The extension of the reduced Clifford algebra and its Brauer class

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Abstract. The *Clifford algebra* C_f of a binary form f of degree d is the k-algebra $k\{x,y\}/I$, where I is the ideal generated by $\{(\alpha x+\beta y)^d - f(\alpha,\beta) \mid \alpha,\beta \in k\}$. C_f has a natural homomorphic image A_f, called the *reduced Clifford algebra*, which is a rank d^2 Azumaya algebra over its center. The center is isomorphic to the coordinate ring of the complement of an explicit Θ-divisor in $\mathrm{Pic}^{d+g-1}_{C/k}$, where C is the curve $(w^d - f(u,v))$ and g is the genus of C ([9]). We show that the Brauer class of A_f can be extended to a class in the Brauer group of $\mathrm{Pic}^{d+g-1}_{C/k}$. We also show that if d is odd, then the algebra A_f is split if and only if the principal homogeneous space $\mathrm{Pic}^1_{C/k}$ of the jacobian of C has a k-rational point.

1 Introduction

Let f be a form of degree d in n variables over a field k. Then the *Clifford algebra* C_f is the k-algebra $k\{x_1,\dots,x_n\}/I$ where $k\{x_1,\dots,x_n\}$ is the free associative algebra in n variables and I is the ideal generated by $\{(\alpha_1 x_1 + \dots + \alpha_n x_n)^d - f(\alpha_1,\dots,\alpha_n) \mid \alpha_1,\dots,\alpha_n \in k\}$. If $d=2$ this is the classical Clifford algebra of a quadratic form. If $d>2$, then this is sometimes called the *generalized* Clifford algebra and has been studied by various authors including Roby [16], Revoy [15], Childs [2] and Van den Bergh [18].

The first case of higher degree (that is, $d=3, n=2$) was studied by Haile in [6, 7]. He showed that the Clifford algebra of a binary cubic form over a field k, with characteristic $\neq 2,3$, is Azumaya with center the affine coordinate ring of an elliptic curve. This elliptic curve is the jacobian of the projective curve C given by the equation $(w^3 - f(u,v))$. Further, this Azumaya algebra extends to the elliptic curve and its Brauer class (using the

Mathematics Subject Classification (2000): 16H05, 16G99, 16K50, 14H50, 14H40, 14K30 diagram 3.2 below) corresponds to $\mathrm{Pic}^1_{C/k} \cong C$. In particular, the Clifford algebra is split if and only if the curve C has a k-rational point.

In the case when $d > 3$, C_f contains a free algebra on two variables and hence cannot be Azumaya. However it has a homomorphic image A_f, called the *reduced Clifford algebra*, which is an Azumaya algebra over its center. The algebra A_f is obtained as the quotient $C_f/(\cap \mathfrak{p})$, where $\mathfrak{p}$ varies over the kernels of all the d-dimensional representations of C_f. In [9], assuming that k is an infinite field and that the characteristic of k does not divide d, we proved that the center is the affine coordinate ring of the complement U of an explicit Θ-divisor in $\mathrm{Pic}^{d+g-1}_{C/k}$. Here C is the plane curve given by the equation $(w^d - f(u,v))$ and g its genus. As an intermediate step, we proved that if the curve C has a k-rational point, then the algebra A_f is isomorphic to the algebra $End(\pi_* \mathcal{P}|_U)(U)$, where $\mathcal{P}$ is the universal line bundle on $C \times \mathrm{Pic}^{d+g-1}_{C/k}$, π is the projection on the second factor and U is the open set as above.

This paper continues the investigation of the Clifford algebra for binary forms of higher degree ($d \geq 3$) which was started in [9]. The results related to the algebra A_f and its center raise two natural questions. The first is whether the sheaf of Azumaya algebras on U extends to $\mathrm{Pic}^{d+g-1}_{C/k}$ and the second is, if it does, then what is its Brauer class. These two questions are the subject of this article. If C has a k-rational point, then we can consider the coherent sheaf $\pi_*(\mathcal{P})$, but this is not a vector bundle on $\mathrm{Pic}^{d+g-1}_{C/k}$. So A_f cannot be extended to $\mathrm{Pic}^{d+g-1}_{C/k}$ as a sheaf of Azumaya algebras using $\pi_*(\mathcal{P})$. However it does extend (using $\pi_*(\mathcal{P})$) to an open set $V \supset U$ as an Azumaya algebra $\mathcal{A}_f$, where the codimension of the complement of V in $\mathrm{Pic}^{d+g-1}_{C/k}$ is at least two. Then by general theorems about the Brauer and the cohomological Brauer groups, we show the following.

Theorem. *The Brauer class of A_f in $\mathrm{Br}(U)$ extends to a class α_f in $H^2_{\text{ét}}(\mathrm{Pic}^{d+g-1}_{C/k}, \mathbb{G}_m) = \mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k})$.*

Now we assume that d is an odd integer so that there is a k-rational line bundle on C of degree $(d+g-1)$. This makes the jacobian $J(C)$ and $\mathrm{Pic}^{d+g-1}_{C/k}$ isomorphic over k. Using this isomorphism, we can pull the class α_f back to a class in $\mathrm{Br}(J(C))$ denoted by α_f again. Using the relation between the Brauer group and the principal homogeneous spaces (see diagram 3.2 below), we prove the following.

Theorem. *The reduced Clifford algebra A_f is split if and only if the principal homogeneous space $\mathrm{Pic}^1_{C/k}$ of $J(C)$ has a k-rational point.*

For any finite field extension k' of k, these theorems using the Tate pairing ([17] or [11]) and the autoduality of jacobians of curves give group homomorphisms $J(C)(k') \to \mathrm{Br}(k')$. It is interesting to know the image of this homomorphism and the subgroup of $\mathrm{Br}(k')$ generated by these images as the form f varies. These questions will be considered in a future article.

Notations and conventions

- All rings have an identity element.
- For any scheme X and a positive integer n, $X^{(n)}$ denotes the n-fold symmetric product of X.
- The terms line bundles (respectively vector bundles) and invertible sheaves (respectively locally free sheaves) are used interchangeably.
- The translation morphism induced by a closed point a of an abelian variety A is denoted by t_a.
- For a k-variety X and an element $\sigma \in \mathrm{Gal}(k'/k)$ for some Galois extension k' of k, we get an induced map $X \times_k k' = X' \to X'$ which is also denoted by σ. In particular, if D is a divisor on X, then the divisor $\sigma^*(D)$ is also written as $D^{\sigma^{-1}}$. If X is a projective curve with a given k-rational embedding in some projective space, then the points $p^{\sigma^{-1}}$ are obtained by letting σ act on the coordinates of the point p.
- All the fields are assumed to be infinite.
- The form f is assumed to have no repeated factors over an algebraic closure of k.
- The degree d of the form f is assumed to be indivisible by the characteristic of the field k.
- For any variety X, the cohomological Brauer group $H^2(X_{\text{ét}}, \mathbb{G}_m)$ is denoted by $\mathrm{Br}'(X)$ and the Brauer group of X is denoted by $\mathrm{Br}(X)$.

