Self-dual Sheaves on Reductive Borel-Serre Compactifications of Hilbert Modular Surfaces

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Abstract

The main theorem of this article asserts that the category of self-dual sheaves compatible with the intersection chain sheaves (for upper/lower middle perversity) on the reductive Borel-Serre compactification $\bar{X}$ of a Hilbert modular surface is nonempty. Also we prove that the direct image of such a sheaf under the canonical map to the Baily-Borel compactification is isomorphic (in the derived category) to the intersection chain sheaf for upper and lower middle perversity. As a consequence of the main theorem, there exist characteristic L-classes of these sheaves in the rational homology of $\bar{X}$. In fact, these classes do not depend on the choice of a self-dual sheaf and hence are invariants of the compactification $\bar{X}$.

1 Introduction

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and let Γ be a torsion-free arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$, the group of $\mathbb{Q}$-rational points of $\mathbf{G}$. Let X be the locally symmetric space $\Gamma\backslash D$, where D is the symmetric space associated to $\mathbf{G}$, that is, the quotient $\mathbf{G}(\mathbb{R})/K$ of the real points of $\mathbf{G}$ by its

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maximal compact subgroup K. The locally symmetric space is in general not compact.

In this paper, we are mainly concerned with two compactifications of X. These are the reductive Borel-Serre compactification introduced by S. Zucker ([22]) and the Baily-Borel compactification ([3]). These are respectively denoted by $\bar{X}$ and $\hat{X}$.

The compactification $\bar{X}$ is obtained from X by adding boundary strata X_P, one for each Γ-conjugacy class of rational parabolic subgroups P of $\mathbf{G}$. (See Section 3 for details.) One of the interesting features of the compactification $\bar{X}$ is that the Hecke operators on X extend to $\bar{X}$. The space $\bar{X}$ is a stratified space and has in general odd codimensional strata as well. Furthermore, it is in general not a Witt space, which means that the intersection chain sheaves for upper (or lower) middle perversity are not self-dual. Thus a natural objective is to construct a cohomology theory which satisfies generalized Poincaré duality.

Now assume that D is Hermitian symmetric and that Γ is a subgroup of the group of biholomorphisms of D. Then the Baily-Borel compactification $\hat{X}$ can be given the structure of a complex projective variety which is normal and is defined over a number field. The boundary strata are obtained as certain projections of the boundary strata X_P above. This gives a canonical, stratified map $\pi : \bar{X} \to \hat{X}$.

Here we consider locally symmetric spaces called *Hilbert modular surfaces.* To describe these, let K be a real quadratic field and $\mathfrak{o}$ be its ring of integers. Let Γ be either $PSL(2, \mathfrak{o})$ or its principal congruent subgroup of some level (see Section 3 below). The discrete group Γ then can be viewed as a subgroup of $\mathrm{PGL}^+(2, \mathbb{R})^2$ and hence acts on $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. We denote the quotient of this action by X. The space X is in fact the quotient of a Hermitian symmetric space and this description can be found in section 3 below. The boundary strata X_P for the space $\bar{X}$ are all circles. Further, the map π collapses these circles to points.

In this paper we consider the question of constructing self-dual sheaves on $\bar{X}$ itself which are compatible with the intersection chain sheaves (in the sense that they have to satisfy strong stalk and costalk conditions, they have to be normalized over the top stratum and they should be self-dual). As mentioned above, the compactification $\bar{X}$ has odd codimensional strata. It is indeed not even a Witt space as the links are tori with nontrivial middle dimensional cohomology. So the (middle perversity) intersection chain sheaves are not self-dual and the problem of recovering generalized Poincaré duality for $\bar{X}$ is nontrivial.

The question of constructing self-dual sheaves on arbitrary non-Witt spaces was investigated by the first named author in [1]. The results of the cited article give an if and only if condition for the existence of a self-dual sheaf compatible with the intersection chain sheaves. In particular, such sheaves can be constructed provided there are Lagrangian subsheaves of the middle dimensional cohomology sheaves on odd codimensional strata corresponding to the associated link bundles. The bilinear form on these cohomology sheaves is the classical intersection form (in the $\mathbb{Q}$-rank 1 case). We denote the category of self-dual perverse sheaves on $\bar{X}$ by $SD(\bar{X})$. Our main theorem is the following:

Main Theorem. *The category $SD(\bar{X})$ of self-dual sheaves compatible with the intersection chain sheaves is nonempty.*

Let $\mathbf{IC}^{\bullet}_{\mathfrak{L}}(\bar{X}, \mathbb{R})$ be an element of this category for a choice of appropriate Lagrangian subsheaf $\mathfrak{L}$. Then there are canonical morphisms $\mathbf{IC}^{\bullet}_{\bar{m}}(\bar{X}, \mathbb{R}) \to \mathbf{IC}^{\bullet}_{\mathfrak{L}}(\bar{X}, \mathbb{R}) \to \mathbf{IC}^{\bullet}_{\bar{n}}(\bar{X}, \mathbb{R})$. Using these maps we show the following:

Theorem. *Let $\pi : \bar{X} \to \hat{X}$ be the canonical collapse map as above. Then we have the following isomorphisms in $D^b(\hat{X})$:*

$$\mathbf{IC}^{\bullet}_{\bar{m}/\bar{n}}(\hat{X}, \mathbb{R}) \cong R\pi_* \mathbf{IC}^{\bullet}_{\bar{m}/\bar{n}}(\bar{X}, \mathbb{R}) \cong R\pi_* \mathbf{IC}^{\bullet}_{\mathfrak{L}}(\bar{X}, \mathbb{R}).$$

As a consequence of the first theorem, one can construct L-classes in $H_{2*}(\bar{X}, \mathbb{Q})$ (following, for example, [6]). The self-dual sheaves constructed above depend on the choice of the Lagrangian subsheaves. However the L-classes for these sheaves do not depend on the choice of the self-dual sheaves. This follows in our situation from results of the last section, and is in general true, see [2].

Now we discuss some related motivating results in connection with our main theorem. The following conjecture was made by M. Rapoport in [15] and he provided motivation for it in [16].

Conjecture. Let X be an arithmetic quotient of a Hermitian symmetric space. Let $\bar{m}$ (resp. $\bar{n}$) denote the lower-middle (resp. upper-middle) perversity function. Then we have $R\pi_* \mathbf{IC}^{\bullet}_{\bar{m}/\bar{n}}(\bar{X}, \mathbf{E}) \cong \mathbf{IC}^{\bullet}_{\bar{m}/\bar{n}}(\hat{X}, \mathbf{E})$ for a local coefficient system $\mathbf{E}$ on X obtained from a representation of $\mathbf{G}$. Here the isomorphism is in the derived category $D^b(\hat{X})$.

A proof of this conjecture has been announced by L. Saper and an abridged version of the ideas used appears in [18]. For Hilbert modular surfaces this has been previously verified or see section 4. Our results can be interpreted as a generalization of this conjecture in this special situation.

The results in this article then raise many interesting questions. For example, whether these L-classes are computable and if any information can be derived from them regarding the underlying real quadratic field K. Also the questions about existence of self-dual sheaves can be asked for reductive Borel-Serre compactifications of locally symmetric spaces associated to other semisimple groups defined over $\mathbb{Q}$. This question for other groups is particularly interesting from the point of view of Rapoport's conjecture. These and related questions will be considered in future articles.

Keeping in mind the possible audience for this paper and their varied background, we have included a section which reviews the material about intersection homology and self-dual perverse sheaves.

2 Intersection homology type invariants for non-Witt spaces

Let $\bar{X}$ be a stratified oriented topological pseudomanifold without boundary. If $\bar{X}$ has strata of even codimension only, then $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})$, the intersection chain sheaf with respect to the lower middle perversity $\bar{m}$, is Verdier self-dual, since $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) = \mathbf{IC}^\bullet_{\bar{n}}(\bar{X})$, the intersection chain sheaf with respect to the upper middle perversity $\bar{n}$. More generally, $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})$ is still self-dual on $\bar{X}$ if $\bar{X}$ is a *Witt space*, i.e. the lower middle perversity middle-dimensional intersection homology of all links of strata of odd codimension vanishes. If $\bar{X}$ is not a Witt space, then the canonical morphism $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \to \mathbf{IC}^\bullet_{\bar{n}}(\bar{X})$ is not an isomorphism and $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})$ is not self-dual. As shown in section 3 below, the reductive Borel-Serre compactification of a Hilbert modular surface is indeed *not* a Witt space.

