly, see [5].

We first consider the solution of (3.29) for $\lambda \notin \sigma_{\rm red} = \{\frac{5}{4}, 0, -\frac{3}{4}\} \cup (-\infty, -1]$, the spectrum of the operator

$$L_{\rm red} = \frac{d^2}{d\xi^2} - (1 - 2w_h(\xi)). \tag{3.30}$$

Clearly,

$$\psi_1(\xi) = C_1\bar{\psi}_1(\xi), \quad \psi_2(\xi) = C_2\bar{\psi}_2(\xi) \tag{3.31}$$

for some constants C_k that depend on λ and $\Delta\Gamma$. Recalling that here $\alpha_k = (\phi_k, \Psi_2)_{L^2}$ and using (3.27), we find

$$\alpha_k = \int_{-\infty}^{\infty} \phi_k(\psi_1 + \psi_2)d\xi = \int_{-\infty}^{\infty} Aw_{h,k}(C_1\bar{\psi}_1 + C_2\bar{\psi}_2)d\xi = AC_k\int_{-\infty}^{\infty} w_{h,k}\bar{\psi}_k d\xi = AC_k\int_{-\infty}^{\infty} w_h\bar{\psi}d\xi \tag{3.32}$$

up to asymptotically small corrections. The quantity

$$\mathcal{R}(\lambda) \equiv \int_{-\infty}^{\infty} w_h\bar{\psi}d\xi, \tag{3.33}$$

is meromorphic for $\lambda \in \mathbb{C}\backslash(-\infty,-1]$, with poles at $\lambda = \frac{5}{4}$ and $\lambda = -\frac{3}{4}$, see [5]. Note that in [5] a more general function, $\mathcal{R}(\lambda;\beta_1,\beta_2)$, has been defined and studied; (3.33) is related to [5] by $\mathcal{R}(\lambda) = 216\mathcal{R}(\lambda;2,2)$. The system (3.28) can be written as

$$\begin{cases} \frac{d^2\psi_1}{d\xi^2} - \left[(1+\lambda) - 2w_{h,1}\right]\psi_1 &= \dfrac{Aw_{h,1}^2\mathcal{R}}{\sqrt{\mu+\lambda}}\left(C_1 + C_2E\right), \\ \frac{d^2\psi_2}{d\xi^2} - \left[(1+\lambda) - 2w_{h,2}\right]\psi_2 &= \dfrac{Aw_{h,2}^2\mathcal{R}}{\sqrt{\mu+\lambda}}\left(C_1E + C_2\right). \end{cases} \tag{3.34}$$

Comparing the equations for $\psi_{1,2}(\xi)$ to (3.29), we obtain the following relations for C_1 and C_2,

$$C_1 = \frac{A\mathcal{R}}{\sqrt{\mu+\lambda}}\left(C_1 + C_2E\right), \quad C_2 = \frac{A\mathcal{R}}{\sqrt{\mu+\lambda}}\left(C_1E + C_2\right), \tag{3.35}$$

or, equivalently,

$$\begin{pmatrix} \dfrac{A\mathcal{R}}{\sqrt{\mu+\lambda}} - 1 & \dfrac{AE\mathcal{R}}{\sqrt{\mu+\lambda}} \\ \dfrac{AE\mathcal{R}}{\sqrt{\mu+\lambda}} & \dfrac{A\mathcal{R}}{\sqrt{\mu+\lambda}} - 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.36}$$

For (3.26) to have non-trivial solutions the determinant of the matrix on the left-hand side of (3.36) must be zero. Isolating $\mathcal{R}(\lambda)$ from the resulting expression and using (2.11) and (3.21), we obtain the equation (3.24) whose zeros are the eigenvalues of the NLEP equation (3.22) up to multiplicity, outside of σ_{red}. These eigenvalues lie on curves $\lambda(\Delta\Gamma)$, parametrized by the pulse separation.

For $\lambda \in \sigma_{\text{red}}$ we analyze the eigenvalue equation case by case. For $\lambda = -\frac{3}{4}$ or $\frac{5}{4}$, equation (3.29) does not have a bounded solution since the right-hand side is not orthogonal to the kernel of the L_{red}, and so these values cannot be eigenvalues. However, the eigenfunction $\frac{d}{d\xi}w_h$ of L_{red} at $\lambda = 0$, is L^2 orthogonal to w_h. The equation (3.32) implies that $\alpha_1 = \alpha_2 = 0$ and the system (3.28) has a double eigenvalue at $\lambda = 0$ with a two-dimensional eigenspace spanned by $\{\frac{d}{d\xi}w_h(\xi-\Gamma_1), \frac{d}{d\xi}w_h(\xi-\Gamma_2)\}$. These eigenvalues do not occur as solutions of (3.24), rather they correspond to exponentially small eigenvalues of the original NLEP system (3.22), whose corresponding eigenfunctions, derived in Lemma 3.7, form the key spectral projection onto the active tangent plane. □

Remark 3.3. Proposition 3.2 is equivalent to Principle Result 5.3 of [16].

We identify conditions on μ and $\mathbf{\Gamma}$ such that the reduced linearized operator, $\tilde{L}_{\mathbf{\Gamma}}$, is spectrally compatible with the manifold $\mathcal{M}$ of two-pulse solutions.

Proposition 3.4. *For each $\Delta\Gamma \in (0,\infty)$, the* NLEP *eigenvalue problem (3.22) has two exponentially small eigenvalues, denoted $\lambda_\pm$, and* 4 *or* 6 *eigenvalues $\lambda_+^{j+}(\Delta\Gamma)$ and $\lambda_-^{j-}(\Delta\Gamma)$, $j_\pm = 1,...,J_\pm, J_\pm = J_\pm(\mu) = 2$ or 3. There exists a unique $\mu_{\text{Hopf}} > 0$ such that for all $\mu > \mu_{\text{Hopf}}$, there is a $\Delta\Gamma^*(\mu)$ and a $\nu > 0$ such that*

$$\text{Re}[\lambda_\pm^{j\pm}(\Delta\Gamma)] < -\nu < 0 \text{ for all } \Delta\Gamma \geq \Delta\Gamma^*(\mu),\ j_\pm = 1,...,J_\pm.$$

For all $\mu > \mu_{\text{TP}} \approx 0.62$ (the tangent point), $\Delta\Gamma^*$ takes the exact form

$$\Delta\Gamma^*(\mu) = \frac{1}{\varepsilon^2\sqrt{\mu}}\log 3, \tag{3.37}$$

while for $\mu \in (\mu_{\text{Hopf}}, \mu_{\text{TP}})$, $\Delta\Gamma^*(\mu)$ increases with decreasing μ, with $\Delta\Gamma^*(\mu) \to \infty$ as $\mu \downarrow \mu_{\text{Hopf}}$.

Since the two pulses of $\Phi_{\mathbf{\Gamma}}(\xi)$ move away from each other ((2.7), (2.8)), this result implies that the spectrum of the NLEP operator $\mathcal{L}$ remains in the stable half-plane for all $t \geq 0$ if $\Delta\Gamma(0) > \Delta\Gamma^*$.