2 Extension of the reduced Clifford algebra

In this section, we consider the Azumaya algebra A_f of the introduction. From the main theorem of [9], we know that the center Z_k of A_f is isomorphic the coordinate ring A_k of the complement of an explicit Θ-divisor in

$\mathrm{Pic}^{d+g-1}_{C/k}$. In fact, this Θ-divisor is the image of the morphism

$$C^{(g-1)} \longrightarrow \mathrm{Pic}^{g-1}_{C/k} \xrightarrow{\otimes \mathcal{O}(1)} \mathrm{Pic}^{d+g-1}_{C/k}. \tag{2.1}$$

Here g is the genus of the curve C and the first arrow is the canonical morphism. We denote by U the scheme *Spec* A_k. The goal of this section is to show that the Brauer class of the Azumaya algebra A_f can be extended as a Brauer class to the scheme $\mathrm{Pic}^{d+g-1}_{C/k}$(that is, it is in the image of the canonical morphism $\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k}) \to \mathrm{Br}(U)$).

First we briefly recall relevant notations and facts from [9]. Let k' be a Galois extension of the base field k so that the curve C has a k'-rational point. Let C' denote the curve $C \times_k k'$. Let U' denote the scheme *Spec* $A_{k'}$ and let $\mathcal{P}$ denote the Poincaré bundle on the scheme $C' \times_{k'} \mathrm{Pic}^{d+g-1}_{C'/k'}$. Let π denote the projection on the second factor $C' \times_{k'} \mathrm{Pic}^{d+g-1}_{C'/k'} \to \mathrm{Pic}^{d+g-1}_{C'/k'}$. Then the sheaf $\pi_*(\mathcal{P})|_{U'}$ is locally free on U' and is of rank d (Prop. 3.3, [9]). Let V' denote the largest open set in $\mathrm{Pic}^{d+g-1}_{C'/k'}$ such that $\pi_*(\mathcal{P})|_{V'}$ is locally free on V'. That V' has to be an open set is immediate from the upper semicontinuity of the function $x \to \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$ for any coherent sheaf $\mathcal{F}$. We have the following lemma.

Lemma 2.1. *The complement of the open set V' in $\mathrm{Pic}^{d+g-1}_{C'/k'}$ has codimension at least 2.*

Proof. We first recall Lemma (5.2) of [13]. Let D be a divisor on a smooth projective curve C of degree $(n-1)+(g-1)$ (g = genus of C) such that $h^0(C, \mathcal{L}(D)) > (n-1)$. Then this lemma asserts the existence of a nonempty open set $\mathcal{U}$ in C such that

$$h^0(\mathcal{L}(D+Q)) = h^0(\mathcal{L}(D))$$

for all closed points $Q \in \mathcal{U}$ and

$$h^0(\mathcal{L}(D+Q)) = h^0(\mathcal{L}(D)) + 1$$

for $Q \notin \mathcal{U}$. For each positive integer n, we consider the morphism

$$\begin{array}{rcl} C' \times \mathrm{Pic}^{(n-1)+(g-1)}_{C'/k'} & \xrightarrow{\varphi_{n-1}} & \mathrm{Pic}^{n+g-1}_{C'/k'} \\ (Q, \mathcal{L}(D)) & \mapsto & \mathcal{L}(D+Q). \end{array} \tag{2.2}$$

Let W^{n-1} denote the closed subset of $\mathrm{Pic}^{(n-1)+(g-1)}_{C'/k'}$ consisting of line bundles $\mathcal{L}(D)$ for which

$$h^0(C_{k(D)}, \mathcal{L}(D)) > n-1.$$

Now we claim that W^1 has codimension at least 2 in $\mathrm{Pic}^{1+(g-1)}_{C'/k'}$. Note that the subvariety W^0 is the image of the morphism

$$(C')^{(g-1)} \longrightarrow \mathrm{Pic}^{g-1}_{C'/k'}.$$

Using Theorem (5.1), [13], we see that the morphism $(C')^{(g-1)} \to W^0$ is birational. The fibre of this map containing a divisor D is the space of effective divisors linearly equivalent to D. So there exists an open set $\mathcal{U}'$ in $\mathrm{Pic}^{g-1}_{C'/k'}$ such that for any line bundle $\mathcal{L}(D)$ on $C_{k(D)}$, $h^0(C_{k(D)}, \mathcal{L}(D)) = 1$. Next consider the restriction of the morphism φ_0 to $C' \times W^0$. Since for any divisor D and any closed point Q on C', $h^0(C_{k(D-Q)}, \mathcal{L}(D-Q)) \geq h^0(C_{k(D)}, \mathcal{L}(D)) - 1$, the subvariety W^1 is in the image of this restriction. However, by Lemma (5.2), [13] cited above, we see that the complement of W^1 in the image of this restriction is nonempty. For example, for a line bundle $\mathcal{L}(D)$ from the open set $\mathcal{U}'$ as above, by this lemma there exist points Q on C' such that $h^0(C_{k(D+Q)}, \mathcal{L}(D+Q)) = 1$. So the line bundle $\mathcal{L}(D+Q)$ is in this complement. Since W^0 is an irreducible variety, the image of this restriction is also irreducible and hence W^1 is of codimension at least one in this image. Further note that the fibres of this restriction (if nonempty) are all one dimensional. So the dimension of the image of the restriction is equal to the dimension of W^0, which is $(g-1)$. It follows that W^1 is of codimension at least 2 in $\mathrm{Pic}^{1+(g-1)}_{C'/k'}$.

Now we claim that W^n has codimension at least 2 in $\mathrm{Pic}^{n+(g-1)}_{C'/k'}$ for any $n \geq 1$. We prove this claim by induction. By above arguments, the claim is true for $n = 1$. Now assume the claim to be true for W^{n-1}.