The present section reviews results of [1], where a theory of intersection homology type invariants for non-Witt spaces is developed. We shall however only discuss simplified versions of the results, adapted to the (real) four-dimensional spaces considered in this paper.

Thus let $\bar{X}^4$ be a four-dimensional oriented pseudomanifold without boundary which is stratified as

$$\bar{X}^4 = \bar{X}_4 \supset \bar{X}_1 \supset \bar{X}_0 \supset \bar{X}_{-1} = \varnothing. \tag{2.1}$$

The strata are indexed by dimension and we shall write $\Sigma = \bar{X}_1$ for the singular set and $\Sigma_1 = \bar{X}_1 - \bar{X}_0$ for the pure 1-dimensional stratum. The connected components of Σ_1 are either circles or open intervals. If $\bar{X}$ is compact, then intervals cannot occur if

$$cl(\Sigma_1) \cap \bar{X}_0 = \varnothing.$$

We will see in section 3 that this condition holds for the reductive Borel-Serre compactification of Hilbert modular surfaces. Thus for such spaces, Σ_1 is a disjoint union of circles and $\bar{X}_0$ consists of the cusp singularities. The intersection chain sheaf $\mathbf{IC}^\bullet_{\bar{p}}(\bar{X})$ on $\bar{X}^4$ for perversity $\bar{p}$ and constant real coefficients is characterized by the following axioms:

(AX0) $\mathbf{IC}^\bullet_{\bar{p}}$ is constructible with respect to stratification (2.1).

(AX1) Normalization: $\mathbf{IC}^\bullet_{\bar{p}}|_{\bar{X}-\Sigma} \cong \mathbb{R}_{\bar{X}-\Sigma}[4]$.

(AX2) Lower bound: $\mathbf{H}^i(\mathbf{IC}^\bullet_{\bar{p}}) = 0$ for $i < -4$.

(AX3) Stalk vanishing conditions: $\mathbf{H}^i(\mathbf{IC}^\bullet_{\bar{p}}|_{\bar{X}-\bar{X}_{3-k}}) = 0$ for $i > \bar{p}(k) - 4$, for $k = 2, 3, 4$.

(AX4) Costalk vanishing conditions:

$$\begin{array}{ll} \mathbf{H}^i(j_1^! \mathbf{IC}^\bullet_{\bar{p}}|_{\bar{X}-\bar{X}_0}) = 0, & i \leq \bar{p}(3) - 3, j_1 : \Sigma_1 \hookrightarrow \bar{X} - \bar{X}_0, \text{ and} \\ \mathbf{H}^i(j_0^! \mathbf{IC}^\bullet_{\bar{p}}) = 0, & i \leq \bar{p}(4) - 3, j_0 : \bar{X}_0 \hookrightarrow \bar{X}. \end{array}$$

We shall denote the derived category of bounded differential complexes of sheaves constructible with respect to (2.1) by $D^b(\bar{X})$. Let us define the category of complexes of sheaves suitable for studying intersection homology type invariants on non-Witt spaces. The objects of this category should satisfy two properties: On the one hand, they should be self-dual, on the other hand, they should be as close to the middle perversity intersection chain sheaves as possible, that is, interpolate between $\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})$ and $\mathbf{IC}^\bullet_{\bar{n}}(\bar{X})$. Given these specifications, we adopt the following definition:

Definition 2.1. Let $SD(\bar{X})$ be the full subcategory of $D^b(\bar{X})$ whose objects $\mathbf{S}^\bullet$ satisfy the following axioms:

(SD1) Normalization: $\mathbf{S}^\bullet$ has an associated isomorphism $\nu : \mathbb{R}_{\bar{X}-\Sigma}[4] \xrightarrow{\simeq} \mathbf{S}^\bullet|_{\bar{X}-\Sigma}$.

(SD2) Lower bound: $\mathbf{H}^i(\mathbf{S}^\bullet) = 0$ for $i < -4$.

(SD3) Stalk vanishing condition for the upper middle perversity $\bar{n}$: $\mathbf{H}^i(\mathbf{S}^\bullet|_{\bar{X}-\bar{X}_{3-k}}) = 0$, for $i > \bar{n}(k) - 4, k = 2, 3, 4$.

(SD4) Self-Duality: $\mathbf{S}^\bullet$ has an associated isomorphism $d : \mathcal{D}\mathbf{S}^\bullet[4] \xrightarrow{\simeq} \mathbf{S}^\bullet$ such that $\mathcal{D}d[4] = d$ and $d|_{\bar{X}-\Sigma}$ is compatible to the orientation under normalization so that

$$\begin{CD}
\mathbb{R}_{\bar X-\Sigma}[4] @>{\nu}>{\simeq}> \mathbf{S}^\bullet|_{\bar X-\Sigma}\\
@A{\text{orient}}A{\simeq}A @A{\simeq}A{d|_{\bar X-\Sigma}}A\\
\mathbb{D}^\bullet_{\bar X-\Sigma} @>{\mathcal{D}\nu^{-1}[4]}>{\simeq}> \mathcal{D}\mathbf{S}^\bullet|_{\bar X-\Sigma}[4]
\end{CD}$$

commutes.

Depending on $\bar X$, the category $SD(\bar X)$ may or may not be empty. If it is non-empty, then its objects can be used to define characteristic classes for $\bar X$. Our main theorem (section 4) shows that if $\bar X$ is the reductive Borel-Serre compactification of a Hilbert modular surface, then indeed $SD(\bar X) \neq \varnothing$. One can show (cf. theorem 2.2 in [1]) that if $\mathbf{S}^\bullet \in SD(\bar X)$, there exist morphisms $\mathbf{IC}^\bullet_{\bar m}(\bar X) \xrightarrow{\alpha} \mathbf{S}^\bullet \xrightarrow{\beta} \mathbf{IC}^\bullet_{\bar n}(\bar X)$ uniquely determined by $\alpha|_{\bar X-\Sigma} = \nu : \mathbb{R}_{\bar X-\Sigma}[4] \xrightarrow{\simeq} \mathbf{S}^\bullet|_{\bar X-\Sigma}$ and $\beta|_{\bar X-\Sigma} = \nu^{-1} : \mathbf{S}^\bullet|_{\bar X-\Sigma} \xrightarrow{\simeq} \mathbb{R}_{\bar X-\Sigma}[4]$, such that the following diagram is commutative:

$$\begin{CD}
\mathbf{IC}^\bullet_{\bar m}(\bar X) @>{\alpha}>> \mathbf{S}^\bullet\\
@A{\simeq}AA @A{\simeq}A{d}A\\
\mathcal{D}\mathbf{IC}^\bullet_{\bar n}(\bar X)[4] @>{\mathcal{D}\beta[4]}>> \mathcal{D}\mathbf{S}^\bullet[4]
\end{CD}$$

(where d is given by (**SD4**)), which clarifies the relation between intersection chain sheaves and objects of $SD(\bar X)$.