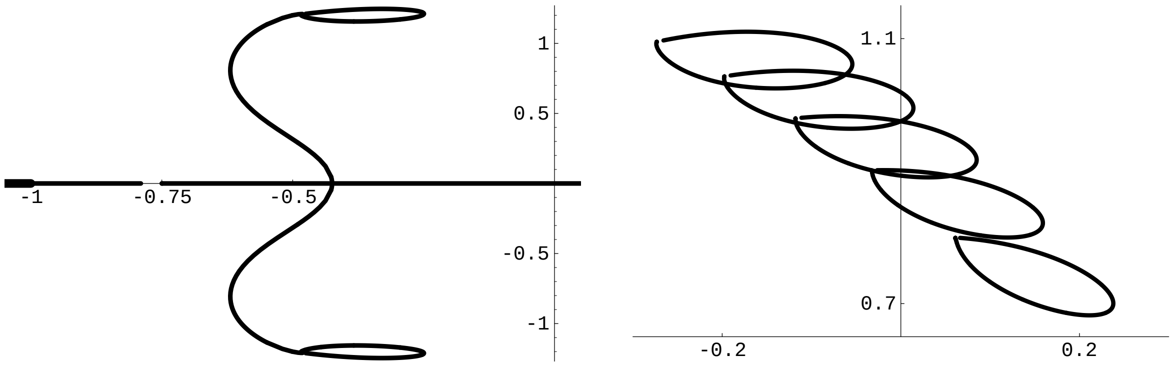

FIG. 3.1. *(a) The orbits of the zeroes $\lambda(\Delta\Gamma)$ of (3.24) plotted parametrically in the complex plane as a function of $\Delta\Gamma$ for $\mu = 1$. The eigenvalues $\lambda_+^{1,2}(\Delta\Gamma)$ are closed loops attached to the homoclinic limit $\lambda \approx -0.48 + 1.20i$; the curves $\lambda_-^{1,2}(\Delta\Gamma)$ approach the homoclinic point in the limit $\Delta\Gamma \to \infty$, but $\lambda_-^{1,2}(\Delta\Gamma)$ collide on the real axis, becoming real as $\Delta\Gamma$ decreases approaching the limits $\lambda_-^{1,2}(\Delta\Gamma) \to -\frac{3}{4}, \frac{5}{4}$ as $\Delta\Gamma \to 0$. The third pair of eigenvalues satisfies $\lambda_\pm^3(\Delta\Gamma) < -\frac{3}{4}$ with $\lambda_-^3(\Delta\Gamma)$ disappearing into the essential spectrum of (3.22) as $\Delta\Gamma$ decreases through a critical value. All eigenvalues $\lambda_\pm^{1,2,j}(\Delta\Gamma)$ have negative real part for $\Delta\Gamma > \Delta\Gamma^*(1)$ given by (3.37). (b) The closed $\lambda_+^1(\Delta\Gamma)$-loops for five values of μ: $\mu = 0.7 > \mu_{\rm TP}$, $\mu = 0.6, 0.5, 0.4 \in (\mu_{\rm Hopf}, \mu_{\rm TP})$ and $\mu = 0.3 < \mu_{\rm Hopf}$. The $\Delta\Gamma$-region $(\Delta\Gamma_+^{1,*}(\mu), \Delta\Gamma_+^{2,*}(\mu))$ in which Re$[\lambda_+^1(\Delta\Gamma)] > 0$ grows as $\mu \in (\mu_{\rm Hopf}, \mu_{\rm TP})$ decreases, so that $\Delta\Gamma^*(\mu) = \Delta\Gamma_+^{2,*}(\mu)$ for $\mu < \mu^* \in (\mu_{\rm Hopf}, \mu_{\rm TP})$.*

Proof: We can distinguish two limits, $\Delta\Gamma \to \infty$ and $\Delta\Gamma \downarrow 0$. The first case represents the situation in which the two pulses of $\Phi_\Gamma(\xi)$ are so far apart that the two-pulse solution can be considered as two one pulse solutions, i.e. the two-pulse solution is in the weak interaction limit. In this limit, (3.24) reduces to

$$\mathcal{R}(\lambda) = 3\frac{\sqrt{\mu + \lambda}}{\sqrt{\mu}}, \tag{3.38}$$

for both $\lambda_\pm(\Delta\Gamma)$. This is the relation that determines the point spectrum of the solitary one-pulse solution of (2.1), independent of the regularization. It was shown in Theorem 5.11 of [5] that there exists a unique $\mu_{\rm Hopf} > 0$ such that all solutions of (3.38) have $\mathrm{Re}(\lambda) < 0$ for $\mu > \mu_{\rm Hopf}$ and that (3.22) always has incompatible eigenvalues if $\mu < \mu_{\rm Hopf}$. Numerical evaluation shows that $\mu_{\rm Hopf} \approx 0.36$. Moreover, (3.38) has 2 or 3 nontrivial eigenvalues, i.e. $\lambda \neq 0$, depending on μ – the third (compatible) eigenvalue is created in an edge bifurcation as μ increases through $\mu_{\rm edge} \approx 0.77$ [5]. There also are 2 or 3 curves $\lambda_+^{j+}(\Delta\Gamma)$ and $\lambda_-^{j-}(\Delta\Gamma)$, i.e. $j_\pm = 1, ..., J_\pm$, $J_\pm(\mu) = 2$, respectively 3, for $\mu < \mu_{\rm edge}$, resp. $> \mu_{\rm edge}$. The eigenvalues $\lambda_\pm^3(\Delta\Gamma)$ are real and $\lambda_\pm^3(\Delta\Gamma) < -\frac{3}{4}$.

For small values of $\Delta\Gamma$ there are two mechanisms to generate incompatible point spectrum, one which occurs for $\mu > \mu_{\rm TP}$ and the other for $\mu \in (\mu_{\rm Hopf}, \mu_{\rm TP})$. The first occurs when the eigenvalues λ_-^1 and λ_-^2 collide and become real. Indeed in the limit $\Delta\Gamma \downarrow 0$, it follows from Proposition 3.2 that $\lambda_+^{1,2}(\Delta\Gamma)$ again approaches a solution of (3.38), i.e. the $\lambda_+^{1,2}(\Delta\Gamma)$-branches are closed curves. On the other hand, $|\mathcal{R}(\lambda_-^{1,2}(\Delta\Gamma))|$ becomes unbounded in this limit. By evaluation of (3.24), we see that $\mathcal{R}(\lambda)$ becomes unbounded as $\lambda_-^1(\Delta\Gamma) \to -\frac{3}{4}$, the stable pole of $\mathcal{R}(\lambda)$, and as $\lambda_-^2(\Delta\Gamma) \to +\frac{5}{4}$, the other, unstable pole of $\mathcal{R}(\lambda)$. The passage of the real eigenvalue $\lambda_-^2(\Delta\Gamma)$ through zero corresponds to $\Delta\Gamma$ given by (3.37) since $\mathcal{R}(0) = 6$ [5]. In particular, the eigenvalue problem (3.22) has incompatible eigenvalues for all $\Delta\Gamma < \log 3/(\varepsilon^2\sqrt{\mu})$, for $\mu > \mu_{\rm TP}$.

In the second case, the $\lambda_+^{1,2}(\Delta\Gamma)$-branches may cross through the imaginary axis. For the tangent point value, $\mu = \mu_{\rm TP}$, the $\lambda_+^{1,2}$ curves are tangent to the imaginary axis. For $\mu \in (\mu_{\rm Hopf}, \mu_{\rm TP})$, a part of the closed, complex conjugate $\lambda_+^{1,2}(\Delta\Gamma)$-curves lies in the unstable half-plane, while the endpoints of the curve, i.e. the eigenvalues associated to the stationary homoclinic one-pulse limit, lies in the stable half-plane, see Figure

3.1(b). More specifically, $\mathrm{Re}[\lambda_+^{1,2}(\Delta\Gamma)] > 0$ for $\Delta\Gamma \in (\Delta\Gamma_+^{1,*}(\mu), \Delta\Gamma_+^{2,*}(\mu))$, where

$$\lim_{\mu\to\mu_{\mathrm{Hopf}}} \Delta\Gamma_+^{1,*}(\mu) = 0, \quad \lim_{\mu\to\mu_{\mathrm{Hopf}}} \Delta\Gamma_+^{2,*}(\mu) = \infty, \quad \lim_{\mu\to\mu_{\mathrm{TP}}} \Delta\Gamma_+^{1,*}(\mu) = \lim_{\mu\to\mu_{\mathrm{TP}}} \Delta\Gamma_+^{2,*}(\mu) \approx \frac{1.32}{\varepsilon^2},$$

so that $\Delta\Gamma^*(\mu) = \Delta\Gamma_+^{2,*}(\mu) > \log 3/(\varepsilon^2\sqrt{\mu})$ for $\mu \in (\mu_{\mathrm{Hopf}}, \mu_+^*)$ for a certain $\mu_+^* \in (\mu_{\mathrm{Hopf}}, \mu_{\mathrm{TP}})$. □

The orbits of the eigenvalues λ of (3.22) as function of $\Delta\Gamma$ can be determined by a direct evaluation of $\mathcal{R}(\lambda)$ [5], see Figure 3.1.

Remark 3.5. Competition instabilities and synchronous oscillatory instabilities were identified for the Gierer-Meinhardt equations in [17, 16], see especially Section 5.2 of [16]. The presence of these two instabilities is related to the two multipliers in the NLEP, as also found here.