Consider the restriction of the morphism φ_{n-1} in 2.2 to $C' \times W^{n-1}$. Now the image of the this restriction contains W^n. But the restriction of φ_{n-1} has fibres (if nonempty) of dimension 1. So the dimension of the image of the restriction of φ_{n-1} is at most dim W^{n-1}. So W^n has dimension at most dim W^{n-1}. This by the induction hypothesis means that W^n has codimension at least 2 in $\mathrm{Pic}^{n+(g-1)}_{C'/k'}$.

This proves the claim. In particular, W^d has codimension at least 2 in $\mathrm{Pic}^{d+g-1}_{C'/k'}$. Now by Corollary 2, p. 50, [14], the sheaf $\pi_*(\mathcal{P})$ restricted to the complement of W^d is locally free of rank d. Since the complement of V' is contained in W^d, the required codimesion is at least 2. (In fact, the complement of V' is equal to W^d and the codimension of W^n is at least $(n+1)$ for any n, but we don't need that here.) $\square$

In order to extend the Brauer class associated to the algebra A_f, we first consider the sheaf $(A_f \otimes_k k')^\sim$ on the open set U'. Recall from Proposition (4.1) and Theorem (6.1), [9] that the algebra $A_f \otimes_k k'$ is isomorphic to $\mathrm{End}_{A_{k'}}(P)$, where P is $A_{k'}$-module $H^0(C' \times_{k'} Spec\ A_{k'}, \pi_*(\mathcal{P}|_{C' \times_{k'} U'}))$. Here $\mathcal{P}$ is the universal (rigidified) line bundle on $C' \times_{k'} \mathrm{Pic}^{d+g-1}_{C'/k'}$ and π is the projection on the second factor. So now we consider the vector bundle $End(\pi_*(\mathcal{P}|_{C' \times_{k'} V'}))$. We will prove that this descends to a sheaf of Azumaya algebras over an open set V (which is obtained by descending V') which contains the open set U.

First we prove a statement which is a refinement of Lemma (2.4) of [9]. We start by recalling the construction of the universal line bundle for degree 0. Let C be a smooth curve over a field k such that $C' = C \times_k k'$ has a k'-rational point. (k'/k is a finite Galois extension.) Let J denote the Jacobian variety $\mathrm{Pic}^0_{C'/k'}$. Fix a k'-rational point P on C'. Denote by Θ_0 the theta divisor in J obtained by translation of the usual Θ-divisor in $\mathrm{Pic}^{g-1}_{C'/k'}$ by $\mathcal{L}([-(g-1)P])$. We have the morphism ([13], Section 2)

$$h : C' \longrightarrow J,$$

which maps a closed point Q to $\mathcal{L}([Q-P])$. For any divisor D in the Jacobian variety, we denote by $\mathcal{L}'(D)$, the line bundle $m^*\mathcal{L}(D) \otimes p^*\mathcal{L}(D)^{-1} \otimes q^*\mathcal{L}(D)^{-1}$ on $J \times_{k'} J$, where m is the multiplication morphism on J and p, q are the projections of $J \times J$ on the first and second factors respectively. Finally, write Θ_0^- for the image of Θ_0 under the morphism $(-1)_J : J \to J$, and $(\Theta_0)_a$ for $t_a\Theta_0 = \Theta_0 + a$. Denote $(\Theta_0^-)_a$ by $(\Theta_0)_a^-$.

With this notation, [13], Lemma 6.8 says the following:

Lemma 2.2. *The sheaf $\mathcal{L} = (h \times (-1)_J)^*\mathcal{L}'(\Theta_0^-)$ is isomorphic to the universal invertible sheaf given by the universal property of J.*

Using the line bundle of the last lemma, we can construct the universal line bundle $\mathcal{P}$ of degree $(d+g-1)$. This construction is well known, see, for example, [1], Chapter 4, Lemma 2.2 and the discussion preceding it. Denote by L_0 the line bundle $\mathcal{L}(-(g-1)P)(-1)$ and by p_1 the projection of $C' \times_{k'} J$ on the second factor. The morphism η is defined as

$$\mathrm{Pic}^{d+g-1}_{C'/k'} \xrightarrow{\ \eta\ } J$$

$$L \quad \mapsto \quad L \otimes L_0$$

With this notation, the line bundle

$$\mathcal{P} = (\mathrm{id}_{C'} \times \eta)^*\mathcal{L} \otimes p_1^*(L_0^{-1})$$

is the universal line bundle of degree $(d+g-1)$.

Let $\sigma \in \mathrm{Gal}(k'/k)$. We denote the induced morphism on $\mathrm{Pic}^{d+g-1}_{C/k}$ and on $C' \times_{k'} \mathrm{Pic}^{d+g-1}_{C'/k'}$ by the same letter.

Proposition 2.3. *The line bundle $\sigma^*(\mathcal{P}) \otimes \mathcal{P}^{-1}$ is isomorphic to the line bundle $L^\sigma = \pi^*\mathcal{L}(\Theta_{[P-P^{\sigma^{-1}}]} - \Theta)$, where π is the projection on the second factor. Here Θ_a denotes the divisor $t_a\Theta = \Theta + a$, the translation of the theta divisor Θ by an element a in the jacobian of C'.*

Proof. The calculations in the proof of Lemma (2.4), [9] show that

$$\begin{aligned}
&\sigma^*(\mathrm{id}_{C'} \times \eta)^*\mathcal{L} \otimes (\mathrm{id}_{C'} \times \eta)^*\mathcal{L}^{-1}\\
\cong\ &(\mathrm{id}_{C'} \times \eta)^*(h \times (-1)_J)^*\\
&(p^*\mathcal{L}((\Theta_0)^-_{[P^{\sigma^{-1}}-P]} - (\Theta_0)^-_{g[P^{\sigma^{-1}}-P]}) \otimes q^*\mathcal{L}((\Theta_0)^-_{[P^{\sigma^{-1}}-P]} - \Theta_0^-)).
\end{aligned}$$

We know from Lemma (2.4), [9] that $(\sigma^*\mathcal{P} \otimes \mathcal{P}^{-1})$ is isomorphic to π^*L for some line bundle L on $\mathrm{Pic}^{d+g-1}_{C'/k'}$. This along with the fact that the line bundles $\mathcal{L}$ and $\mathcal{P}$ differ by the pull-back of a line bundle on C' gives that

$$\begin{aligned}
&\sigma^*\mathcal{P} \otimes \mathcal{P}^{-1}\\
\cong\ &\sigma^*(\mathrm{id}_{C'} \times \eta)^*\mathcal{L} \otimes (\mathrm{id}_{C'} \times \eta)^*\mathcal{L}^{-1}\\
\cong\ &(\mathrm{id}_{C'} \times \eta)^*(h \times (-1)_J)^*q^*\mathcal{L}((\Theta_0)^-_{[P^{\sigma^{-1}}-P]} - \Theta_0^-)).
\end{aligned}$$