To understand the structure of $SD(\bar X)$ (how can one construct objects in $SD(\bar X)$?), one introduces the notion of a *Lagrangian structure*. The reason why $\mathbf{IC}^\bullet_{\bar m}(\bar X)$ is not self-dual is that the sheaf

$$\mathcal{O} = \mathbf{H}^{-3}(Ri_*\mathbb{R}_{\bar X-\Sigma}[4])[3] \in D^b(\bar X - \bar X_0),$$

$i : \bar X - \Sigma \hookrightarrow \bar X - \bar X_0$, is not trivial (note $\bar n(\operatorname{codim}\Sigma_1) - \dim \bar X = -3$). Its support is Σ_1, where it is a local system with stalks $H^1(L_x;\mathbb{R})$, $x \in \Sigma_1$, with L_x the link of the component of Σ_1 containing x. The sheaf $\mathcal{O}$ is isomorphic to the algebraic mapping cone of the canonical morphism $\mathbf{IC}^\bullet_{\bar m}(\bar X - \bar X_0) \to$

$\mathbf{IC}^{\bullet}_{\bar{n}}(\bar{X} - \bar{X}_0)$: We have the distinguished triangle

$$\begin{tikzcd}
\mathbf{IC}^{\bullet}_{\bar{m}}(\bar{X} - \bar{X}_0) \arrow[rr] & & \mathbf{IC}^{\bullet}_{\bar{n}}(\bar{X} - \bar{X}_0) \arrow[dl] \\
& \mathcal{O} \arrow[ul, "{[1]}"] &
\end{tikzcd} \tag{2.2}$$

Dualizing (2.2), one sees that $\mathcal{O}$ is self-dual: $\mathcal{D}\mathcal{O}[5] \cong \mathcal{O}$ (the duality-dimension is one off).

Definition 2.2. A *Lagrangian structure* (along Σ_1) is a morphism $\mathcal{L} \longrightarrow \mathcal{O}$, $\mathcal{L} \in D^b(\bar{X} - \bar{X}_0)$, which induces injections on stalks and has the property that some distinguished triangle on $\mathcal{L} \longrightarrow \mathcal{O}$ is an algebraic nullcobordism (in the sense of [6]) for $\mathcal{O}$.

This means the following: Some distinguished triangle on $\phi : \mathcal{L} \longrightarrow \mathcal{O}$ has to be of the form

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "\phi"] & & \mathcal{O} \arrow[dl] \\
& \mathcal{D}\mathcal{L}[5] \arrow[ul, "\gamma"', "{[1]}"] &
\end{tikzcd}$$

and we require $\mathcal{D}\gamma[5] = \gamma[-1]$.

Equivalently, every stalk $\mathcal{L}_x, x \in \Sigma_1$, is a Lagrangian (i.e. maximally isotropic) subspace of $\mathcal{O}_x$ with respect to the pairing $\mathcal{O}_x \otimes \mathcal{O}_x \to \mathbb{R}$ induced by the self-duality of $\mathcal{O}$. If $\mathcal{L}_1 \to \mathcal{O}$ and $\mathcal{L}_2 \to \mathcal{O}$ are two Lagrangian structures, then a *morphism of Lagrangian structures* is a commutative square in $D^b(\bar{X} - \bar{X}_0)$:

$$\begin{tikzcd}
\mathcal{L}_1 \arrow[r] \arrow[d] & \mathcal{O} \arrow[d, "\mathcal{O}(f)"] \\
\mathcal{L}_2 \arrow[r] & \mathcal{O}
\end{tikzcd}$$

where $f \in \mathrm{Hom}_{D^b(\bar{X}-\Sigma)}(\mathbb{R}_{\bar{X}-\Sigma}[4], \mathbb{R}_{\bar{X}-\Sigma}[4])$ and $\mathcal{O}(f) = \mathbf{H}^{-3}(Ri_* f)[3]$. Thus Lagrangian structures form a category $\mathrm{Lag}(\Sigma_1)$. The relevance of $\mathrm{Lag}(\Sigma_1)$ vis-à-vis $SD(\bar{X})$ is explained as follows:

1. Extracting Lagrangian structures from self-dual sheaves: There exists a covariant functor

$$\Lambda : SD(\bar{X} - \bar{X}_0) \longrightarrow \mathrm{Lag}(\Sigma_1).$$

This means that every self-dual perverse sheaf has naturally associated Lagrangian structures.

2. Lagrangian structures as building blocks for self-dual sheaves: Let $\mathrm{Const}(\bar{X} - \Sigma)$ denote the category whose single object is the constant sheaf $\mathbb{R}_{\bar{X}-\Sigma}$ on the top stratum $\bar{X} - \Sigma$ and whose morphisms are sheaf maps $\mathbb{R}_{\bar{X}-\Sigma} \to \mathbb{R}_{\bar{X}-\Sigma}$. Let

$$\mathrm{Const}(\bar{X} - \Sigma) \rtimes \mathrm{Lag}(\Sigma_1)$$

denote the twisted product of categories whose objects are pairs $(\mathbb{R}_{\bar{X}-\Sigma}, \mathcal{L} \xrightarrow{\phi} \mathcal{O})$, $\phi \in \mathrm{Lag}(\Sigma_1)$, and whose morphisms are pairs with the first component a morphism $f \in \mathrm{Hom}_{D^b(\bar{X}-\Sigma)}(\mathbb{R}_{\bar{X}-\Sigma}, \mathbb{R}_{\bar{X}-\Sigma})$ and the second component a commutative square

$$\begin{tikzcd}
\mathcal{L}_1 \arrow[r] \arrow[d] & \mathcal{O} \arrow[d, "\mathcal{O}(f)"] \\
\mathcal{L}_2 \arrow[r] & \mathcal{O}
\end{tikzcd}$$

There exists a covariant functor

$$\begin{array}{rccc} \boxplus : & \mathrm{Const}(\bar{X} - \Sigma) \rtimes \mathrm{Lag}(\Sigma_1) & \longrightarrow & SD(\bar{X} - \bar{X}_0), \\ & (\mathbb{R}_{\bar{X}-\Sigma}, \mathcal{L}) & \mapsto & \mathbb{R}_{\bar{X}-\Sigma} \boxplus \mathcal{L}, \end{array}$$

that is, a Lagrangian structure along Σ_1 naturally gives rise to a self-dual sheaf on $(\bar{X} - \Sigma) \cup \Sigma_1 = \bar{X} - \bar{X}_0$, and thus, using the Deligne-extension, to a self-dual sheaf on $\bar{X}$.

It is shown in [1] that

$$\mathrm{Const}(\bar{X} - \Sigma) \rtimes \mathrm{Lag}(\Sigma_1) \underset{(i^*,\Lambda)}{\overset{\boxplus}{\rightleftarrows}} SD(\bar{X} - \bar{X}_0)$$

induces an equivalence of categories (i^* denotes restriction to $\bar{X} - \Sigma$). Summarizing, one obtains

Theorem 2.3. *There is an equivalence of categories*

$$SD(\bar{X}^4) \simeq \mathrm{Const}(\bar{X} - \Sigma) \rtimes \mathrm{Lag}(\Sigma_1).$$

This result will be used in section 4 to construct self-dual perverse sheaves and characteristic classes for the reductive Borel-Serre compactification of Hilbert modular surfaces.

3 Geometry of reductive Borel-Serre compactifications

3.1 Hilbert modular surfaces and their compactifications

First we fix some notation. Let $K = \mathbb{Q}[\sqrt{D}]$ be a real quadratic field and let σ be its nontrivial $\mathbb{Q}$-automorphism. Let V be the $\mathbb{Q}$-vector space of 2×2 skew-Hermitian matrices over K:

$$V = \{v \in M_2(K) \mid {}^t v = -v^\sigma\}.$$

The determinant defines a quadratic form Q on $V_\mathbb{R}$ which has signature $(2,2)$. We let $\mathbf{G}$ be the group $SO^0(Q)$, the group of special orthogonal transformations for the form Q. Its group of real points, $\mathbf{G}(\mathbb{R})$, is isomorphic $SO^0(2,2)$ and its maximal compact subgroup is $K_0 = SO^0(2) \times SO^0(2)$. The center of $\mathbf{G}$ is a finite group and hence $A_\mathbf{G}$, the identity component of the real points of the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{G}$ is trivial. The quotient $\mathbf{G}(\mathbb{R})/K_0 A_\mathbf{G}$ is homeomorphic to $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. See, for example, Chap. 3, [11].