Remark 3.6. Proposition 3.4 implies that $\Phi_\Gamma(\xi)$ is not spectrally compatible with the manifold $\mathcal{M}$ if $\Delta\Gamma(0) < \Delta\Gamma^*(\mu)$. However this lower bound on the admissible pulse separation distance does not limit the semi-strong character of the pulse interaction in $\Phi_\Gamma(\xi)$, since the U-component of $\Phi_\Gamma(\xi)$ evolves on the slow $\varepsilon^2\xi$ space scale. To quantify the lower bound on pulse separation we determine the corresponding maximum value of the minimum $U_{\min}(t)$ of the inhibitor U between the two pulses $\Gamma_{1,2}$ – see also Figure 1.1. Since $U_{\min}(t;\mu)$ decreases monotonically in time (by (2.11) and (2.13)), we find that a spectrally compatible two-pulse solution must satisfy

$$U_{\min}(0) < U^*_{\min}(\mu) = \frac{A(\Delta\Gamma^*)}{\cosh \varepsilon^2\sqrt{\mu}\Delta\Gamma^*/2} = \frac{3}{16}\sqrt{3\mu},$$

if $\mu > \mu_{\mathrm{TP}}$ (3.37). In the context of Figure 1.1, in which $\mu = 5$, it follows that $U_{\min}(0)$ must be less than 0.72.... The evolution shown there is thus governed by Theorem 1.1.

Remark 3.7. The lower bound (3.37) on the pulse separation distance does not contradict the pulse-splitting behavior observed in the Gierer-Meinhardt equation [7], in which a stable two-pulse solution is observed with an $\mathcal{O}(1)$ pulse separation distance at the onset of splitting. It is shown in [7] that pulse splitting only occurs for $\mu = \mathcal{O}(1/\varepsilon^4)$. For these values of μ, $\Delta\Gamma^*(\mu) = \mathcal{O}(1)$ (3.37), which implies that the two V-pulses of $\Phi_\Gamma(\xi)$ are no longer well-separated. Thus, the lower bound (3.37) agrees with the analysis of [7], since it implies that μ must be $\mathcal{O}(1/\varepsilon^4)$ in order to have two-pulse solutions that are not well-separated.

3.3. The Resolvent Estimates and the Semi-group. To establish estimates on the semigroup generated by the reduced linearization $\tilde{L}$ we begin with preliminary bounds on the resolvents of L_{11} and $\mathcal{L}$ in the norms defined in Section 2.1. A key point is that the resolvent of L_{11} is strongly contractive on zero-mass functions.

Lemma 3.1. *Let $\lambda \in \mathbf{C}$ be an $\mathcal{O}(1)$ distance from $\sigma(L_{11})$ and set $g = (L_{11} - \lambda)^{-1} f$. Then the following estimates hold uniformly in λ,*

$$\varepsilon\|g\|_{L^2} + \varepsilon^{-1}\|\partial_\xi g\|_{L^2} \le c\varepsilon^2 \|f\|_{\widehat{L}^\infty}. \tag{3.39}$$

Moreover for small total mass, $\overline{f}$ we have the improved estimate,

$$\varepsilon\|g\|_{L^2} + \varepsilon^{-1}\|\partial_\xi g\|_{L^2} \le c\left(\varepsilon^2|\overline{f}| + \varepsilon^4 \left\|<x>f\right\|_{L^1}\right). \tag{3.40}$$

Proof: We take the Fourier transform of the equation $(L_{11} - \lambda)g = f$ obtaining,

$$\widehat{g}(k) = \frac{1}{\sqrt{2\pi}} \frac{\varepsilon^4 \widehat{f}(k)}{k^2 + \varepsilon^4(\lambda+\mu)}. \tag{3.41}$$

Assuming that $f \in \widehat{L}^\infty$, the bound

$$\left(\int_{-\infty}^{\infty} \left| \frac{\varepsilon^4}{k^2 + \varepsilon^4(\lambda + \mu)} \right|^2 dk \right)^{\frac{1}{2}} \leq c\varepsilon, \tag{3.42}$$

for some $c > 0$, shows that $\|g\|_{L^2} \leq c\varepsilon \|f\|_{\widehat{L}^\infty}$. Replacing $\widehat{f}$ with $ik\widehat{f}$ in (3.41) and calculating an integral similar to (3.42) gives $\|\partial_\xi g\|_{L^2} \leq c\varepsilon^3 \|f\|_{\widehat{L}^\infty}$. Together these results yield (3.39). In the case that f has small mass, the identify $\widehat{f}(0) = \overline{f}$ and the fact that the norm $\| <x> f\|_{L^1}$ controls the L^∞ norm of the k-derivative of the Fourier transform of f, imply that $\widehat{f}$ is uniformly Lipschitz and small at zero, and so we have the estimate

$$|\widehat{f}(k)| \leq c \frac{|\overline{f}| + |k|}{1 + |k|} \| <x> f\|_{L^1}. \tag{3.43}$$

This inequality, used in (3.41) leads to the bound (3.40). $\square$

We define $\mathcal{V}$ to be the eigenspace associated to the two exponentially small eigenvalues $\lambda_\pm^*$ of $\mathcal{L}^\dagger$, the adjoint of $\mathcal{L}$.

Lemma 3.2. *Assume that $\lambda \in \mathbf{C}$ is an $\mathcal{O}(1)$ distance from $\sigma(\mathcal{L})\backslash\{\lambda_+, \lambda_-\}$. Then we have the following estimate, uniformly in λ, and for $\Gamma \in \mathcal{K}$.*

$$\|(\mathcal{L} - \lambda)^{-1} f\|_{H^1} \leq c\|f\|_{L^2}, \tag{3.44}$$

for all $f \perp \mathcal{V}$.

Proof: The NLEP operator $\mathcal{L}$ is a finite rank perturbation of L_{22}, a self-adjoint Schrödinger operator, and hence is Fredholm. Moreover, away from its point spectrum, $\mathcal{L} - \lambda$ is boundedly invertible with $\mathcal{O}(1)$ inverse, uniformly in $\Delta\Gamma$ for $\Gamma \in \mathcal{K}$. If $f \perp \mathcal{V}$ then $\mathcal{L} - \lambda$ is uniformly invertible for λ in a neighborhood of $\lambda_\pm$. To obtain uniformity in λ for large $|\lambda|$ we observe that the resolvent of $\mathcal{L}$ can be explicitly constructed in terms of the resolvent of the self-adjoint operator L_{22} and that this later quantity decays like $\left(\text{ dist } (\lambda, \sigma(L_{22})\right)^{-1}$. That the resolvent of $\mathcal{L}$ maps into H^1 follows from a classic argument by contradiction. $\square$

To study the resolvent of $\tilde{L}$ we project off the eigenspace $\{\Psi_+, \Psi_-\}$ associated to its small eigenvalues, $\lambda_\pm$. We introduce the space $X_{\mathbf{\Gamma}} = \left\{ \vec{U} \middle| \|\vec{U}\|_X < \infty \text{ and } \pi_{\mathbf{\Gamma}} \vec{U} = 0 \right\}$, where the spectral projection is given in terms of the adjoint eigenfunctions $\Psi_\pm^\dagger$ by

$$\pi_{\mathbf{\Gamma}} \vec{U} = \frac{(\vec{U}, \Psi_-^\dagger)}{(\Psi_-, \Psi_-^\dagger)} \Psi_- + \frac{(\vec{U}, \Psi_+^\dagger)}{(\Psi_+, \Psi_+^\dagger)} \Psi_+. \tag{3.45}$$

The complimentary projection is $\tilde{\pi}_{\mathbf{\Gamma}} = I - \pi_{\mathbf{\Gamma}}$. Assuming the spectral compatibility of Φ_Γ, the space $X_{\mathbf{\Gamma}}$ is associated to temporally decaying solutions of the semigroup generated by $\tilde{L}_{\mathbf{\Gamma}}$, while $\tilde{X}_{\mathbf{\Gamma}} = \mathcal{R}\pi_{\mathbf{\Gamma}}$ is the eigenspace associated to the two exponentially small eigenvalues $\lambda_\pm$. To characterize the projections we need asymptotics for these eigenfunctions.

Lemma 3.3. *The small eigenvalue eigenfunctions have the following asymptotic form*

$$\Psi_\pm = \begin{pmatrix} 0 \\ \phi_1' \pm \phi_2' \end{pmatrix} + \text{ exponentially small }, \tag{3.46}$$

$$\Psi_\pm^\dagger = \begin{pmatrix} 0 \\ \phi_1' \pm \phi_2' \end{pmatrix} + \mathcal{O}\left(\varepsilon^4\right), \tag{3.47}$$

in the X-norm.