We have the relations

$$(h \times (-1)_J) \circ q = q \circ (-1)_J \quad \text{and} \quad (\mathrm{id}_{C'} \times \eta) \circ q = q \circ \eta.$$

So we get

$$\begin{aligned}
&(\mathrm{id}_{C'} \times \eta)^*(h \times (-1)_J)^*q^*\mathcal{L}((\Theta_0)^-_{[P^{\sigma^{-1}}-P]} - \Theta_0^-))\\
\cong\ &q^*\eta^*(-1)_J^*\mathcal{L}((\Theta_0)^-_{[P^{\sigma^{-1}}-P]} - \Theta_0^-))\\
\cong\ &q^*\eta^*\mathcal{L}((\Theta_0)_{[P-P^{\sigma^{-1}}]} - \Theta_0))\\
\cong\ &\pi^*\mathcal{L}((\Theta)_{[P-P^{\sigma^{-1}}]} - \Theta)).
\end{aligned}$$

Combining last two identities, we have

$$\sigma^*(\mathcal{P}) \otimes \mathcal{P}^{-1} \cong \pi^*\mathcal{L}((\Theta)_{[P-P^{\sigma^{-1}}]} - \Theta)).$$

$\square$

Next we consider the question of descent for the sheaf of Azumaya algebras $End(\pi_*(\mathcal{P}|_{C'\times_{k'}V'}))$. In order to do this, we need to consider the open set V' of Lemma 2.1 and its descent.

Corollary 2.4. *The open set V' is $\mathrm{Gal}(k'/k)$-invariant and hence descends to some open set V in $\mathrm{Pic}^{d+g-1}_{C/k}$. Furthermore, the complement of V in $\mathrm{Pic}^{d+g-1}_{C/k}$ is of codimension at least 2.*

Proof. If π is the projection of $C' \times_{k'} \mathrm{Pic}^{d+g-1}_{C'/k'}$ on the second factor, then we have

$$\pi_*(\sigma^*\mathcal{P}) \cong \pi_*(\mathcal{P} \otimes L^\sigma) \cong \pi_*(\mathcal{P}) \otimes \mathcal{L}(\Theta_{[P-P^{\sigma^{-1}}]} - \Theta).$$

Here the last isomorphism follows from the last proposition. In particular, $\pi_*(\mathcal{P})$ is locally free on $\sigma^{-1}(V')$ as well. Now it follows from the definition of V' that it is $\mathrm{Gal}(k'/k)$-invariant. So it descends to some open set V in $\mathrm{Pic}^{d+g-1}_{C/k}$.

The second part now follows from Lemma 2.1. □

For each $\sigma \in \mathrm{Gal}(k'/k)$, we define a morphism of sheaves which is the composition

$$\varphi_\sigma : \sigma^*(End(\pi_*(\mathcal{P})|_{V'})) \longrightarrow End(\sigma^*(\pi_*(\mathcal{P})|_{V'})) \longrightarrow End(\pi_*(\mathcal{P})|_{V'}). \quad (2.3)$$

Here the first morphism is canonical and the second is constructed as the sequence of morphisms

$$\begin{aligned} End(\sigma^*(\pi_*(\mathcal{P})|_{V'})) &\longrightarrow End(\pi_*(\sigma^*(\mathcal{P})|_{V'})) \\ &\longrightarrow End(\pi_*(\mathcal{P} \otimes L^\sigma)|_{V'}) \\ &\longrightarrow End(\pi_*(\mathcal{P})|_{V'} \otimes (\mathcal{L}(\Theta_{[P-P^{\sigma^{-1}}]} - \Theta))|_{V'}) \\ &\longrightarrow End(\pi_*(\mathcal{P})|_{V'}), \end{aligned}$$

where the first three isomorphisms above are either canonical or given by Prop. 2.3. The last isomorphism uses the isomorphism

$$End(\mathcal{L}(\Theta_{[P-P^{\sigma^{-1}}]} - \Theta)|_{V'}) \longrightarrow \mathcal{O}_{V'},$$

where the identity morphism on the left side is mapped to 1 in $\mathcal{O}_{V'}$. We have the following proposition.

Proposition 2.5. *The collection $\{\varphi_\sigma\}$ is the descent data for the sheaf of algebras $End(\pi_*(\sigma^*(\mathcal{P})|_{V'}))$. Let $\mathcal{A}_f$ be the sheaf of Azumaya algebras obtained by this descent data. Then the algebra $\mathcal{A}_f(U)$ is isomorphic to the reduced Clifford algebra A_f.*

Proof. First we will prove that the descent data $\{\varphi_\sigma\}$ restricted to the open set U' gives A_f as the descended module. This will then imply the second part of the statement. Recall that there is an isomorphism (Proposition (4.1) and Theorem (6.1), [9])

$$\phi : A_f \otimes_k k' \longrightarrow End(\pi_*(\mathcal{P})|_{U'}).$$

The morphism $\phi \circ \sigma \circ \phi^{-1}$ is a σ-linear automorphism. This gives an $A_{k'}$-linear automorphism

$$\phi_{\sigma^{-1}} : \sigma^*(End(\pi_*(\mathcal{P})|_{U'})) \longrightarrow End(\pi_*(\mathcal{P})|_{U'}).$$

Note that the choice $\phi_{\sigma^{-1}}$ as notation for the map associated to any σ in $\mathrm{Gal}(k'/k)$ reflects the contravariance of the functor $Spec(-)$. Since $A_f \otimes_k k'$ descends to A_f, the collection $\{\phi_\sigma\}$ defines the descent data for the algebra $End(\pi_*(\mathcal{P})|_{U'})$.