The group $SL(2,K)$ acts on V by conjugation. For an element $\alpha \in SL(2,K)$, $v \mapsto {}^t\alpha^\sigma v \alpha$. Note that this action preserves the form Q. Let $\mathfrak{o}$ be the ring of integers in K. By Γ we denote either $PSL(2,\mathfrak{o})$ or a subgroup of $PSL(2,\mathfrak{o})$ which depends on a prime ideal $\mathfrak{p}$ in $\mathfrak{o}$ and is given by

$$\Gamma = \begin{cases} \{A \in SL(2,\mathfrak{o}) \mid A \equiv I \bmod (\mathfrak{p})\}/\{\pm 1\}, & \text{if } -1 \equiv 1 \bmod (\mathfrak{p}) \\ \{A \in SL(2,\mathfrak{o}) \mid A \equiv I \bmod (\mathfrak{p})\} & \text{otherwise.} \end{cases}$$

where I is the identity matrix. The group Γ is torsion-free. By the remarks above, we have a map $\Gamma \to \mathbf{G}(\mathbb{R})$ which is injective. Now we define the *Hilbert modular surface* to be the quotient $\Gamma\backslash\mathbf{G}(\mathbb{R})/K_0 A_\mathbf{G} \cong \mathbb{H} \times \mathbb{H}/\Gamma$. Note that in the last expression, the action of Γ on $\mathbb{H} \times \mathbb{H}$ may be assumed to be induced by the two embeddings of K into $\mathbb{R}$ (and so of Γ into $PGL^+(2,\mathbb{R})^2$). It is easy to see that the group $\mathbf{G}$ is semisimple and that its $\mathbb{Q}$-rank is 1.

For a locally symmetric space, there are three compactifications that are widely used: Borel-Serre compactification, reductive Borel-Serre compactification, and (if the space is Hermitian) Baily-Borel compactification. In this paper, we will use the latter two compactifications, denoted by $\bar{X}$ and $\hat{X}$, respectively.

The reductive Borel-Serre compactification $\bar{X}$ is obtained (as a set) from X by adding a union of nonsingular *boundary strata* X_P, one for each Γ-conjugacy class of rational parabolic subgroups P of $\mathbf{G}$.

To describe the boundary strata X_P, we adopt the following notation. Let P be a rational parabolic subgroup of $\mathbf{G}$. Then a rational Langlands decomposition is denoted by $P = M_P A_P \mathfrak{U}_P$, where $\mathfrak{U}_P$ is the unipotent radical of P, $M_P A_P$ is a Levi subgroup, and A_P is the connected component of the real points of the $\mathbb{Q}$-split torus S_P in the center of $M_P A_P$. Let K_P be the maximal compact subgroup of M_P and Γ_M be the projection of $\Gamma \cap P$ into M_P. Then the boundary stratum X_P is the quotient $\Gamma_M \backslash M_P / K_P$ of M_P/K_P by the discrete group Γ_M.

Since the group $\mathbf{G}$ has $\mathbb{Q}$-rank 1, the parabolic subgroups are maximal. So there are no closure relations among the boundary strata. For more general cases, see [9]. For the topology of Borel-Serre compactifications, see [22, 18].

3.2 $\bar{X}$ as a stratified space

Recall from Theorem (8.10), [9] that the stratification of the reductive Borel-Serre compactification using the boundary strata X_P is locally trivial. This means that a boundary stratum X_P has a *link bundle*, which is a locally trivial fiber bundle whose fibers are the links of points in the strata. The link bundle is the boundary of a regular neighborhood of X_P in $\bar{X}$. Determining that bundle will be an important step for our purposes.

It is well-known that for any Γ-conjugacy class of rational parabolic subgroups, the corresponding boundary stratum X_P is S^1, the circle, and the link bundle is a torus bundle over S^1. However, we need more information about the link bundles. This is done by first carrying out all the computations for a concrete example and then using it for any other conjugacy class. First we consider a special case of a specific parabolic subgroup in $\mathbf{G}$. Recall that we have the vector space V and the form Q on V is the determinant form. Let P be the isotropy subgroup of the $\mathbb{Q}$-rational isotropic flag $\{e_1\}$. Then we have the following proposition.

Proposition 3.1. *The reductive Borel-Serre stratum X_P associated to the conjugacy class of P is a circle and its link bundle is the Borel-Serre stratum Y_P associated to the conjugacy class of P. Furthermore, Y_P is a torus-bundle over X_P whose fiber is the torus $\mathbb{R}^2/L$, where L is as in Eq. (3.3) below. The class of the bundle Y_P is given by the automorphism of the torus as in Eq. (3.4) below for an element u which is a generator for the unit group in $\mathfrak{o}$ or the subgroup of the unit group whose elements are* $1 \bmod (\mathfrak{p})$ *for some ideal* $\mathfrak{p} \subset \mathfrak{o}$*.*

Proof. The *Borel-Serre* stratum associated to the conjugacy class of P is a torus bundle over X_P. The fibers of the canonical map

$$\pi : Y_P \to X_P$$

are precisely the nilmanifolds N_P. The link bundle for the stratum X_P is in fact diffeomorphic to Y_P. This follows using the fact that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1 in the proof of Theorem (8.10) of [9]. We now write this explicitly in order to get the gluing map for the torus bundle. It is easy to see that the preimage of $\Gamma \cap P = \Gamma_P$ in $SL(2, \mathfrak{o})$ under the canonical morphism is given by

$$\left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in SL(2,\mathfrak{o}) \left| \begin{array}{l} ad = 1, \text{ and} \\ a,\ d = 1 \bmod (\mathfrak{p}), \\ c \in \mathfrak{p} \text{ if } \Gamma \neq PSL(2,\mathfrak{o}) \end{array} \right. \right\}. \tag{3.1}$$

Let $a = a_0 + a_1\sqrt{D},\ c = c_0 + c_1\sqrt{D}$. The we can write Γ_P as

$$\left\{ A = \begin{bmatrix} aa^\sigma & a_0c_1 - a_1c_0 & a_0c_0 - a_1c_1D & \frac{cc^\sigma}{2D} \\ 0 & \frac{a_0^2+a_1^2D}{N(a)} & \frac{-2a_0a_1D}{N(a)} & -\frac{(a_0c_1+a_1c_0)}{N(a)} \\ 0 & \frac{-2a_0a_1}{N(a)} & \frac{a_0^2+a_1^2D}{N(a)} & \frac{a_0c_0+a_1c_1D}{N(a)D} \\ 0 & 0 & 0 & (aa^\sigma)^{-1} \end{bmatrix} |a,\ c \text{ as in (3.1)} \right\},$$

where $N(a)$ is the norm of a and must be ± 1 since a is a unit in $\mathfrak{o}$. So we can describe Y_P as the quotient of $\mathbb{R}^3$ with coordinates (b, μ, η) by the action of Γ_P which is given as follows. Let A be an element in Γ_P as above, then

$$A \cdot \begin{bmatrix} b \\ \mu \\ \eta \end{bmatrix} = \begin{bmatrix} (a_0^2 + a_1^2D)b - 2a_0a_1D \\ (a_0^2 + a_1^2D)\mu - 2a_0a_1D\eta + \frac{c^\sigma a^\sigma - ca}{2\sqrt{D}} \\ -2a_0a_1\mu + (a_0^2 + a_1^2D)\eta + \frac{c^\sigma a^\sigma + ca}{2D} \end{bmatrix}. \tag{3.2}$$

The corresponding map $\pi : Y_P \to X_P$ is given by projection onto the b coordinate. Note that the only condition on c (for Γ_P) is that it is either in $\mathfrak{p}$ or in $\mathfrak{o}$ depending on Γ. Now any ideal in $\mathfrak{o}$ (including itself) is a $\mathbb{Z}$-module of rank 2. Then we consider the rank 2 $\mathbb{Z}$-lattice L in $\mathbb{R}^2$ given by

$$\left\{ \left(\frac{c^\sigma - c}{2\sqrt{D}}, \frac{c^\sigma + c}{2D}\right) \in \mathbb{R}^2 \,|c \in \mathfrak{p} \text{ or } \mathfrak{o} \right\}. \tag{3.3}$$

Next note that the lattice L is invariant under the following linear transformation of $\mathbb{R}^2$ (with coordinates given by (μ, η)) for any a as in Γ_P.

$$\begin{bmatrix} \mu \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} a_0^2 + a_1^2D & -2a_0a_1D \\ -2a_0a_1 & a_0^2 + a_1^2D \end{bmatrix} \begin{bmatrix} \mu \\ \eta \end{bmatrix}. \tag{3.4}$$

Now it follows from Eq. (3.2) that the fiber nilmanifold of the map π is given by the torus $\mathbb{R}^2/L$ which makes the link bundle Y_P into a torus bundle over the circle X_P.