Proof: The expansion for the eigenfunctions follows from classical results. For the adjoint eigenfunctions we present the case for a single pulse, the generalization to two-pulses is straightforward. The adjoint operator is given by

$$\tilde{L}_{\mathbf{\Gamma}}^{\dagger} = \begin{pmatrix} L_{11} & -\dfrac{\phi_1^2}{A^2} \\ 2\varepsilon^{-2}\phi_1 \otimes \delta_{\Gamma_1} & L_{22} \end{pmatrix}, \tag{3.48}$$

where L_{11} and L_{22} are given in (3.7). Writing $\Psi^{\dagger} = (\Psi_1^{\dagger}, \Psi_2^{\dagger})^t$, and taking λ_+^* exponentially small, we solve for the second component of $\Psi^{\dagger}$, noting that ϕ_1 is in the range of L_{22} since it is orthogonal to its kernel, ϕ_1',

$$\Psi_2^{\dagger} = \beta\phi_1' - 2\varepsilon^{-2}\Psi_1^{\dagger}(\Gamma_1)L_{22}^{-1}\phi_1, \tag{3.49}$$

where β is a free parameter. Solving for $\Psi_1^{\dagger}$ we have

$$\Psi_1^{\dagger} = \frac{\beta}{A^2}L_{11}^{-1}\phi_1^2\phi_1' - \frac{2\Psi_1^{\dagger}(\Gamma_1)}{\varepsilon^2 A^2}L_{11}^{-1}\left(\phi_1^2 L_{22}^{-1}\phi_1\right). \tag{3.50}$$

The function $\phi_1^2\phi_1'$ has zero mass, so from (3.40) we have

$$\|L_{11}^{-1}\phi_1^2\phi_1'\|_{L^\infty} \leq c\varepsilon^4. \tag{3.51}$$

It can be verified that $\phi_1^2 L_{22}^{-1}\phi_1$ is a positive $\mathcal{O}(1)$ function, thus we know $L_{11}^{-1}\left(\phi_1^2 L_{22}^{-1}\phi_1\right)\Big|_{\xi=\Gamma_1}$ is nonzero and $\mathcal{O}(1)$. Evaluating (3.50) at $\xi = \Gamma_1$ and solving for $\Psi_1^{\dagger}(\Gamma_1)$ shows that $\Psi_1^{\dagger}(\Gamma_1) = \mathcal{O}(\varepsilon^6)$. Substituting this back into (3.50) and choosing $\beta = 1$ yields the equivalent of (3.47) in the one-pulse case. $\square$

With these results we may estimate the resolvent of $\tilde{L}_{\mathbf{\Gamma}}$ restricted to $X_{\mathbf{\Gamma}}$.

Proposition 3.8. *Let λ be an $\mathcal{O}(1)$ distance from $\sigma(\tilde{L})\backslash\{\lambda_+, \lambda_-\}$ and denote $G = (\tilde{L} - \lambda)^{-1}F$. For $F \in X_{\mathbf{\Gamma}}$, we have the following estimates on the resolvent of $\tilde{L}$, holding uniformly in λ, and in $\Gamma \in \mathcal{K}$,*

$$\|G\|_X \leq c\left(\varepsilon^2\|F_1\|_{L^1} + \|F_2\|_{L^2}\right). \tag{3.52}$$

If in addition the mass of F_1 is small, then we have the improved estimate

$$\|G\|_X \leq c\left(\varepsilon^2|\overline{F}_1| + \varepsilon^4\| <x> F_1\|_{L^1} + \|F_2\|_{L^2}\right). \tag{3.53}$$

Proof: By analogy with the eigenvalue problem we solve for G_1

$$G_1 = (L_{11} - \lambda)^{-1}F_1 + \alpha_1 H(\lambda, \xi - \Gamma_1) + \alpha_2 H(\lambda, \xi - \Gamma_2), \tag{3.54}$$

where H is given by (3.16), and $\alpha_k = (G_2, \phi_k)_{L^2}$, for $k = 1, 2$. The second component of G satisfies

$$(L_{22} - \lambda)G_2 = F_2 + \frac{V_0^2}{A^2}\left((L_{11} - \lambda)^{-1}F_1 + \alpha_1 H(\xi - \Gamma_1) + \alpha_2 H(\xi - \Gamma_2)\right). \tag{3.55}$$

Approximating the product $V_0^2 H$ as in the eigenvalue problem, we find the equation

$$(\mathcal{L} - \lambda)\, G_2 = F_2 + \frac{V_0^2}{A^2}(L_{11} - \lambda)^{-1}F_1, \tag{3.56}$$

where the NLEP operator $\mathcal{L}$ is defined in (3.23).

From the asymptotics on $\Psi_\pm^\dagger$ the condition $F \in X_{\mathbf{\Gamma}}$ is equivalent to the right-hand side of (3.56) being orthogonal to $\mathcal{V}$. From Proposition 3.1 the point spectrum of $\tilde{L}$, less its exponentially small eigenvalues, agrees with the point spectrum of $\mathcal{L}$, less its exponentially small eigenvalues, up to $\mathcal{O}(\varepsilon^2)$. So λ is an $\mathcal{O}(1)$ distance from $\sigma(\mathcal{L})\backslash\{\lambda_\pm\}$ and the estimate (3.44) applied to (3.56) yields

$$\begin{aligned}\|G_2\|_{H^1} &\leq c\left(\|F_2\|_{L^2} + \|V_0^2(L_{11}-\lambda)^{-1}F_1\|_{L^2}\right),\\ &\leq c\left(\|F_2\|_{L^2} + \|(L_{11}-\lambda)^{-1}F_1\|_{L^\infty}\right).\end{aligned} \tag{3.57}$$

From (3.39) and (1.2) we find that

$$\|G_2\|_{H^1} \leq c\left(\|F_2\|_{L^2} + \varepsilon^2\|F_1\|_{L^1}\right). \tag{3.58}$$

If F_1 has small mass, then applying (3.40) we have the improved estimate

$$\|G_2\|_{H^1} \leq c\left(\|F_2\|_{L^2} + \varepsilon^2|\overline{F}_1| + \varepsilon^4\left\|<x>F_1\right\|_{L^1}\right). \tag{3.59}$$

From (3.54) and (3.39) we find that

$$\|G_1\|_{L^2} \leq c\left(\varepsilon\|F_1\|_{L^1} + \|G_2\|_{L^2}\|H(\lambda)\|_{L^2}\right), \tag{3.60}$$

but since

$$|\widehat{H}(k,\lambda)| \leq c\left|\frac{\varepsilon^2}{k^2+\varepsilon^4(\lambda+\mu)}\right|, \tag{3.61}$$

we have $\|H(\lambda)\|_{L^2} \leq c\varepsilon^{-1}$ and we obtain

$$\|G_1\|_{L^2} \leq c\left(\varepsilon\|F_1\|_{L^1} + \varepsilon^{-1}\|F_2\|_{L^2}\right). \tag{3.62}$$

A similar argument yields

$$\|\partial_\xi G_1\|_{L^2} \leq c\left(\varepsilon^3\|F_1\|_{L^1} + \varepsilon\|F_2\|_{L^2}\right), \tag{3.63}$$

which verifies (3.52)

If F_1 has small mass, then applying (3.40) to (3.57) yields the improved estimate

$$\|G_2\|_{H^1} \leq c\left(\|F_2\|_{L^2} + \varepsilon^2|\overline{F}_1| + \varepsilon^4\left\|<x>F_1\right\|_{L^1}\right). \tag{3.64}$$

Following the arguments laid out in (3.60-3.63) yields (3.53). □

Since $\tilde{L}$ is an analytic operator we can generate its semigroup from the Laplace transform of the resolvent. We fix the contour C in the complex plane as depicted in Figure 3.3 and generate the semigroup S associated to $\tilde{L}\Big|_{X_{\mathbf{\Gamma}}}$ via the contour integral

$$S(t)F = \frac{1}{2\pi i}\int_C e^{\lambda t}(\lambda - \tilde{L})^{-1}F\,d\lambda, \tag{3.65}$$

where we assume that $F \in X_{\mathbf{\Gamma}}$. The semi-group inherits the following properties from the resolvent.

