Math 132 **Section 2**

Fall 2002

Examples of Graphing Rational Functions

Here are two examples to illustrate graphing rational functions further. We saw a few examples in class and these are supposed to augment those.

Example: Graph the rational function

$$y = f(x) = \frac{2x^2 + x - 1}{x^2 - 1}.$$

First note that the given function is neither odd nor even. So there is no associated symmetry in the graph.

Asymptotes:

Vertical Asymptotes: The only possibilities are when $x^2 = 1$. This happens when $x = 1$ and $x = -1$.

$x = 1$:

$$\lim_{x \to 1^+} \frac{2x^2 + x - 1}{x^2 - 1} = \infty, \quad \lim_{x \to 1^-} \frac{2x^2 + x - 1}{x^2 - 1} = -\infty$$

So $x = 1$ is a vertical asymptote. (Note that for this to happen we needed only one of the limits to be $\pm\infty$. But it is useful and necessary to know these limits when sketching the graph.)

$x = -1$:

Note that when $x = -1$, $2x^2 + x - 1 = 2(-1)^2 + (-1) - 1 = 0$. So the numerator in $f(x)$ has a fctor of $(x + 1)$. In fact,

$$2x^2 + x - 1 = (x + 1)(2x - 1).$$

So

$$\lim_{x \to -1^-} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x \to -1^+} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x \to -1} \frac{2x^2 + x - 1}{x^2 - 1}$$

and these are equal to

$$\lim_{x \to -1} \frac{(x + 1)(2x - 1)}{(x - 1)(x + 1)} = \lim_{x \to -1} \frac{2x - 1}{x - 1} = \frac{3}{2}.$$

So $x = -1$ is not a vertical asymptote and $y = f(x)$ has a removable discontinuity at $x = -1$.

Horizontal Asymptotes: We compute the limits as $x \to \pm\infty$.

$$\lim_{x\to\infty} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x\to\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{x - \frac{1}{x^2}} = 2.$$

Similarly,

$$\lim_{-\infty} \frac{2x^2 + x - 1}{x^2 - 1} = 2.$$

So $y = 2$ is the only horizontal asymptote. Note that

$$\frac{2x^2 + x - 1}{x^2 - 1} = 2 + \frac{x + 1}{x^2 - 1}.$$

So there is no oblique asymptote.

Information using y': First using the quotient rule

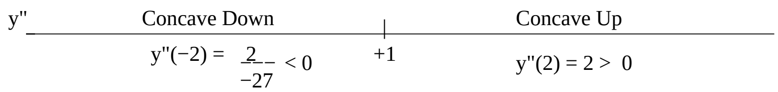
$$\begin{aligned}
y' &= \frac{(x^2 - 1)(2(2x) + 1) - (2x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\
&= \frac{(x + 1)((x - 1)(4x + 1) - (2x^2 + x - 1)(2x))}{((x - 1)(x + 1))^2} \\
&= \frac{(x + 1)(4x^2 - 3x - 1 - 4x^2 + 2x)}{(x - 1)^2(x + 1)^2} \\
&= -\frac{(x + 1)(x + 1)}{(x - 1)^2(x + 1)^2} \\
&= -\frac{1}{(x - 1)^2} \quad \text{as } x = -1 \text{ is not in the domain}
\end{aligned}$$

Since $y' < 0$ everywhere in the domain, $y = f(x)$ is a decreasing function. Note that $x = 1$ is not in the domain and so $y = f(x)$ has no critical points.

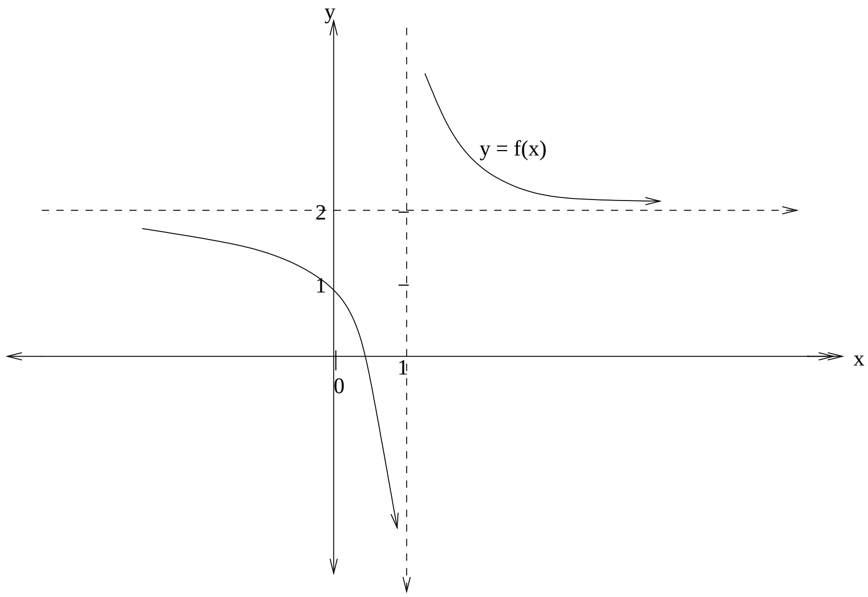
Information using y": We compute y''.

$$y'' = \frac{d}{dx}\left(-\frac{1}{(x - 1)^2}\right) = \frac{2}{(x - 1)^3}.$$

So we have the following picture:



Note that the function $f(x)$ has no inflection points. So finally we can put all the information together and get the following graph:



Graph of $y = f(x)$

Here is another example.