To compute the gluing map of this torus bundle first note that the elements a which appear as entries in elements of Γ_P form a group isomorphic to $\mathbb{Z}$. Also the map

$$a \mapsto \begin{bmatrix} a_0^2 + a_1^2 D & -2a_0a_1 \\ -2a_0a_1 D & a_0^2 + a_1^2 D \end{bmatrix}$$

is a group homomorphism. So we let $u = u_0 + u_1\sqrt{D}$ be a generator of this group and note that the corresponding isomorphism of the torus $\mathbb{R}^2/L$ given by Eq. (3.4) determines the monodromy of the torus bundle Y_P/X_P. $\square$

Now we consider an arbitrary parabolic subgroup Q of $\mathbf{G}$. By Witt's theorem (see, for example, Chap. IV, Theorem (1.5), [21]), P and Q are conjugate by an element g of $\mathbf{G}(\mathbb{Q})$. It follows that their unipotent radicals $\mathfrak{U}_P$, $\mathfrak{U}_Q$ are isomorphic over $\mathbb{Q}$ and so are their Levi quotients L_P and L_Q. These will result in isomorphic groups M_Q (resp. K_Q) and M_P (resp. K_P). We have the commutative diagram

$$\begin{tikzcd} P \arrow[r, "g^{-1}(-)g"] \arrow[d] & Q \arrow[d] \\ L_P \arrow[r, "g^{-1}(-)g"] \arrow[d] & L_Q \arrow[d] \\ M_P \arrow[r, "g^{-1}(-)g"] & M_Q \end{tikzcd} \tag{3.5}$$

As L_P, L_Q are isomorphic over $\mathbb{Q}$, we get isomorphic lifts of A_P (resp. S_P, M_P) and A_Q (resp. S_Q, M_Q). Next we consider the arithmetic subgroup ${}^g\Gamma = g\Gamma g^{-1}$ of $\mathbf{G}(\mathbb{Q})$. Let

$$Y = ({}^g\Gamma \cap P)\backslash P/K_PA_P \qquad \text{and} \qquad X = {}^g\Gamma_{M_P}\backslash M_P/K_P,$$

where ${}^g\Gamma_{M_P}$ is the projection ${}^g\Gamma \cap P$ into M_P. Note that the projection $P \to M_P$ induces a projection $Y \to X$.

Proposition 3.2. *Let Y_Q (resp. X_Q) be the (resp. reductive) Borel-Serre stratum associated to the Γ-conjugacy class of a parabolic subgroup Q. The space Y constructed above is a fiber bundle over X and we have the following isomorphism of fiber bundles:*

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d] & Y_Q \arrow[d] \\ X \arrow[r, "\varphi"] & X_Q \end{tikzcd} \tag{3.6}$$

Proof. That Y_Q is the link bundle of the stratum X_Q follows as in the case of the parabolic subgroup P using the proof of Theorem (8.10), [9]. By starting with the arithmetic subgroup ${}^g\Gamma$ instead of Γ, we see that Y is a fiber bundle over X (in fact, the link bundle for the stratum X for this choice of the arithmetic subgroup). The rest of the proposition follows by using the isomorphisms induced from the conjugation isomorphism $P \to Q$ as in diagram 3.5. ☐

The proposition above says that for the question of determining Lagrangian subsheaves in the cohomology sheaf $\mathbf{H}^1$ of the link bundle Y_Q/X_Q, we may consider the fiber bundle Y/X. This, in turn, allows us to do all the computations for the parabolic subgroup P, but possibly with different arithmetic subgroups. The goal for the remainder of this section is to relate the fiber bundles Y/X and Y_P/X_P. Note that since $g \in \mathbf{G}(\mathbb{Q})$, the arithmetic subgroups Γ and ${}^g\Gamma$ of $\mathbf{G}$ are commensurable. We will use this fact in the following discussion repeatedly. Now we introduce some notation. Let $\Gamma^1 = {}^g\Gamma \cap \Gamma$ (for g as above). Then Γ^1 is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. We denote by Y_1 (resp. X_1) the (resp. reductive) Borel-Serre boundary stratum

$$\Gamma^1_P\backslash P/K_P A_P \text{ (resp. } \Gamma^1_{M_P}\backslash M_P/K_P).$$

Note that we have the usual canonical projection $Y_1 \to X_1$ and that Y_1 is the link bundle for the stratum X_1. This last claim follows as before, see the proof of Prop. 3.2.

Proposition 3.3. *We have the following commutative diagram of fiber bun-*

dles in which all the horizontal maps are covering maps.

$$\begin{tikzcd} Y \arrow[d] & Y_1 \arrow[l] \arrow[r, "f_Y"] \arrow[d] & Y_P \arrow[d] \\ X & X_1 \arrow[l] \arrow[r, "f_X"] & X_P \end{tikzcd}$$

Proof. It is sufficient to prove the existence of the commutative square for one of the two squares as Γ_1 is of finite index in both Γ and ${}^g\Gamma$. The commutativity of the right square follows from the commutative diagram

$$\begin{tikzcd} P/K_PA_P \arrow[r] \arrow[d] & Y_P \arrow[d] \\ M_P/K_P \arrow[r] & X_P \end{tikzcd}$$

and by noting that $\Gamma_P^1 \subset \Gamma_P$ and $\Gamma_{M_P}^1 \subset \Gamma_{M_P}$.

Now it is clear that Γ_P^1 (resp. $\Gamma_{M_P}^1$) is of finite index in Γ_P (resp. Γ_{M_P}). Recall that M_P/K_P is diffeomorphic to $\mathbb{R}$ and Γ_{M_P} acts by translation. So $\Gamma_{M_P}^1$ acts by translations as well and so f_X is a covering map. Next, consider the map f_Y. From section 7 of [9] or section 3.4 of [22], we have a diffeomorphism

$$\mathfrak{U}_P \times D_P \to P/K_PA_P,\ D_P = M_P/K_P.$$

The action of P on the left side which makes this map P-equivariant is given by Lemma 7.8 of [9]. We need to consider only two consequences of this action: 1) $\mathfrak{U}_P$ acts only on the first factor and does so by translation (7.9.1 of [9]) and 2) M_P acts on the second factor by translation (this is obvious). A simple computation shows that $\Gamma_P \cap A_P = \{e\}$. This, together with the two consequences of the action, shows that the action of Γ_P on P/K_PA_P is properly discontinuous and free, and hence that $P/K_PA_P \to Y_P$ is a covering map. Note that the same consequences and the fact that $\Gamma_P^1 \cap A_P = \{e\}$ say that the map $P/K_PA_P \to Y_1$ is also a covering map. Since the first map $P/K_PA_P \to Y_P$ factors through this map, we have that $Y_1 \to Y_P$ is a covering map.

To finish the proof, we consider the fiber bundles Y_1/X_1 and Y/X. The argument above will apply in this case as well once we have proved that ${}^g\Gamma_P \cap A_P = \{e\}$ since the arithmetic subgroup for the fiber bundle Y/X is ${}^g\Gamma$.