Proposition 3.9. *Let $\mu > \mu_{\mathrm{Hopf}}$ and $\Delta\Gamma > \Delta\Gamma^*(\mu)$ be given and let $\nu > 0$ be as given by Proposition 3.4. The solution $\vec{U}$ of $\vec{U} = S(t)F$, where $F \in X_{\mathbf{\Gamma}}$, satisfies*

$$\|\vec{U}\|_X \leq Me^{-\nu t}\left(\varepsilon^2\|F_1\|_{L^1} + \|F_2\|_{L^2}\right), \tag{3.66}$$

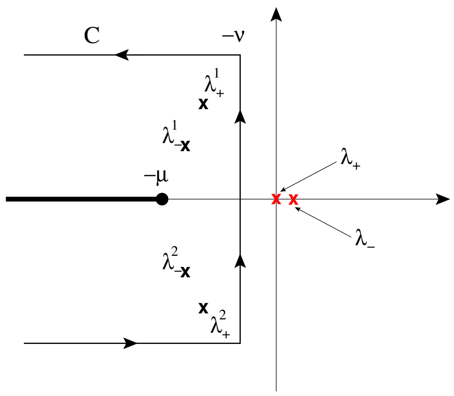

FIG. 3.2. *The spectrum, $\sigma(\tilde{L})$, of the reduced operator as determined by Proposition 3.3, and the contour C used to generate the semi-group S. Depiction is for the case $\mu > \mu_{\rm Hopf}$ and $\Delta\Gamma > \Delta\Gamma^*(\mu)$ for which $\lambda_\pm^{1,2}$ are within the left-half plane and $J_\pm = 2$, see Proposition 3.4. The eigenspace corresponding to the small point spectrum $\{\lambda_\pm\}$ is projected away and is not contained within the contour.*

for some $M > 0$ independent of $\Delta\Gamma > \Delta\Gamma^*(\mu)$. If in addition F_1 has small mass, then we have the improved estimate

$$\|\vec{U}\|_X \leq Me^{-\nu t}\left(\varepsilon^2|\overline{F}_1| + \varepsilon^4\| <x> F_1\|_{L^1} + \|F_2\|_{L^2}\right). \tag{3.67}$$

Proof: By Proposition 3.4, the conditions on μ and $\mathbf{\Gamma}$ imply that $\sigma(\tilde{L})\backslash\{\lambda_+, \lambda_-\}$ is contained within the interior of the contour C, and $\text{dist}(\sigma(\tilde{L}), C) = \mathcal{O}(1)$. The estimates on the semigroup follow directly from the contour integral representation (3.65) of $S(t)$, the resolvent estimates (3.52-3.53), and the uniformity of these estimates over the contour C. $\square$

4. Nonlinear Stability via the Renormalization Group method. We adapt the renormalization group method developed in [14] to the singular perturbation setting of the Gierer-Meinhardt equations. We assume at time t_0 that our initial data $\vec{U}_0$ satisfies

$$\|\Phi_{\mathbf{\Gamma}_*} - \vec{U}_0\| \leq \delta, \tag{4.1}$$

for some $\mathbf{\Gamma}_* \in \mathcal{K}$. The following Proposition, adapted from Proposition 2.2 of [14], permits us to choose our base point $\mathbf{\Gamma}_0$ about which we develop our local coordinate system.

PROPOSITION 4.1. *Fix $\delta \ll 1$. Given $\vec{U}_0$ and $\mathbf{\Gamma}_* \in \mathcal{K}$ satisfying $\|W_*\|_X \leq \delta$, for $W_* \equiv \Phi_{\mathbf{\Gamma}_*} - \vec{U}_0$, then there exists $M > 0$, independent of $\vec{U}_0$ and $\mathbf{\Gamma}_*$, and a smooth function $\mathcal{H} : X \mapsto \mathcal{K}$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_* + \mathcal{H}(W_*)$ satisfies*

$$W_0 \equiv \vec{U}_0 - \Phi_{\mathbf{\Gamma}} \in X_{\mathbf{\Gamma}}. \tag{4.2}$$

Moreover, if $W_* \in X_{\tilde{\mathbf{\Gamma}}}$ for some $\tilde{\mathbf{\Gamma}} \in \mathcal{K}$ then

$$|\mathbf{\Gamma} - \mathbf{\Gamma}_*| \leq M_0\|W_*\|_X|\mathbf{\Gamma}_* - \tilde{\mathbf{\Gamma}}|. \tag{4.3}$$

Proof: Since

$$W_0 = W_* + \Phi_{\mathbf{\Gamma}} - \Phi_{\mathbf{\Gamma}_*}, \tag{4.4}$$

the condition (4.2) is equivalent to

$$0 = \pi_{\mathbf{\Gamma}} W_0 = \pi_{\mathbf{\Gamma}} \left(W_* + \Phi_{\mathbf{\Gamma}} - \Phi_{\mathbf{\Gamma}_*} \right). \tag{4.5}$$

Since $\Psi^{\dagger}_{\pm}$, are approximately spanned by $(0, \phi_1')^t$ and $(0, \phi_2')^t$, and $\Phi_{\mathbf{\Gamma},2} = V_0 + \mathcal{O}(\varepsilon^2)$, our equations $\Lambda = (\Lambda_1, \Lambda_2)^t$, are equivalent, up to $\mathcal{O}(\varepsilon^2)$, to

$$\Lambda_1(\mathbf{\Gamma}, W_*) \equiv \Big(W_{2,*} + V_0(\mathbf{\Gamma}) - V_0(\mathbf{\Gamma}_*), \phi_1'(\cdot, \Gamma_1) \Big)_{L^2} = 0, \tag{4.6}$$

$$\Lambda_2(\mathbf{\Gamma}, W_*) \equiv \Big(W_{2,*} + V_0(\mathbf{\Gamma}) - V_0(\mathbf{\Gamma}_*), \phi_2'(\cdot, \Gamma_2) \Big)_{L^2} = 0. \tag{4.7}$$

Since $\Lambda(\mathbf{\Gamma}_*, 0) = 0$ and the $\mathbf{\Gamma}$ gradient of Λ given by

$$\nabla_{\mathbf{\Gamma}} \Lambda \Big|_{(\mathbf{\Gamma} = \mathbf{\Gamma}_*, W_* = 0)} = \begin{pmatrix} -\|\phi_1'\|_{L^2} & 0 \\ 0 & -\|\phi_1'\|_{L^2} \end{pmatrix} + \mathcal{O}(\varepsilon^2), \tag{4.8}$$

is uniformly invertible, the implicit function theorem guarantees the existence of a smooth function $\mathcal{H}$ which provides the solution of (4.2) and in a neighborhood about the manifold $\mathcal{M}$ defined in (2.6). The interval of existence of $\mathcal{H}$ may be chosen uniformly in $\mathbf{\Gamma}$ since the solution of (4.2) behaves smoothly as $\Delta\Gamma \to \infty$.

If in addition we have $W_* \in X_{\tilde{\mathbf{\Gamma}}}$, then $\Big(W_{2,*}, \phi_k'(\tilde{\Gamma}_k) \Big)_{L^2} = \mathcal{O}(\varepsilon^4)$ for $k = 1, 2$. We see that

$$\left| \Big(W_{2,*}, \phi_k'(\Gamma_k) \Big)_{L^2} \right| \leq \left| \Big(W_{2,*}, \phi_k'(\tilde{\Gamma}_k) - \phi_k'(\Gamma_k) \Big)_{L^2} \right| \leq M_0 \|W_*\|_{L^2} |\tilde{\mathbf{\Gamma}} - \mathbf{\Gamma}|. \tag{4.9}$$

□

4.1. The Projected Equations. To begin the RG procedure we freeze $\mathbf{\Gamma} = \mathbf{\Gamma}_0$ in $X_{\mathbf{\Gamma}_0}$, where $\mathbf{\Gamma}_0$ is the base point provided by the proposition above, and change variables as

$$\vec{U}(t) = \Phi_{\mathbf{\Gamma}} + W, \tag{4.10}$$

where $W \in X_{\mathbf{\Gamma}_0}$, and $\mathbf{\Gamma} = \mathbf{\Gamma}(t)$. Comparing to (3.2), we write the evolution for the remainder W as

$$W_t + \frac{\partial \Phi}{\partial \mathbf{\Gamma}} \dot{\mathbf{\Gamma}} = \mathbf{R} + \tilde{L}_{\mathbf{\Gamma}_0} W + \Big(L_{\mathbf{\Gamma}} - \tilde{L}_{\mathbf{\Gamma}_0} \Big) W + \mathcal{N}(W), \tag{4.11}$$

$$W(\xi, 0) = W_0, \tag{4.12}$$

where $W_0 = W_* + \Phi_{\mathbf{\Gamma}_0} - \Phi_{\mathbf{\Gamma}_*}$. The terms $\Delta L \equiv L_{\mathbf{\Gamma}} - \tilde{L}_{\mathbf{\Gamma}_0}$ include both the approximations made to the linear operator and the secular growth implicit in the sliding of $\mathbf{\Gamma}$ away from $\mathbf{\Gamma}_0$.