We claim that the morphisms $\varphi_\sigma|_{U'}$ and ϕ_σ are the same. Since both the maps are $A_{k'}$-linear, it is sufficient to check the claim on the generators. Recall from [9] that there are two generators $\alpha_u,\ \alpha_v$ which satisfy the following property. Let P denote the global sections of the direct image of the universal line bundle on U'. Then the locally free sheaf $\pi_*(\mathcal{P})|_{U'}$ is isomorphic to $(P \otimes_{A_{k'}} A_{k'}[u,v])^\sim$ where $u,\ v$ are degree 1 elements. In particular, there is a graded $S_{k'} = (A_{k'}[u,v,w]/(w^d - f(u,v)))$-action on the graded module $P \otimes_{A_{k'}} A_{k'}[u,v]$. It was shown in Proposition (4.1), [9] that w acts as $u \circ \alpha_u + v \circ \alpha_v$. Also recall from the cited proposition that $\phi^{-1}(\alpha_u) = x_1 \otimes 1,\ \phi^{-1}(\alpha_v) = x_2 \otimes 1$, where $x_1,\ x_2$ are the generators of the Clifford algebra. It follows that for any $\sigma \in \mathrm{Gal}(k'/k)$, $\phi_\sigma(1 \otimes \alpha_u) = \alpha_u$ and $\phi_\sigma(1 \otimes \alpha_v) = \alpha_v$. Here we are writing the elements of the module using $\sigma^*(End(\pi_*(\mathcal{P})|(U'))) = A^\sigma_{k'} \otimes_{A_{k'}} End(\pi_*(\mathcal{P})|(U'))$.

Next we consider $\varphi_\sigma^{-1}(\alpha_u)$ and $\varphi_\sigma^{-1}(\alpha_v)$. Let $(\mathcal{L}(\Theta_{[P-P^{\sigma^{-1}}]} - \Theta))|_{U'} = \tilde{M}$ for some invertible $A_{k'}$-module M. Then we have

$$(\mathcal{P} \otimes_{A_{k'}} L^\sigma)|_{\pi^{-1}(U')} \cong (A_{k'}[u,v] \otimes_{A_{k'}} (P \otimes_{A_{k'}} M))^\sim$$

and

$$\sigma^*(\mathcal{P})|_{\pi^{-1}(U')} \cong (A_{k'}[u,v] \otimes_{A_{k'}} (P \otimes_{A_{k'}} A^\sigma_{k'}))^\sim.$$

For a quasi-coherent sheaf $\mathcal{F}$ on $X = \mathrm{Proj}(S)$, denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\oplus_k H^0(X, \mathcal{F}(k))$. Since the line bundles $(\mathcal{P} \otimes_{A_{k'}} L^\sigma)|_{\pi^{-1}(U')}$ and $\sigma^*(\mathcal{P})|_{\pi^{-1}(U')}$ are isomorphic on $C \times_{k'} A_{k'}$, we have that the graded $S_{k'}$-modules $\Gamma_*((\mathcal{P} \otimes_{A_{k'}} L^\sigma)|_{\pi^{-1}(U')})$ and $\Gamma_*((\sigma^*(\mathcal{P})|_{\pi^{-1}(U')})$ are isomorphic. By using a proof similar to Corollary (3.2), [9], we have the horizontal graded $S_{k'}$-module isomorphisms

$$\begin{tikzcd}
\Gamma_*(\mathcal{P} \otimes_{A_{k'}} L^\sigma)|_{\pi^{-1}(U')}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & A_{k'}[u,v] \otimes_{A_{k'}} (P \otimes_{A_{k'}} M) \\
\Gamma_*(\sigma^*(\mathcal{P})|_{\pi^{-1}(U')}) \arrow[r, "\cong"] & A_{k'}[u,v] \otimes_{A_{k'}} (P \otimes_{A_{k'}} A^\sigma_{k'})
\end{tikzcd}$$

So the actions of $1 \otimes \alpha_u$ and $1 \otimes \alpha_v$ commute with the isomorphism

$$End_{A_{k'}}(P \otimes_{A_{k'}} M) \longrightarrow End_{A_{k'}}(P \otimes_{A_{k'}} A^\sigma_{k'}).$$

Now following the sequence of isomorphisms in the definition 2.3, we see that $\varphi_\sigma^{-1}(\alpha_u) = 1 \otimes \alpha_u$ and $\varphi_\sigma^{-1}(\alpha_v) = 1 \otimes \alpha_v$. Here we are again using the identification $\sigma^*\mathcal{F} \cong \mathcal{O}^\sigma_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on a k'-variety X' which descends to a variety X over k. This finishes the proof of the second part.

Now for any elements $\sigma,\ \tau \in \mathrm{Gal}(k'/k)$, consider $\varphi_\sigma \circ \tau\varphi_\sigma \circ \varphi_{\tau\sigma}^{-1}$. To prove the first part, we must show that this isomorphism is identity. For a quick proof, we use the fact that this morphism restricted to U' is same as the morphism $\phi_\sigma \circ \tau\phi_\sigma \circ \phi_{\tau\sigma}^{-1}$ which is the identity morphism. This follows since the collection $\{\phi_\sigma\}$ is the descent data. By Prop. (2.3), Chap. 4, [12], any automorphism of an Azumaya algebra is locally, for the Zariski topology, an inner automorphism. So we may restrict the automorphism $\varphi_\sigma \circ \tau\varphi_\sigma \circ \varphi_{\tau\sigma}^{-1}$ to an affine Zariski open subset $Spec\ R$ of U' and assume that the automorphism is given by $a \mapsto uau^{-1}$ for some $u \in \Gamma(Spec\ R, End(\pi_*\mathcal{P}|_{Spec\ R}))^*$. But since this automorphism is identity on a nonempty open subset of $Spec\ R$, it follows that $u \in Q(R)$, the fraction field of R. But since an Azumaya algebra is a finite module over its center and R is a normal domain, it follows that $u \in R$. So the automorphisms $\varphi_\sigma \circ \tau\varphi_\sigma \circ \varphi_{\tau\sigma}^{-1}$ restricted to $Spec\ R$ and hence on V' is identity. $\square$

Before we consider the question of extending the Brauer class, we record the following fact in relation to the Brauer and the cohomological Brauer group of $\mathrm{Pic}^{d+g-1}_{C/k}$.

Theorem 2.6. *([8, 4]) For a principal homogeneous space X of an abelian variety A over an arbitrary field k, the natural inclusion of groups* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ *is a bijection.*

Proof. If X is an abelian variety, then this is proved in [8]. If X is some nontrivial principal homogeneous space of X, then this follows from Prop. 3 of [4]. $\square$

As a consequence, the Brauer and the cohomological groups of $\mathrm{Pic}^{d+g-1}_{C/k}$ are the same and we make no distinction between them in the following. The Azumaya algebra $\mathcal{A}_f$ of Prop. 2.5 gives a class in $\mathrm{Br}(V)$. We denote this class by α_f. We have the following theorem.