Suppose that $\gamma \in {}^g\Gamma_P \cap A_P$. Then since Γ^1 is of finite index in Γ, some power of γ lies in $\Gamma^1 \cap A_P$. But since $\Gamma^1 \subset \Gamma$ and $\Gamma_P \cap A_P = \{e\}$, we get that $\gamma^n = e$, for some nonnegative integer n. Since Γ is a torsion-free group, $n = 1$. So ${}^g\Gamma_P \cap A_P = \{e\}$. This finishes the proof. $\square$

Corollary 3.4. *The reductive Borel-Serre strata for* Γ*-conjugacy classes of parabolic subgroups are always circles and their associated link bundles are torus bundles over these circles.*

Proof. We have already seen that the statement is true for a specific choice P (Prop. 3.1). Now for any other conjugacy class, the proposition follows from Prop. 3.3. $\square$

Finally, we can construct a stratification on $\bar{X}$. Recall that X has finitely many (zero dimensional) singularities, see for example, chapter 5 of [11]. Now we consider the filtration of $\bar{X}$ by closed subspaces as

$$\bar{X} = \bar{X}_4 \supset \bar{X}_1 \supset \bar{X}_0 \supset \bar{X}_{-1} = \varnothing,$$

where

$$\bar{X}_1 = \{\cup \text{ boundary strata } X_Q\} \cup \{\text{singularities on } X\}$$

and X_0 denotes the set of (zero dimensional) singularities on X. That this makes $\bar{X}$ into a (locally trivial) stratified space follows from Theorem (8.10) of [9].

For the following sections, we introduce some notation. Let $U_k = \bar{X} - \bar{X}_{4-k}$ and let $i_k : U_k \hookrightarrow U_{k+1}$ be this inclusion. Also, j_k denotes the inclusion $U_{k+1} - U_k \hookrightarrow U_{k+1}$. As usual, $\bar{n}$, $\bar{m}$ denote the upper and lower middle perversities respectively.

4 Self-dual Sheaves on $\bar{X}$

The goal of this section is to prove the main theorem which asserts the existence of self-dual sheaves on $\bar{X}$. First we consider the question of existence of Lagrangian structures on odd codimensional strata. This will then prove the main theorem by the equivalence of categories, see theorem 2.3.

In section 2, we saw necessary and sufficient conditions for the existence of self-dual sheaves on $\bar{X}$ that are compatible with the intersection chain sheaves. In this section, first we transform the general criterion to conditions on the monodromy for the link bundles. Recall that odd codimensional boundary strata in $\bar{X}$ are circles and their associated link bundles are torus bundles. We have the following lemma:

Lemma 4.1. *Let X_Q be a boundary circle associated to the conjugacy class of a parabolic subgroup Q and let Y_Q be the associated link bundle. Let T^2 denote the fiber torus of this link bundle. Then a Lagrangian structure on X_Q exists if and only if the action of a generator of the fundamental group of X_Q on $H^1(T^2, \mathbb{R})$ has a real eigenvalue.*

Proof. First recall the notion of a Lagrangian subsheaf. Let $\mathbf{H}$ be a locally constant sheaf on a manifold M with a nondegenerate pairing

$$\mathbf{H} \otimes \mathbf{H} \to \mathbb{R}_M.$$

Then a subsheaf $\mathbf{E} \subset \mathbf{H}$ is Lagrangian if, for every $x \in M$, the stalk $\mathbf{E}_x$ is a Lagrangian subspace of $\mathbf{H}_x$ with respect to the pairing above. Let j denote the inclusion $X_Q \to \bar{X} - \bar{X}_0$ and i_3 be the inclusion as in section 3.2. By remark 2.4 of [1], a Lagrangian structure on X_Q exists if and only if there exists a Lagrangian subsheaf $\mathbf{E} \subset \mathbf{H}$, where $\mathbf{H}$ is the cohomology sheaf $\mathbf{H}^{-3}(j^* R i_{3*} \mathbb{R}[4])$ on X_Q. In our situation, the sheaf $\mathbf{H}^{-3}(j^* R i_{3*} \mathbb{R}[4])$ on X_Q is precisely the cohomology sheaf $\mathbf{H}^1(R\pi_* \mathbb{R}_{Y_Q})$ with the usual cup product as the pairing required as above. Here π is the map $Y_Q \to X_Q$. Let $B(\cdot, \cdot)$ denote this pairing.

Now any locally trivial fiber bundle over the unit interval I is trivial. So we view the fiber bundle Y_Q as obtained by gluing the ends $\{0\} \times I$ and $\{1\} \times I$ of the trivial fiber bundle $I \times T^2$ by a gluing map $\varphi : T^2 \to T^2$ (so that Y_Q is the mapping torus of φ). The map φ lifts to a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. To prove the proposition, we need to analyze the gluing map φ.

Let V denote the 2 dimensional $\mathbb{R}$ vector space $H^1(T^2, \mathbb{R})$. The map φ induces the same map on V up to conjugation. Now a Lagrangian subsheaf as above will exist if and only if there exists a nonzero vector $v \in V$ such that

$$B(v, v) = 0, \text{ and } \varphi(v) = \lambda v \text{ for some } \lambda \in \mathbb{R}.$$

In our case, for any vector $w \in V$, $B(w, w) = 0$ is automatic since the pairing is anti-symmetric. So the required condition holds if and only if φ has a real eigenvalue. ∎

Next we address the question of existence of Lagrangian structures on the boundary strata X_Q. Recall that we have computed explicitly the gluing map for the link bundle Y_P over X_P, where P is the isotropy subgroup of the isotropic rational flag $\{e_1\}$. See Prop. 3.1. We have the following lemma.

Lemma 4.2. *There exists a Lagrangian structure on the boundary stratum X_P, where P is the parabolic subgroup as above.*

Proof. By Lemma 4.1, we need to check that the gluing map for Y_P has a real eigenvalue. Recall from Prop. 3.1 that the gluing map (up to conjugation) is given by the linear map

$$\begin{bmatrix} u_0^2 + u_1^2 D & -2u_0u_1 \\ -2u_0u_1D & u_0^2 + u_1^2 D \end{bmatrix}$$

for some unit $u = u_0 + u_1\sqrt{D}$ in $\mathfrak{o}$. Using the fact that the the norm of u is ± 1, it is easy to see that the eigenvalues of this matrix are

$$(u_0 + u_1\sqrt{D})^2 \text{ and } (u_0 - u_1\sqrt{D})^2,$$

which are real. So Lagrangian structures exist for the stratum X_P. □

In the remainder of this section, we consider an arbitrary stratum X_Q. In order to analyze the action of the fundamental group of X_Q on the cohomology of the link N_Q (which is the fiber of the link bundle Y_Q), we use a theorem of B. Kostant, which, combined with a theorem of K. Nomizu (or independently by W. T. van Est), describes the cohomology of N_Q as a direct sum of weight spaces for the action of $L_Q(\mathbb{C})$. Here L_Q denotes the Levi quotient of the parabolic subgroup Q.

To use this theorem, we first compute the cohomology $H^*(N_P, \mathbb{R})$ with the action of $L_P(\mathbb{C})$, where P is the parabolic subgroup of section 3. We fix the torus

$$T(\mathbb{C}) = \left\{ A = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & d & b & 0 \\ 0 & \frac{b}{D} & d & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \middle| d^2 - \frac{b^2}{D} = 1 \right\}$$

in $L_P(\mathbb{C})$. Since we are in the $\mathbb{Q}$-rank 1 case, we can fix $P(\mathbb{C})$ as a Borel subgroup. Further we fix as a Cartan subgroup $H(\mathbb{C}) = Z_G(T(\mathbb{C})) = T(\mathbb{C})$. We let β_1, β_2 be the characters of $T(\mathbb{C})$ defined by

$$\beta_1(A) = \alpha, \ \beta_2(A) = d + \frac{b}{\sqrt{D}},$$

where A is an element of $T(\mathbb{C})$ as above. A simple calculation (or see section (23.4), [4]) shows that the roots $\Phi = \Phi(H(\mathbb{C}), \mathfrak{g}(\mathbb{C}))$ are

$$\pm(\beta_1 - \beta_2),\ \pm(\beta_1 + \beta_2).$$

The choice of the Borel subgroup $P(\mathbb{C})$ determines a positive root system Φ^+ and a basis $\Delta \subset \Phi^+$. In fact, it is easy to compute Φ^+ to get

$$\Phi^+ = \{\beta_1 + \beta_2,\ \beta_1 - \beta_2\}.$$

So we get that the half sum of the positive roots is

$$\rho = \rho_G = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta = \beta_1.$$