To enforce $W \in X_{\mathbf{\Gamma}_0}$ we impose the non-degeneracy condition $\frac{\partial}{\partial t} \pi_0 W = 0$, where $\pi_0 = \pi_{\mathbf{\Gamma}_0}$ is given by (3.45). Since π_0 is independent of time, the non-degeneracy condition is equivalent to $\pi_0 W_t = 0$, and moreover as π_0 commutes with $\tilde{L}_{\mathbf{\Gamma}_0}$ it follows that $\pi_0 \tilde{L}_{\mathbf{\Gamma}_0} W = \tilde{L}_{\mathbf{\Gamma}_0} \pi_0 W = 0$. The non-degeneracy condition is thus equivalent to the two equations

$$\left(\frac{\partial \Phi}{\partial \mathbf{\Gamma}} \dot{\mathbf{\Gamma}}, \Psi^{\dagger}_{\pm} \right)_{L^2} = \Big(\mathbf{R} + \Delta L \; W + \mathcal{N}(W), \Psi^{\dagger}_{\pm} \Big)_{L^2}. \tag{4.13}$$

From the form of the semi-strong pulse solutions, and assuming momentarily that $\dot{\mathbf{\Gamma}} = \mathcal{O}(\varepsilon^2)$, we calculate

$$\frac{\partial \Phi_{\mathbf{\Gamma}}}{\partial \mathbf{\Gamma}} \dot{\mathbf{\Gamma}} = \begin{pmatrix} 0 \\ \phi_1' \dot{\Gamma}_1 + \phi_2' \dot{\Gamma}_2 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\varepsilon^3) \\ \mathcal{O}(\varepsilon^4) \end{pmatrix}, \tag{4.14}$$

component-wise in the L^2 norm. Using the form of the adjoint eigenvector (3.47) and (3.6), the equations (4.13) may be written

$$\begin{pmatrix} \|\phi_1'\|_{L^2}^2 + \mathcal{O}(\varepsilon^4) & \|\phi_2'\|_{L^2}^2 + \mathcal{O}(\varepsilon^4) \\ \|\phi_1'\|_{L^2}^2 + \mathcal{O}(\varepsilon^4) & -\|\phi_2'\|_{L^2}^2 + \mathcal{O}(\varepsilon^4) \end{pmatrix} \dot{\mathbf{\Gamma}} = \begin{pmatrix} \left(\mathbf{R} + \Delta L\ W + \mathcal{N}(W), \Psi_+^\dagger\right)_{L^2} \\ \left(\mathbf{R} + \Delta L\ W + \mathcal{N}(W), \Psi_-^\dagger\right)_{L^2} \end{pmatrix}. \tag{4.15}$$

Again using the asymptotic form of the adjoint eigenfunctions $\Psi_\pm^\dagger$ we may neglect the contribution from $\Psi_{\pm,1}^\dagger$ in the inner products on the right-hand side of (4.15). In particular, from the L^1 bounds on R_1 from (2.18), we have

$$(R_1, \Psi_{\pm,1}^\dagger)_{L^2} \leq \|R_1\|_{L^1} \|\Psi_{\pm,1}^\dagger\|_{L^\infty} = \mathcal{O}(\varepsilon^5). \tag{4.16}$$

Inverting the matrix on the left-hand side, and using the expansions for $\Psi_{\pm,2}^\dagger$ we arrive at the equations of motion for $\mathbf{\Gamma}$ which show explicitly the coupling between the remainder W and the pulse evolution,

$$\dot{\Gamma}_k = -\frac{\left(R_2 + [\Delta L\ W]_2 + \mathcal{N}_2, \phi_k'(\cdot; \mathbf{\Gamma}_0)\right)_{L^2}}{(\phi_k', \phi_k')_{L^2}} + \mathcal{O}(\varepsilon^5, \varepsilon^4 \|W\|_X). \tag{4.17}$$

To simplify the equation for the evolution of the remainder, W, we introduce the reduced residual

$$\tilde{\mathbf{R}} = \tilde{\pi}_{\mathbf{\Gamma}} \left(\mathbf{R} - \frac{\partial \Phi_{\mathbf{\Gamma}}}{\partial \mathbf{\Gamma}} \dot{\mathbf{\Gamma}} \right), \tag{4.18}$$

and observe from the asymptotic description (2.17) of R_2, that the projection removes the leading order term from the second component of the residual. By Lemma 2.1, the reduced residual enjoys the estimates

$$\|\tilde{R}_1\|_{L^1} \leq \mathcal{O}(\varepsilon), \tag{4.19}$$
$$\|\tilde{R}_2\|_{L^2} \leq \mathcal{O}(\varepsilon^4). \tag{4.20}$$

The evolution for the remainder W is now given by

$$W_t = \tilde{\mathbf{R}} + \tilde{L}_0 W + \tilde{\pi}_0 \left(\Delta L\ W + \mathcal{N} \right), \tag{4.21}$$
$$W(\xi, t_0) = W_0, \tag{4.22}$$

where $\tilde{L}_0 = \tilde{L}_{\mathbf{\Gamma}_0}$ and $\tilde{\pi}_0 = I - \pi_{\mathbf{\Gamma}_0}$. The point of the reduction of the Gierer-Meinhardt (2.3) to the projected residual equation (4.21), in the case of two-pulse dynamics, is that the asymptotically relevant and the asymptotically negligible terms are now evident. The evolution for W is controlled by the first two terms on the right-hand side of (4.21), we will show that the last two terms are asymptotically irrelevant, until $\mathbf{\Gamma} - \mathbf{\Gamma}_0$ is so large that the secularity implicit in ΔL forces an update of $\mathbf{\Gamma}_0$.

4.2. Decay of the Remainder. We identify the duration of each renormalization interval, and quantify the decay of the remainder W over this interval. To control the dynamics we introduce the quantities

$$T_0(t) = \sup_{t_0 < s < t} e^{\nu(s - t_0)} \|W(s)\|_X, \tag{4.23}$$
$$T_1(t) = \sup_{t_0 < s < t} |\mathbf{\Gamma}(s) - \mathbf{\Gamma}_0|. \tag{4.24}$$

The first enforces the decay of the remainder, W, the second measures the distance the pulse positions have moved from their frozen base point. The variation of constants formula applied to (4.21) yields the solution

$$W(\xi, t) = S(\Delta t) W_0 + \int_{t_0}^{t} S(t - s) \left(\tilde{R} + \tilde{\pi}_0 \left(\Delta L\ W + \mathcal{N} \right) \right) ds, \tag{4.25}$$

where we have introduced $\Delta t = t - t_0$.

To estimate the distance that the pulse locations, $\mathbf{\Gamma}$, have moved from the base point, $\mathbf{\Gamma}_0$ we examine the equations (4.17). We break ΔL in secular and reductive parts, $\Delta L = \Delta L_s + \Delta L_r$ where $\Delta L_s = L_{\mathbf{\Gamma}} - L_{\mathbf{\Gamma}_0}$ and $\Delta L_r = L_{\mathbf{\Gamma}_0} - \tilde{L}_{\mathbf{\Gamma}_0}$, and remark that,

$$
\begin{aligned}
\|[\Delta L\, W]_2\|_{L^2} &\leq \|\, [\Delta L_s\, W]_2\, \|_{L^2} + \|\, [\Delta L_r\, W]_2\, \|_{L^2}, && (4.26)\\
&\leq c\left(|\mathbf{\Gamma} - \mathbf{\Gamma}_0| + \varepsilon^2\right)\|W\|_{L^2}, && (4.27)\\
&\leq c(T_1(t) + \varepsilon^2) e^{-\nu(t-t_0)} T_0(t), && (4.28)
\end{aligned}
$$

where the estimates on ΔL_s and ΔL_r are described in more detail below. From the form (3.4) of the regularized nonlinearity it is straightforward to obtain the estimate

$$
|(\mathcal{N}_2, \phi_k')_{L^2}| \leq c\|W\|_X^2. \tag{4.29}
$$

With these bounds in hand, the drift of the pulses is controlled by their speed,

$$
\begin{aligned}
T_1(t) \leq \int_{t_0}^{t_0+\Delta t} |\dot{\mathbf{\Gamma}}(s)| ds &\leq \int_{t_0}^{t_0+\Delta t} c\left(\|R_2\|_{L^2} + (\varepsilon^2 + T_1(t)) e^{-\nu(s-t_0)} T_0(t) + e^{-2\nu(s-t_0)} T_0^2(t)\right) ds, && (4.30)\\
&\leq c(\varepsilon^2 \Delta t + (\varepsilon^2 + T_1) T_0 + T_0^2). && (4.31)
\end{aligned}
$$