Example: Graph the rational function

$$y = f(x) = \frac{x^2 - 4}{x - 1}.$$

First note that the given function is neither odd nor even. So there is no associated symmetry in the graph.

Asymptotes:

Vertical Asymptotes: The only possibility is when $x = 1$.

$$\lim_{x \to 1^+} \frac{x^2 - 4}{x - 1} = -\infty, \quad \lim_{x \to 1^-} \frac{x^2 - 4}{x - 1} = \infty.$$

So $x = 1$ is a vertical asymptote. (Note that for this to happen we needed only one of the limits to be $\pm\infty$. But it is useful and necessary to know these limits when sketching the graph.)

Horizontal Asymptotes: We compute the limits as $x \to \pm\infty$.

$$\lim_{x \to \infty} \frac{x^2 - 4}{x - 1} = \lim_{x \to \infty} \frac{x - \frac{4}{x}}{1 - \frac{1}{x}} = \infty.$$

Similarly,

$$\lim_{-\infty} \frac{x^2 - 4}{x - 1} = -\infty.$$

So there are no horizontal asymptotes. On the other hand note that

$$\frac{x^2 - 4}{x - 1} = x + 1 - \frac{3}{x - 1}. \tag{1}$$

So $y = x + 1$ is an oblique asymptote.

Information using y': We can use Equation 1 to compute the derivative.

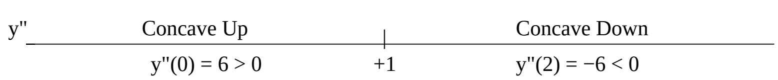
$$y' = 1 + \frac{3}{2(x - 1)^2} = \frac{x^2 - 2x + 4}{(x - 1)^2}.$$

Note that the numerator $x^2 - 2x + 4 = (x - 1)^2 + 3$ is always greater that zero. Also the denominator is always greater than zero. So the derivative y' is always positive and hence $y = f(x)$ is an increasing function. Also since $x = 1$ is not in the domain, there are no critical points for the function.

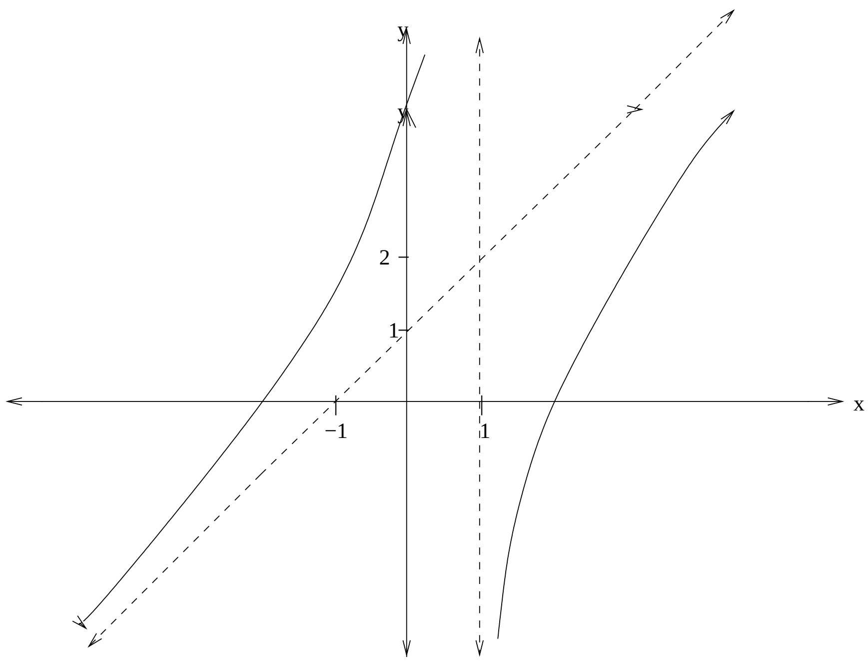
Information using y”: We compute y'' usinf the formula for y'.

$$y'' = -\frac{6}{(x - 1)^3}.$$

So we have the following picture:



Note that the function $f(x)$ has no inflection points. So finally we can put all the information together and get the following graph:



Graph of $y = f(x)$