Next the Weyl group W of G is $W = N_G(T)/Z_G(T) = \mathbb{Z}/2 \times \mathbb{Z}/2$, where the two generators $\sigma,\ \tau$ are given by the conjugation by the respective matrices

$$\begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{D}}{2} & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{D}} & 0 & 0 & \frac{1}{2\sqrt{D}} \\ 0 & 1 & \sqrt{D} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & \sqrt{D} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2\sqrt{D}} & 0 & 0 & -\frac{1}{\sqrt{D}} \\ 0 & \frac{1}{2} & -\frac{\sqrt{D}}{2} & 0 \end{bmatrix}.$$

For each $w \in W$, we set

$$\Phi^+(w) = \{\beta \in \Phi^+ \mid w^{-1}\beta \in \Phi^-\}.$$

Then $|\Phi^+(w)| = l(w)$ is the length of w. In our case, we compile this information in the following table:

w	$\Phi^+(w)$	$l(w)$	$w(\rho)$	$w(\rho) - \rho$
τ	$\{\beta_1 - \beta_2\}$	1	β_2	$\beta_2 - \beta_1$
σ	$\{\beta_1 + \beta_2\}$	1	$-\beta_2$	$-\beta_2 - \beta_1$
$\tau\sigma$	$\{\beta_1 + \beta_2,\ \beta_1 - \beta_2\}$	2	$-\beta_1$	$-2\beta_1$
id	$\varnothing$	0	β_1	0

It's easy to see that the action of $H(\mathbb{C})$ on $\mathfrak{n}_P(\mathbb{C})$ (Lie algebra of the unipotent radical $\mathfrak{U}_P$ of P) has weights $\beta_1 + \beta_2$ and $\beta_1 - \beta_2$. So finally we have the set

$$\begin{aligned} W_P^1 &= \{w \in W \mid \Phi^+(w) \subset \Phi(H(\mathbb{C}), \mathfrak{n}_P(\mathbb{C}))\} \\ &= W, \end{aligned}$$

where the last equality follows by using the table above. For any dominant weight μ, let V_μ be the irreducible $L_P(\mathbb{C})$-module with highest weight μ. We have the following theorem.

Theorem 4.3. *([7], [14], [13] Theorem 5.14) There is an isomorphism*

$$H^k(\mathfrak{n}_P, \mathbb{C}) \cong H^k(N_P, \mathbb{C})$$

between the Lie algebra cohomology of $\mathfrak{n}_P$ and the de Rham cohomology of N_P. Furthermore, the $L_P(\mathbb{C})$ representation $H^*(\mathfrak{n}_P, \mathbb{C})$ is algebraic and is given by

$$H^*(\mathfrak{n}_P, \mathbb{C}) = \bigoplus_{w \in W_P^1} V_{w(\rho)-\rho}[-l(w)]$$

where $[-l(w)]$ indicates that the module appears in degree $l(w)$, the length of w.

Since the trivial representation is defined over $\mathbb{Q}$, another consequnce of Theorem (5.14) of [13] says that the representation $H^*(\mathfrak{n}_P, \mathbb{C})$ is also defined over $\mathbb{Q}$. Using the theorem above, we have the following proposition.

Proposition 4.4. *Let Q be any parabolic subgroup in $\mathbf{G}$. Then there exist Lagrangian structures for the associated reductive Borel-Serre stratum X_Q.*

Proof. By Lemma 4.1, we need to verify that the map induced on H^1 by the gluing map for the link bundle Y_Q/X_Q has a real eigenvalue. By Prop. 3.2, it is sufficient to verify this property for the link bundle Y/X of that proposition. So we need to verify that the action of a particular element of $L_P(\mathbb{R})$ on $H^1(N_P, \mathbb{R})$ has a real eigenvalue. By Theorem 4.3 and the table above, the representation of $L_P(\mathbb{R})$ on H^1 of the link has highest weights

$$\beta_2 - \beta_1, \ -\beta_2 - \beta_1.$$

However recall that $L_P = M_P A_P$ is diagonalizable (over $\mathbb{R}$) and hence the highest weight representations are one-dimensional. In fact, an element

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & d & b & 0 \\ 0 & \frac{b}{D} & d & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix}$$

of $L_P(\mathbb{R})$ acts on $H^1(N_P, \mathbb{R})$ with eigenvalues

$$\beta_2 - \beta_1 = \frac{d + \frac{b}{\sqrt{D}}}{\alpha}, \text{ and } -\beta_2 - \beta_1 = \frac{d - \frac{b}{\sqrt{D}}}{\alpha}$$

for some $d, \ b \in \mathbb{R}$. Since these eigenvalues are real, the proposition is proved. ∎

Remark 4.5. Note that the eigenvalues of the action of the generator of the monodromy group for any link bundle are never equal to 1. This follows from the proof of the proposition above.

Now we can prove the main theorem.

Main Theorem. *Let $\bar{X}$ be the reductive Borel-Serre compactification of the (non-compact) Hilbert modular surface X associated to a real quadratic field K and the arithmetic group Γ as above. Then the category $SD(\bar{X})$ of self-dual sheaves compatible with intersection chain sheaves is nonempty. In particular, there exists an object of $D^b(\bar{X})$ which is Verdier self-dual and which satisfies the intersection chain sheaves type axioms.*

Proof. We continue to use the notation introduced in section 3.2. Using Theorem (2.10) of [1], it follows that the theorem follows if the category $Lag(U_4 - U_3)$ of Lagrangian structures on $U_4 - U_3$ is nonempty. But this set is precisely the set of boundary strata X_Q, one for each Γ-conjugacy class of parabolic subgroups Q. By Prop. 4.4, the category $Lag(X_Q)$ is nonempty. In particular, $Lag(U_4 - U_3)$ is nonempty. This proves the theorem. □

5 Relation to intersection homology of the Baily-Borel compactification

Let $\hat{X}$ denote the Baily-Borel compactification of the Hilbert modular surface X. Topologically, it is obtained from the reductive Borel-Serre compactification $\bar{X}$ by collapsing each component of the one-dimensional stratum $\bar{X}_1$ to a point. Thus if $\mathcal{P}$ denotes the set of Γ-conjugacy classes of parabolic subgroups, then $\hat{X}$ is a two-strata space

$$\hat{X}^4 = \hat{X}_4 \supset \hat{X}_0,$$

where $\hat{X}_0$ contains (in addition to the singularities of X) , a point x_Q, one for each $Q \in \mathcal{P}$ which is the image of $X_Q \subset \bar{X}_1$ ($X_Q \cong S^1$) under the collapse map

$$\pi : \bar{X} \longrightarrow \hat{X}.$$

Therefore π is a stratified map (in the sense of Whitney stratified spaces). We shall investigate the functor $R\pi_* : SD(\bar{X}) \to D^b(\hat{X})$, that is, we investigate the pushforward $R\pi_* \mathbf{IC}^\bullet_{\mathcal{L}}$, where $\mathbf{IC}^\bullet_{\mathcal{L}}$ is any self-dual sheaf in $SD(\bar{X})$ determined by a Lagrangian structure $\mathcal{L}$ along $\bar{X}_1$. We will see that $R\pi_*$ restricts to a functor $SD(\bar{X}) \to SD(\hat{X})$. The latter category however contains (up to isomorphism) only one object $\mathbf{IC}^\bullet_{\bar{m}}(\hat{X}) \cong \mathbf{IC}^\bullet_{\bar{n}}(\hat{X})$, as $\hat{X}$ has only strata of even codimension. Moreover, we will independently verify Saper's theorem [18] (Rapoport's conjecture [15]) for the Hilbert modular surface case. Saper's result asserts that for X an arithmetic quotient of a Hermitian symmetric space,

$$R\pi_* \mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \cong \mathbf{IC}^\bullet_{\bar{m}}(\hat{X}) \cong R\pi_* \mathbf{IC}^\bullet_{\bar{n}}(\bar{X})$$

in $D^b(\hat{X})$ (for certain specific cases, the conjecture was proved by Zucker ($G = \mathrm{Sp}(4)$), Goresky-MacPherson ($G = \mathrm{Sp}(k), k = 4, 6, 8$) ([10]), Saper-Stern ($\mathbb{Q}$-rank $G = 1$)) ([17]).