Theorem 2.7. *The class α_f in $\mathrm{Br}(V)$ lies in the image of the restriction homomorphism*

$$\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k}) \longrightarrow \mathrm{Br}(V) \tag{2.4}$$

induced by the inclusion $V \to \mathrm{Pic}^{d+g-1}_{C'/k'}$. Furthermore, there is a unique class in $\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k})$ which maps to α_f.

Proof. First we have the commutative diagram of abelian groups:

$$\begin{tikzcd}
\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k}) \arrow[r] \arrow[d] & \mathrm{Br}(V) \arrow[d] \\
\mathrm{Br}'(\mathrm{Pic}^{d+g-1}_{C/k}) \arrow[r] & \mathrm{Br}'(V)
\end{tikzcd}$$

The vertical maps in this diagram are injective by Theorem (2.5), [12] (with the left vertical map an isomorphism). So we denote the image of α_f by the same letter. Now we know that the class α_f^d is trivial (since $\mathcal{A}_f$ is of dimension d^2). In particular, the order n of α_f divides d. Recall that the degree d of the form f is assumed to be not divisible by p, the characteristic of the field k. So all the primes l dividing n are distinct from p. For such l, from Theorem (6.1), Corollary (6.2), [5], we have that the map

$$\mathrm{Br}'(\mathrm{Pic}^{d+g-1}_{C/k})(l) \longrightarrow \mathrm{Br}'(V)(l)$$

given by 2.4 is an isomorphism provided V has codimension at least 2 in $\mathrm{Pic}^{d+g-1}_{C/k}$ and $\mathrm{Pic}^{d+g-1}_{C/k}$ is a smooth scheme over k. Here for an abelian group A, $A(l)$ denotes the l-primary component of A. That these conditions are satisfied follows from Corollary 2.4 and Proposition 2.5. Since n is not divisible by p, it follows (using Theorem 2.6) that the class α_f in $\mathrm{Br}'(V)$ is in the image of the map 2.4.

For the second part, we only need to observe that the canonical map $\mathrm{Br}'(\mathrm{Pic}^{d+g-1}_{C/k}) \to H^2(Spec\ k(\xi), \mathbb{G}_m)$ is injective, where ξ is the generic point of $\mathrm{Pic}^{d+g-1}_{C/k}$. This is shown, for example, in Example (2.22), Chapter III, [12]. $\square$

In the following, we will denote this extended class in $\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k})$ by $\tilde{\alpha}_f$.

3 When is this Brauer class trivial?

In this section, we consider the case when d is an odd integer. In this case, we will find a precise arithmetic condition on a principal homogeneous space of the jacobian of the curve C for the extended class $\tilde{\alpha}_f$ in $\mathrm{Br}'(\mathrm{Pic}^{d+g-1}_{C/k})$ to be trivial.

The genus of the curve C is $(d-1)(d-2)/2$. So $d+g-1 = d(d-1)/2$. Since d is odd, d divides $d+g-1$. In the following, we denote by $J(C)$ the jacobian of C. We have a k-rational isomorphism

$$J(C) \longrightarrow \mathrm{Pic}^{d+g-1}_{C/k} \tag{3.1}$$

given by $L \mapsto L \otimes \mathcal{O}((d-1)/2)$. We fix this isomorphism throughout this section to identify these two varieties. Let G denote the Galois group of an algebraic closure $\bar{k}$ of k. Let $\mathrm{Br}_1(J(C))$ denote the kernel of the homomorphism $\mathrm{Br}(J(C)) \to \mathrm{Br}(\overline{J(C)})$, where $\overline{J(C)}$ denotes the variety $J(C) \otimes_k \bar{k}$. Let $\bar{k}(\xi)$ denote the rational function field of $\overline{J(C)}$. Finally, let $\mathrm{Pic}(\overline{J(C)})$ and $\mathrm{Div}(\overline{J(C)})$ denote the Picard group and the group of divisors on $\overline{J(C)}$ respectively. Recall from [11] (or from [10] for curves) that we have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{Br}(k) \arrow[r] \arrow[d, "\sim"] & \mathrm{Br}_1(J(C)) \arrow[d, "\nu"] & H^1(G, \mathrm{Pic}(\overline{J(C)})) \arrow[d, "\delta"] & \\
\mathrm{Br}(k) \arrow[r] & H^2(G, \bar{k}(\xi)^*) \arrow[r, "\rho"] \arrow[d] & H^2(G, \bar{k}(\xi)^*/\bar{k}^*) \arrow[r] \arrow[d] & H^3(G, \bar{k}^*) \\
 & H^2(G, \mathrm{Div}(\overline{J(C)})) \arrow[r, "\sim"] & H^2(G, \mathrm{Div}(\overline{J(C)})) &
\end{tikzcd} \tag{3.2}$$

The last column is obtained from the exact sequence of G-modules

$$0 \longrightarrow \bar{k}(\xi)^*/\bar{k}^* \longrightarrow \mathrm{Div}(\overline{J(C)}) \longrightarrow \mathrm{Pic}(\overline{J(C)}) \longrightarrow 0,$$

while the second row is obtained from the exact sequence

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(\xi)^* \longrightarrow \bar{k}(\xi)^*/\bar{k}^* \longrightarrow 0.$$

Note that the Brauer class $\tilde{\alpha}_f$ of the last section can be pulled back to $\mathrm{Br}(J(C))$ via the k-rational isomorphism mentioned above. We denote this pull-back by the same letter. The Brauer class corresponding to A_f is split if C has a k-rational point. So we know that this class is split by a finite field extension k'/k such that $C(k')$ is nonempty and hence that $\tilde{\alpha}_f \in \mathrm{Br}_1(J(C))$. We relate $\rho\nu(\alpha_f)$ to the image of a class corresponding to a principal homogeneous space of $J(C)$.