For T_0 small enough we can eliminate T_1 from the right-hand side, and neglecting T_0 in the sum $T_0 + \Delta t$, we obtain

$$
T_1 \leq c(\varepsilon^2 \Delta t + T_0^2). \tag{4.32}
$$

Turning to bounds on the remainder, we estimate the irrelevant terms first. The secular term takes the form

$$
\Delta L_s = \begin{pmatrix} 0 & 2\varepsilon^{-2}\Big(V_0(\cdot;\mathbf{\Gamma}) - V_0(\cdot;\mathbf{\Gamma}_0)\Big) \\ V_{12}(\cdot;\mathbf{\Gamma}) - V_{12}(\cdot,\mathbf{\Gamma}_0) & V_{22}(\cdot;\mathbf{\Gamma}) - V_{22}(\cdot;\mathbf{\Gamma}_0) \end{pmatrix}, \tag{4.33}
$$

where V_{12} and V_{22} denote the potentials in the $\tilde{L}_{12}$ and $\tilde{L}_{22}$ components of $\tilde{L}$. Since each potential V_0, V_{12}, and V_{22} decays rapidly away from the pulse locations, the difference between the potential centered at pulse locations Γ_0 and at Γ scales like $\Gamma - \Gamma_0$ in any reasonable norm. In particular

$$
\|V_0(\cdot,\Gamma) - V_0(\cdot,\Gamma_0)\|_{H^1} + \|V_0(\cdot,\Gamma) - V_0(\cdot,\Gamma_0)\|_{L^1} \leq c|\Gamma - \Gamma_0|, \tag{4.34}
$$

and similarly for V_{12} and V_{22}. Using these estimates it follows directly that

$$
\begin{aligned}
\|[\Delta L_s\, W]_1\|_{L^1} &\leq c\varepsilon^{-2}\|V_0(\cdot,\Gamma) - V_0(\cdot,\Gamma_0)\|_{L^1}\|W_2\|_{L^\infty}, && (4.35)\\
&\leq c\varepsilon^{-2} T_1 \|W\|_X. && (4.36)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|[\Delta L_s\, W]_2\|_{L^2} &\leq c\|V_{12}(\cdot,\Gamma) - V_{12}(\cdot,\Gamma_0)\|_{H^1}\|W_1\|_X + \|V_{22}(\cdot,\Gamma) - V_{22}(\cdot,\Gamma_0)\|_{H^1}\|W_2\|_X, && (4.37)\\
&\leq c T_1 \|W\|_X. && (4.38)
\end{aligned}
$$

Combining these estimates with the unweighted semi-group estimate (3.66) we find

$$
\|S(t-s)\tilde{\pi}_0(\Delta L_s\; W(s))\|_X \leq M e^{-\nu(t-s)} T_1(s) \|W\|_X. \tag{4.39}
$$

The small mass version of the semi-group estimate plays a key role in controlling the reductive term, $\Delta L_r\, W$, given by

$$\Delta L_r = \begin{pmatrix} 0 & 2\varepsilon^{-2}\Big(V_0 - \delta_{\Gamma_1}\otimes\phi_1(\mathbf{\Gamma}_0) - \delta_{\Gamma_2}\otimes\phi_2(\mathbf{\Gamma}_0)\Big) + \mathcal{O}(1) \\ \mathcal{O}(\varepsilon^2 V_0^2) & \mathcal{O}(\varepsilon^2 V_0) \end{pmatrix}, \tag{4.40}$$

where here the $\mathcal{O}$ indicates point-wise estimates. The first component of the reductive term can be decomposed as

$$[\Delta L_r\, W]_1 = \varepsilon^{-2}\left(\Sigma_1 + \Sigma_2\right), \tag{4.41}$$

where

$$\Sigma_k = \phi_k - \delta_{\mathbf{\Gamma}_1}\otimes\phi_k + \mathcal{O}(\varepsilon^2\phi_k). \tag{4.42}$$

That is, each Σ_k decays at an $\mathcal{O}(1)$ exponential rate away from $x=\Gamma_k$, and moreover at leading order each component $\Sigma_k W_2$ is mass-free for $k=1,2$. In particular the total mass of the 1 component of $\Delta L_r\, W$ arises only from the higher order corrections,

$$|\overline{[\Delta L_r\, W]_1}| \leq c\|W_2\|_X. \tag{4.43}$$

In the weighted norms we estimate

$$\| {<}x-\Gamma_k{>}\Sigma_k W_2\|_{L^2} \leq \varepsilon^{-2}\Big(\| {<}x-\Gamma_k{>}\delta_{\Gamma_k}\|_{L^2} + \| {<}x-\Gamma_k{>}\phi_k\|_{L^2}\Big)\|W_2\|_{L^\infty} \leq c\varepsilon^{-2}\|W\|_X, \tag{4.44}$$

for $k=1,2$. For the second component we observe that

$$\|[\Delta L_r W]_2\|_{H^1} \leq c\varepsilon^2\|W\|_{H^1}, \tag{4.45}$$

so from the weighted semigroup estimate (3.67) we find

$$\|S(t-s)\tilde{\pi}_0(\Delta L_r\; W(s))\|_X \leq Me^{-\nu(t-s)}\varepsilon^2\|W\|_X, \tag{4.46}$$

independent of the pulse spacing $\Delta\Gamma$.

Finally for the nonlinear term, given by (3.4), it is easy to verify,

$$\begin{aligned}\|S(t-s)\tilde{\pi}_0\mathcal{N}\|_X &\leq Me^{-\nu(t-s)}\Big(\|W_2^2\|_{L^1} + \|W_2^2\|_{L^2} + \|W_1^2\|_{L^2}\Big), &(4.47)\\ &\leq Me^{-\nu(t-s)}\|W\|_X^2. &(4.48)\end{aligned}$$

From the bounds on the reduced residual (4.19-4.20) and the semi-group estimate we obtain

$$\|S(t-s)\tilde{R}\|_X \leq M\varepsilon^3 e^{-\nu(t-s)}. \tag{4.49}$$

Taking the X-norm of variation of constants solution for W, (4.25), and using the estimates outlined above we obtain

$$\|W(t)\|_X \leq M\left(e^{-\nu\Delta t}\|W(t_0)\|_X + \int_{t_0}^{t} e^{-\nu(t-s)}\left[\varepsilon^3 + \left(\varepsilon^2 + T_1(s)\right)\|W(s)\|_X + \|W(s)\|_X^2\right]\, ds\right). \tag{4.50}$$

To estimate the decay of $\|W(t')\|_X$ for $t'\in(t_0,t)$ we evaluate (4.50) at $t=t'$, multiply by $e^{\nu(t'-t_0)}$, and take the sup over $t'\in(t_0,t)$ obtaining

$$\begin{aligned}T_0(t) &\leq M\left(T(t_0) + \int_{t_0}^{t}\Big[\varepsilon^3 e^{\nu(s-t_0)} + \left(\varepsilon^2 + T_1(t)\right)T_0(t) + e^{-\nu s}T_0(t)^2\Big]\, ds\right), &(4.51)\\ &\leq M\Big(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t} + \left(\varepsilon^2 + T_1(t)\right)\Delta t\, T_0(t) + T_0(t)^2\Big). &(4.52)\end{aligned}$$

From (4.32) we may eliminate T_1 from the T_0 estimate,

$$T_0(t) \leq M\Big(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t} + \varepsilon^2(\Delta t + (\Delta t)^2)T_0(t) + T_0(t)^2 + \Delta t\, T_0^3\Big) \tag{4.53}$$

For $\Delta t \ll \min\left\{\varepsilon^{-1}, T_0^{-1}\right\}$ the term $M\varepsilon^2((\Delta t)^2+\Delta t) < \frac{1}{2}$ and we may eliminate the linear term in T_0 from the right-hand side, in addition we may absorb the cubic term in T_0 into the quadratic one. With these reductions (4.53) becomes

$$T_0 \leq 2M\Big(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t} + T_0^2\Big). \tag{4.54}$$