Theorem 5.1. *If* $\mathbf{IC}^\bullet_{\mathcal{L}} \in SD(\bar{X})$ *and*

$$\mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \xrightarrow{\alpha} \mathbf{IC}^\bullet_{\mathcal{L}} \xrightarrow{\beta} \mathbf{IC}^\bullet_{\bar{n}}(\bar{X})$$

are the canonical morphisms (cf. section 2), then

(i) $R\pi_*\alpha$ *and* $R\pi_*\beta$ *are isomorphisms, and*

(ii) $R\pi_* \mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \cong \mathbf{IC}^\bullet_{\bar{m}}(\hat{X})$. *In particular, we have* $R\pi_* \mathbf{IC}^\bullet_{\mathcal{L}} \cong \mathbf{IC}^\bullet_{\bar{m}}(\hat{X})$ *and* $R\pi_* \mathbf{IC}^\bullet_{\bar{n}}(\bar{X}) \cong \mathbf{IC}^\bullet_{\bar{m}}(\hat{X})$.

Proof. To prove *(i)*, we shall focus on α. The statement for β will then follow from duality. We start out with the distinguished triangle

$$\begin{tikzcd} \mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \arrow[rr, "\alpha"] & & \mathbf{IC}^\bullet_{\mathcal{L}} \arrow[dl] \\ & \mathcal{L} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

see [1]. Its pushforward is

$$\begin{tikzcd} R\pi_*\mathbf{IC}^\bullet_{\bar{m}}(\bar{X}) \arrow[rr, "R\pi_*\alpha"] & & R\pi_*\mathbf{IC}^\bullet_{\mathcal{L}} \arrow[dl] \\ & R\pi_*\mathcal{L} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

and we have to show that $R\pi_*\mathcal{L} \cong \mathbf{0}$, the zero complex. Over $\hat{X} - \{x_Q | Q \in \mathcal{P}\}$, π is the identity map and $\mathcal{L}$ is supported over $\coprod_{Q\in\mathcal{P}} X_Q$, the preimages of $\{x_Q\}$. Hence it suffices to check the vanishing of the cohomology stalks of $R\pi_*\mathcal{L}$ over the points x_Q. There exists an open neighborhood $U \subset \hat{X}$ of x_Q such that $\pi^{-1}(U) \subset \bar{X}$ is an open tube-neighborhood of X_Q and

$$\mathbf{H}^i(R\pi_*\mathcal{L})_{x_Q} \cong \mathcal{H}^i(U; R\pi_*\mathcal{L}) \cong \mathcal{H}^i(\pi^{-1}(U); \mathcal{L}).$$

Consider the closed inclusion $j : X_Q \hookrightarrow \pi^{-1}(U)$. Since $j_!$ is extension by zero and $\mathcal{L}|\pi^{-1}(U)$ is supported over X_Q, we have $\mathcal{L}|\pi^{-1}(U) \cong j_! j^*\mathcal{L}|\pi^{-1}(U)$. As j is a closed inclusion, $j_! = j_*$. Therefore,

$$\mathcal{H}^i(\pi^{-1}(U); \mathcal{L}) \cong \mathcal{H}^i(\pi^{-1}(U); j_* j^*\mathcal{L}) \cong \mathcal{H}^i(X_Q; j^*\mathcal{L}) \cong \mathcal{H}^i(S^1; j^*\mathcal{L}).$$

The cohomology of the circle with local coefficient system $j^*\mathcal{L}$ is computed from a cochain complex which has nonzero groups of rank one in precisely two adjacent dimensions and only one nontrivial coboundary operator $m-1$, where $m \in \mathbb{R}$ is the monodromy of $j^*\mathcal{L}$. Now $\mathcal{L}$ is a subsheaf of the sheaf on S^1 determined by the monodromy $\varphi^* : H^1(T^2) \to H^1(T^2)$ from section 3, where we also saw that φ^* does not have 1 among its eigenvalues (Remark 4.5). Thus $m \neq 1$, the cochain complex is acyclic, and $\mathcal{H}^i(S^1; j^*\mathcal{L}) = 0$ for all i.

We prove statement *(ii)* by demonstrating that $R\pi_*\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})$ satisfies the Goresky-MacPherson axiomatics for the middle perversity intersection chain sheaf on $\hat{X}$. The statement being clear on the top stratum, it is again enough to concentrate on the singularities $\{x_Q\}$. Let U be as before and put $T = \pi^{-1}(U)$. We observe that since $\bar{m}(3) = 0$ and

$$\mathbf{IC}^\bullet_{\bar{m}}(T) \cong \tau_{\leq \bar{m}(3)-4} Ri_*\mathbb{R}_{T-X_Q}[4]$$

we have $\mathbf{IC}^\bullet_{\bar{m}}(T) \cong \mathbf{IC}^\bullet_{\bar{0}}(T)$, the zero-perversity intersection chain complex. Furthermore,

$$\mathbf{IC}^\bullet_{\bar{0}}(T) \cong \mathbb{R}_T[4],$$

since T is a normal pseudomanifold. We obtain for the stalk-cohomology:

$$\begin{aligned}
\mathbf{H}^i(R\pi_*\mathbf{IC}^\bullet_{\bar{m}}(\bar{X}))_{x_Q} &\cong \mathcal{H}^i(U; R\pi_*\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})) \\
&\cong \mathcal{H}^i(T; \mathbf{IC}^\bullet_{\bar{m}}(T)) \\
&\cong \mathcal{H}^i(T; \mathbb{R}_T[4]) \\
&\cong H^{i+4}(T) \\
&\cong H^{i+4}(S^1)
\end{aligned}$$

(T is homotopy equivalent to S^1). The last group vanishes when $i > -3$. The stalk condition for the middle perversity is thus satisfied. The costalk condition may be checked using duality as follows: If $j : \{x_Q\} \hookrightarrow \hat{X}$ is the inclusion, then

$$\begin{aligned}\mathbf{H}^i(j^! R\pi_* \mathbf{IC}^\bullet_{\bar{m}}(\bar{X})) &\cong H^i(\mathcal{D}_{\{x_Q\}} j^* R\pi_! \mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(\bar{X})) \\ &\cong H^{i+4}(\mathcal{D}_{\{x_Q\}} j^* R\pi_* \mathbf{IC}^\bullet_{\bar{n}}(\bar{X})) \\ &\cong H^{i+4}(\mathcal{D}_{\{x_Q\}} j^* R\pi_* \mathbf{IC}^\bullet_{\bar{m}}(\bar{X})) \\ &\cong \mathrm{Hom}(\mathbf{H}^{-i-4}(R\pi_* \mathbf{IC}^\bullet_{\bar{m}}(\bar{X}))_{x_Q}, \mathbb{R})\end{aligned}$$

which vanishes by the stalk condition when $i \leq -2$. ☐

Corollary 5.2. *The direct image $R\pi_*$ defines a functor*

$$R\pi_* : SD(\bar{X}) \longrightarrow SD(\hat{X}).$$

Corollary 5.3. *Let $L_i(\mathbf{IC}^\bullet_{\mathcal{L}}) \in H_i(\bar{X}; \mathbb{Q})$ be the i-th L-class of the self-dual sheaf $\mathbf{IC}^\bullet_{\mathcal{L}} \in SD(\bar{X})$, let $L_i(\hat{X}) \in H_i(\hat{X}; \mathbb{Q})$ be the Goresky-MacPherson L-class of the Baily-Borel compactification and let $\pi_* : H_i(\bar{X}) \to H_i(\hat{X})$ be the map induced by π on homology. Then*

$$\pi_* L_i(\mathbf{IC}^\bullet_{\mathcal{L}}) = L_i(\hat{X}).$$

Proof. On the one hand $\pi_* L_i(\mathbf{IC}^\bullet_{\mathcal{L}}) = L_i(R\pi_* \mathbf{IC}^\bullet_{\mathcal{L}})$ (see [6]), and on the other hand $R\pi_* \mathbf{IC}^\bullet_{\mathcal{L}} \cong \mathbf{IC}^\bullet_{\bar{m}}(\hat{X})$ by theorem 5.1. ☐

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