In fact, consider the principal homogeneous space $\mathrm{Pic}^1_{C/k}$ of $J(C)$. This gives a cohomology class β in $H^1(G, J(C)(\bar{k}))$. Throughout this section, we fix the k-rational duality isomorphism between $J(C)$ and its dual $J(C)^\vee$ constructed as follows. Recall that we have a k-rational divisor Θ in $\mathrm{Pic}^{d+g-1}_{C/k}$. Then the translate $\Theta_{[\mathcal{O}(-(d-1)/2)]}$ of the Θ-divisor by $\mathcal{O}(-(d-1)/2)$ is a k-rational divisor in $J(C)$. Then the morphism

$$\begin{array}{ccc} J(C) & \xrightarrow{\varphi_{\mathcal{L}(\Theta_{[\mathcal{O}(-(d-1)/2)]})}} & J(C)^\vee \\ x & \longrightarrow & t_x^*\mathcal{L}(\Theta_{[\mathcal{O}(-(d-1)/2)]}) \otimes (\mathcal{L}(\Theta_{[\mathcal{O}(-(d-1)/2)]}))^{-1} \end{array} \tag{3.3}$$

is an isomorphism defined over $k$and is a homomorphism of abelian varieties. This follows, for example, from Theorem (6.6), [13]. Using this isomorphism, we get a cohomology class in $H^1(G, J(C)^\vee(\bar{k}))$. We denote this class again by β. Next we have the exact sequence

$$0 \longrightarrow J(C)^\vee(\bar{k}) \xrightarrow{i} \mathrm{Pic}(\overline{J(C)}) \longrightarrow \mathrm{NS}(\overline{J(C)}) \longrightarrow 0,$$

where $\mathrm{NS}(\overline{J(C)})$ denotes the Neron-Severi group of the variety $\overline{J(C)}$. Recall that the cohomology group $H^0(G, \mathrm{NS}(\overline{J(C)}))$ is in general nonzero. However we still have the following:

Lemma 3.1. *The cohomology class $i_*(\beta)$ in $H^1(G, \mathrm{Pic}(\overline{J(C)}))$ is zero if and only if the class β is zero.*

Proof. Suppose that the class $i_*(\beta) = 0$. Then using the diagram

$$\begin{tikzcd} H^0(G, \mathrm{NS}(\overline{J(C)})) \arrow[r] & H^1(G, J(C)^\vee(\bar{k})) \arrow[r, "i_*"] \arrow[d, "\sim"'] & H^1(G, \mathrm{Pic}(\overline{J(C)})) \\ & H^1(G, J(C)(\bar{k})) & \end{tikzcd}$$

we see that the class β is in the image of the first (horizontal) map.

Consider the multiplication by (-1) automorphism of $J(C)$. Using the duality isomorphism, we have induced maps (-1) on $J(C)(\bar{k})$ and $J(C)^\vee(\bar{k})$. Thus the map induced on $H^1(G, J(C)^\vee(\bar{k}))$ is also (-1). Also this induces a map on $\mathrm{NS}(\overline{J(C)})$ which from a direct calculation is the identity morphism. So we have the diagram

$$\begin{tikzcd} H^0(G, \mathrm{NS}(\overline{J(C)})) \arrow[r] \arrow[d, "\mathrm{id}"'] & H^1(G, J(C)^\vee(\bar{k})) \arrow[d, "(-1)"] \\ H^0(G, \mathrm{NS}(\overline{J(C)})) \arrow[r] & H^1(G, J(C)^\vee(\bar{k})) \end{tikzcd}$$

which gives that $\beta = -\beta$ or $2\beta = 0$. By using the duality isomorphism, we see that the class $\beta \in H^1(G, J(C)(\bar{k}))$ is such that $2\beta = 0$. However the class $2\beta \in H^1(G, J(C)(\bar{k}))$ corresponds to the principal homogeneous space $\mathrm{Pic}^2_{C/k}$. Thus the triviality of the class 2β implies that there is a k-rational line bundle of degree 2 on C. Since we have a degree d, k-rational line bundle on C and d is odd, we get a degree 1, k-rational line bundle on C. This means that the cohomology class β is trivial. $\square$

Proposition 3.2. *$\rho\nu(\tilde{\alpha}_f) = \delta i_*(\beta)$ in $H^2(G, \bar{k}(\xi)^*/\bar{k}^*)$.*

Proof. First we compute $\rho\nu(\tilde{\alpha}_f)$. To do this, we use the k-rational isomorphism of 3.1 and do the computations in $\mathrm{Pic}^{d+g-1}_{C/k}$. Let $\bar{V}$ denote the open set $V' \times_{k'} \bar{k}$. Let i_ξ denote the inclusion *Spec* $\bar{k}(\xi) \to \bar{V}$. The maps $\{\varphi_\sigma\}$ defined in 2.3 give maps

$$\tilde{\varphi_\sigma} : \sigma^* i_\xi^*(End(\pi_*(\mathcal{P})|_{\bar{V}})) \longrightarrow i_\xi^*(End(\pi_*(\mathcal{P})|_{\bar{V}})),$$

which by a choice of basis give isomorphisms $M_d(\bar{k}(\xi)) \xrightarrow{\tilde{\varphi_\sigma}} M_d(\bar{k}(\xi))$. It is easy to see that for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ these maps are induced by the maps

$$i_\xi^* \sigma^*(\pi_*\mathcal{P}|_{\bar{V}}) \longrightarrow i_\xi^*(\pi_*\mathcal{P}|_{\bar{V}} \otimes L^\sigma|_{\bar{V}}) \longrightarrow i_\xi^*(\pi_*\mathcal{P}|_{\bar{V}}).$$

These maps can be interpreted as being obtained by applying the functor π_* to the restriction of the maps

$$(\mathrm{id}_{C'} \times i_\xi)^* \sigma^*(\mathcal{P}) \longrightarrow (\mathrm{id}_{C'} \times i_\xi)^*(\mathcal{P} \otimes \pi^* L^\sigma) \longrightarrow (\mathrm{id}_{C'} \times i_\xi)^*(\mathcal{P}).$$

to $\bar{C} \times_{\bar{k}} \bar{V}$ $(\bar{C} = C' \times_{k'} \bar{k})$. For each line bundle L^σ, we fix its divisor of a section given by $(\bar{\Theta}_{[P-P^{\sigma^{-1}}]} - \bar{\Theta})$, where P is a point in $\bar{C}(\bar{k})$ and $\bar{\Theta}$ denotes the Θ-divisor as constructed above assuming that $k = \bar{k}$. Then using the definitions of the maps above, it follows that the inner automorphism $\tilde{\varphi}_\sigma \circ \sigma\tilde{\varphi}_\tau \circ (\tilde{\varphi}_{\sigma\tau})^{-1}$ of $M_d(\bar{k}(\xi))$ is conjugation by a function whose divisor is

$$\bar{\Theta}_{[P-P^{\sigma^{-1}}]} + \bar{\Theta}_{[P^{\sigma^{-1}} - P^{\sigma^{-1}\tau^{-1}}]} - \bar{\Theta}_{[P-P^{\sigma^{-1}\tau^{-1}}]} - \bar{\Theta}.$$