The quadratic equation in T_0

$$0 = T_0(t_0) + \varepsilon^3 e^{\nu\Delta t} - \frac{1}{2M}T_0 + T_0^2, \tag{4.55}$$

has two positive real roots so long as $T_0(t_0) + \varepsilon^3 e^{\nu\Delta t} \ll 1$. The smaller of these roots, r_0 takes the form

$$r_0 = 2M(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t}) + \mathcal{O}\Big(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t}\Big)^2, \tag{4.56}$$

while the larger is

$$r_1 = \frac{1}{2M} + \mathcal{O}\Big(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t}\Big). \tag{4.57}$$

Thus if $T_0(t_0) \ll 1$ and $\varepsilon^3 e^{\nu\Delta t} \ll 1$ then there is an **excluded region**, either $0 < T_0 < r_0$ or $r_1 < T_0 < \infty$. Since $T_0(t_0) < r_0$ and T_0 is continuous in t, we see that

$$T_0(t) \leq r_0 \leq M(T_0(t_0) + \varepsilon^3 e^{\nu\Delta t}) \tag{4.58}$$

so long as

$$\Delta t \leq \frac{3\beta|\log\varepsilon|}{\nu} \tag{4.59}$$

for any fixed $\beta < 1$. This condition on Δt prevents the secularity from dominating the linear operator, in particular it is a stronger condition on Δt than that imposed after equation (4.53). This implies that

$$\|W(t)\|_X \leq M\Big(e^{-\nu(t-t_0)}\|W(t_0)\|_X + \varepsilon^3\Big), \qquad \text{for } t \in \left(t_0, t_0 + \frac{3\beta|\log\varepsilon|}{\nu}\right) \tag{4.60}$$

and in particular for $t_1 = t_0 + \Delta t$ we have

$$\|W(t_1)\|_X \leq M\Big(\varepsilon^{3\beta}\|W(t_0)\|_X + \varepsilon^3\Big). \tag{4.61}$$

4.3. The RG Iteration. We break the time evolution into a series of initial value problems, tracking the decay of the remainder over the long-time scale of many RG iterations. We fix $\beta < 1$ and $\Delta t = \frac{3\beta|\log\varepsilon|}{\nu}$. The renormalization times are defined sequentially

$$t_n = t_{n-1} + \Delta t. \tag{4.62}$$

We break the evolution of W into disjoint intervals $I_n = [t_n, t_{n+1})$. On each interval I_n we solve the initial value problem (4.21) with initial data $W(t_n) \in X_{\mathbf{\Gamma}_n}$, with the quantities $T_{0,n}$ and $T_{1,n}$ corresponding to (4.23-4.24) over I_n. The renormalization map, $\mathcal{G}$, takes the initial data, $W_{n-1} = W(t_{n-1})$ for the initial value problem on interval I_{n-1} and returns the initial data $W_n = W(t_n)$ for the initial value problem on the interval I_n,

$$\mathcal{G}W_{n-1} = W_n. \tag{4.63}$$

Arguing inductively, the initial data and the new base point $\mathbf{\Gamma}_n$ are obtained from $W(t_n^-)$, the end-value of the evolution of W over I_{n-1}, by applying Proposition 4.1. Indeed we know that $W(t_n^-) \in X_{\mathbf{\Gamma}_{n-1}}$ and so from (4.3) we have

$$|\mathbf{\Gamma}_n - \mathbf{\Gamma}(t_n^-)| \le M_0 \|W(t_n^-)\|_X |\mathbf{\Gamma}(t_n^-) - \mathbf{\Gamma}(t_{n-1})| \le M_0 \|W(t_n^-)\|_X T_{1,n-1}(t). \tag{4.64}$$

From the estimates on Δt, and $T_{1,n-1}$, we bound the jump in $\mathbf{\Gamma}$ at renormalization by

$$|\mathbf{\Gamma}_n - \mathbf{\Gamma}(t_n^-)| \le M_0 \left(|\log \varepsilon| \varepsilon^2 + T_{0,n-1}^2 \right) \|W(t_n^-)\|_X. \tag{4.65}$$

The solution at time $t = t_n$ is independent of the decomposition,

$$\vec{U}(t_n) = \Phi_{\mathbf{\Gamma}(t_n^-)} + W(t_n^-) = \Phi_{\mathbf{\Gamma}_n} + W_n, \tag{4.66}$$

and we may bound the jump in W at each renormalization

$$\|W(t_n^-) - W(t_n)\|_X = \|\Phi_{\mathbf{\Gamma}(t_n^-)} - \Phi_{\mathbf{\Gamma}_n}\|_X \le c|\mathbf{\Gamma}_n - \mathbf{\Gamma}(t_n^-)| \le M_0 \left(|\log \varepsilon| \varepsilon^2 + T_{0,n-1}^2 \right) \|W(t_n^-)\|_X, \tag{4.67}$$

where we used the fact that U_0 is $\mathcal{O}(1)$ X-Lipschitz in $\mathbf{\Gamma}$, as follows from (3.5) and (1.2). From (4.58), using the equality $T_{0,n-1}(t_{n-1}) = \|W_{n-1}\|_X$, we have the estimate

$$T_{0,n-1} \le M_1 \left(\|W_{n-1}\|_X + \varepsilon^{3(1-\beta)} \right). \tag{4.68}$$

Combining the estimates (4.67) and (4.68) with (4.61), we obtain a bound on $\mathcal{G}W_{n-1} = W_n$,

$$\|\mathcal{G}W_{n-1}\|_X \le \left(1 + M_0 \left[|\log \varepsilon| \varepsilon^2 + M_1^2 \left(\|W(t_{n-1})\|_X + \varepsilon^{3(1-\beta)} \right)^2 \right] \right) M \left(\varepsilon^{3\beta} \|W(t_{n-1})\|_X + \varepsilon^3 \right) \tag{4.69}$$

Neglecting the terms involving positive powers of ε within the first parenthesis on the left-hand side, we may bound $\|W(t_n)\|_X$ by η_n, solution of the map

$$\eta_n = M(1 + M_2 \eta^2)(\varepsilon^{3\beta} \eta_{n-1} + \varepsilon^3), \tag{4.70}$$

with $\eta_0 = \|W(\cdot, t_0)\|_X$, and $M_2 = M_0 M_1^2$. It is easy to see for $\eta_0 = \mathcal{O}(1)$ and ε sufficiently small that

$$\eta_n \to \frac{M}{1 - \varepsilon^{3\beta} M} \varepsilon^3, \tag{4.71}$$

as $n \to \infty$. Since $\|W(\cdot, t_n)\|_X \le \eta_n$, the estimate (4.61) yields the result (1.4) in Theorem 1.1.

4.4. Long-time Asymptotics. To recover the asymptotic pulse motion, we consider the situation where t is sufficiently large that $\|W\|_X \le M\varepsilon^3$. In this regime we see from (4.32) that $T_1 \le c\varepsilon^2 |\log \varepsilon|$, and hence from (4.27) that

$$\|\Delta L \; W\|_{L^2} \le c\varepsilon^2 |\log \varepsilon| \|W\|_{L^2} \le c\varepsilon^5 |\log \varepsilon|. \tag{4.72}$$

Moreover from the form (3.4) of the nonlinearity we readily verify that

$$\|\mathcal{N}_2(W)\| \le c\|W\|_X^2 = \mathcal{O}(\varepsilon^6). \tag{4.73}$$

In this regime the estimates (4.72) and (4.73) on the secularity and the nonlinearity show that the remainder W has an asymptotically small influence on the pulse evolution equations (4.17) which reduce to

$$\dot{\Gamma}_k(t) = -\frac{\Big(R_2, \phi_k'(\cdot;\mathbf{\Gamma}(t))\Big)_{L^2}}{\|\phi_k'\|_{L^2}^2} + \mathcal{O}(|\log\varepsilon|\varepsilon^5). \tag{4.74}$$

Furthermore, the asymptotic form (2.17) for the second component of the remainder shows that

$$\dot{\Gamma}_k(t) = \varepsilon^2\hat{c}(\mathbf{\Gamma})\frac{\Big(\phi_1' - \phi_2', \phi_k'\Big)_{L^2}}{\|\phi_k'\|_{L^2}^2} + \mathcal{O}(\varepsilon^4) = (-1)^{k+1}\varepsilon^2\hat{c}(\mathbf{\Gamma}) + \mathcal{O}(\varepsilon^4), \tag{4.75}$$

where $\hat{c}(\mathbf{\Gamma})$ is – by construction – the position dependent formal pulse speed given by (2.8). In particular the pulse separation $\Delta\Gamma = \Gamma_1 - \Gamma_2$ grows as given by (1.5) while the amplitudes increase according to (2.11).

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