(We need to choose representative matrices for $\tilde{\varphi}_\sigma$ in order to get this scalar.) Denote by $\bar{\Theta}_0$ the pull-back of the divisor $\bar{\Theta}$ under the morphism of 3.1 (interpreted over $\bar{k}$). Then in fact the cocycle $(\rho\nu(\tilde{\alpha}_f))_{\sigma,\tau}$ is given by

$$(\bar{\Theta}_0)_{[P-P^{\sigma^{-1}}]} + (\bar{\Theta}_0)_{[P^{\sigma^{-1}} - P^{\sigma^{-1}\tau^{-1}}]} - (\bar{\Theta}_0)_{[P-P^{\sigma^{-1}\tau^{-1}}]} - (\bar{\Theta}_0). \tag{3.4}$$

Next consider the class β in $H^1(G, \overline{J(C)}(\bar{k}))$ corresponding to the principal homogeneous space $\mathrm{Pic}^1_{C/k}$ of $J(C)$. To write the corresponding cocycle, fix a closed point P on C. Then the corresponding cocycle is

$$\beta_\sigma = [P^{\sigma^{-1}} - P].$$

By using the isomorphism in 3.4, we get the cocycle for β in $H^1(G, \overline{J(C)}^\vee(\bar{k}))$ as

$$\beta_\sigma = [(\bar{\Theta}_0)_{[P-P^{\sigma^{-1}}]} - \bar{\Theta}_0]. \tag{3.5}$$

Since the divisor $\bar{\Theta}_0$ is Galois invariant, it follows that for any $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$, $(\delta i_*(\beta))_{\sigma,\tau} = (\rho\nu(\tilde{\alpha}_f))_{\sigma,\tau}$. □

Remark 3.3. Recall that Prop. 2.3 was proved for $\sigma \in \mathrm{Gal}(k'/k)$, where k'/k was assumed to be a finite extension. However the same argument works for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Also the descent data $\{\varphi_\sigma\}$ was defined for $\sigma \in \mathrm{Gal}(k'/k)$. Again we may lift these maps for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ to get the descent data. These observations will be used in the proof of the theorem below.

Main Theorem. *The cohomology class $\tilde{\alpha}_f \in \mathrm{Br}(J(C))$ is trivial if and only if the class $\beta \in H^1(G, \overline{J(C)}(\bar{k}))$ is trivial.*

Proof. We use the commutative diagram 3.2. If $\tilde{\alpha}_f \in \mathrm{Br}(J(C))$ is trivial, then the class $\delta i_*(\beta)$ is also trivial by Proposition 3.2. Since δ is injective, $i_*(\beta) = 0$. By Lemma 3.1, we see that the class $\beta = 0$.

Now let the class $\beta = 0$. Using the cocycle 3.5 for $\beta \in H^1(G, \overline{J(C)}^{\vee}(\bar{k}))$, we see by triviality of β that there exists a divisor D on $\overline{J(C)}$ such that for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$, the divisors

$$[(\bar{\Theta}_0)_{[P-P^{\sigma^{-1}}]} - \bar{\Theta}_0] \sim [D^{\sigma} - D],$$

where $\sim$ is the notation for linear equivalence. Pulling this back to $\mathrm{Pic}^{d+g-1}_{\bar{C}/\bar{k}}$, we see that there exists a line bundle $L = L(D)$ on the curve $\bar{C}$ such that for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ the line bundles L^{σ} of Proposition 2.3 are isomorphic to $\sigma^* L \otimes L^{-1}$. In particular, the universal line bundle $\mathcal{P}$ on $\mathrm{Pic}^{d+g-1}_{\bar{C}/\bar{k}}$ satisfies the property

$$\sigma^*\mathcal{P} \otimes \mathcal{P}^{-1} \cong \pi^*(\sigma^* L \otimes L^{-1}).$$

This means we have an isomorphism $\sigma^*(\mathcal{P} \otimes \pi^*(L^{-1})) \cong \mathcal{P} \otimes \pi^*(L^{-1})$ for any σ. For the line bundle L^{-1}, we fix an isomorphism $End(L^{-1}) \to \mathcal{O}_{\mathrm{Pic}^{d+g-1}_{\bar{C}/\bar{k}}}$, where the identity endomorphism is sent to the section 1. Then it is easy to check that we have a commutative diagram

$$\begin{tikzcd}
\sigma^*(End((\pi_*(\mathcal{P}) \otimes L^{-1})|_{\bar{V}})) \arrow[r, "\varphi_{\sigma}'"] \arrow[d] & End((\pi_*(\mathcal{P}) \otimes L^{-1})|_{\bar{V}}) \arrow[d] \\
\sigma^*(End(\pi_*(\mathcal{P})|_{\bar{V}})) \arrow[r, "\varphi_{\sigma}"] & End(\pi_*(\mathcal{P})|_{\bar{V}})
\end{tikzcd} \tag{3.6}$$

where the isomorphism of the top line is constructed using the isomorphism $\sigma^*(\mathcal{P} \otimes \pi^*(L^{-1})) \cong \mathcal{P} \otimes \pi^*(L^{-1})$. Note that the choice of this isomorphism does not affect the commutativity of this diagram. Now the morphisms $\{\varphi_{\sigma}'\}$ define descent data for the sheaf $End((\pi_*(\mathcal{P}) \otimes L^{-1})|_{\bar{V}})$. But since this data is obtained from the descent data for the line bundle $\mathcal{P} \otimes \pi^*(L^{-1})$, the sheaf $End((\pi_*(\mathcal{P}) \otimes L^{-1})|_{\bar{V}})$ descends to $End(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on V (which is the descent of $\bar{V}$). By commutativity of the diagram 3.6, it follows that the Azumaya algebra $\mathcal{A}_f$ of Proposition 2.5 is split and so its Brauer class is trivial. But this class is $\alpha_f \in \mathrm{Br}'(V)$. So the class $\tilde{\alpha}_f$ in $\mathrm{Br}(\mathrm{Pic}^{d+g-1}_{C/k})$ (of Theorem 2.7) and hence in $\mathrm{Br}(J(C))$ is trivial. $\square$

Corollary 3.4. *The reduced Clifford algebra C_f is split over k if and only if the principal homogeneous space $\mathrm{Pic}^1_{C/k}$ of $J(C)$ has a k-rational point.*

Proof. In view of the main theorem, we only need to observe that the canonical map $\mathrm{Br}'(\mathrm{Pic}^{d+g-1}_{C/k}) \to H^2(Spec\ k(\xi), \mathbb{G}_m)$ is injective and that this map factors through $\mathrm{Br}'(U)$. This is shown, for example, in Example (2.22), Chapter III, [12]. □

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