

STRONG INFLUENCE OF A SMALL FIBER ON SHEAR STRESS IN FIBER-REINFORCED COMPOSITES

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Abstract. In stiff fiber-reinforced materials, the high shear stress concentration occurs in the narrow region between fibers. Much attention has been devoted to the blow-up estimate for stress associated with the distance between neighboring fibers. As an optimal result, it has been known that the stress increases at the rate of $1/\sqrt{\epsilon}$ as the distance ϵ between adjacent fibers approaches 0. This paper reveals a strong influence of small geometric changes, as well as the distance, on the blow-up so that the growth of stress is significantly accelerated by adding a small fiber in-between fibers. Specifically, we establish an explicit dependency of the stress with respect to a geometric change: if a fiber F_2 of small diameter d is located in-between fibers F_1 and F_3 , and ϵ_i is the distance between F_2 and F_i for $i = 1, 3$, then the blow-up rate is exactly $1/\sqrt{d\epsilon_i}$ in the narrow region between F_2 and F_i . Moreover, the same estimate also holds between F_1 and F_2 even when a part of F_2 overlaps with F_3 . The magnification factor $1/\sqrt{d}$ in our estimates yields the dramatic increase in stress caused by a small geometric change.

Key words. high shear stress, gradient estimates, blow-up, stiff fiber-reinforced material, perfect conductor, small fiber in-between fibers.

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1. Introduction. Stiff fiber-reinforced composites have very low strength in longitudinal shear and this has been explained by high stress concentrations in the narrow regions between neighboring fibers [6]. Much attention has been focused on the estimate for high stress concentration, especially associated with the distance between adjoining fibers, refer to [3, 4, 9, 14, 15]. An optimal blow-up rate in terms of the distance has been obtained in a way to employ a conductivity equation inspired by a standard anti-plane shear model, so that the stress in the narrow region increases toward infinity at the rate of $\frac{1}{\sqrt{\epsilon}}$ as the distance ϵ between the fibers approaches 0.

In this paper, we intend to disclose another crucial factor which also plays a very influential role in the blow-up. It is a small geometric change, such as a tin additive fiber, in the narrow region between fibers. The growth of stress is significantly accelerated by adding a small fiber in-between fibers, see Figure 1.1 and 1.2. Specifically, this paper provides an optimal dependency of the stress with respect to the geometric change: if a fiber F_2 of small diameter d is located in-between fibers F_1 and F_3 , and ϵ_i is the distance between F_2 and F_i for $i = 1, 3$, then the blow-up rate is exactly $1/\sqrt{d\epsilon_i}$ in the narrow region between F_2 and F_i . Moreover, the same estimate also holds between F_1 and F_2 even when a part of F_2 overlaps with F_3 . The magnification factor $1/\sqrt{d}$ in these estimates yields the remarkable increase in stress due to a small geometric change. Moreover, according to geometrical features that appear in Figure 1.1 and 1.2, Theorem 1.1, 1.2, 1.3 and 1.4 are presented for *Case A*, *B*, *C* and *D*, respectively.

In a standard anti-plane shear model, the out-of-plane elastic displacement u satisfies the two dimensional conductivity equation: for a given harmonic function H

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in \mathbb{R}^2 ,

$$\begin{cases} \nabla \cdot \left\{ \left(1 + \sum_{l=1}^L (a_l - 1) \chi(D_l) \right) \nabla u \right\} = 0 \text{ in } \mathbb{R}^2 \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \rightarrow \infty \end{cases}$$

supposed that the constant conductivity a_l is different from 1. Here, the stress tensor and the cross-section of stiff fibers correspond to the electric field ∇u and the embedded conductors D_l for $l = 1, \dots, L$, respectively. Consequently, the primary objective is to estimate the gradient of the solution ∇u to the conductivity equation.

There have been numerous investigations of the blow-up phenomena associated with the distance between fibers. The earlier works pointed out that the extreme conductivities (or shear moduli) a_i are indispensable to the blow-up phenomena. When inclusions' conductivities are away from zero and infinity, it has been shown by Li and Vogelius [12] that the stress is bounded even though the distance is arbitrary small, see also [9], and it was generalized to elliptic systems by Li and Nirenberg [11]. In [3, 4], for general conductivities including extreme values, Ammari et al. have established the optimal bounds of the gradient of solutions to the two dimensional conductivity equation, under the assumption that a pair of inclusions are of circular shape. Specifically, these bounds have yielded that ∇u generally becomes unbounded as ϵ tends to zero and the blow-up rate is $1/\sqrt{\epsilon}$, where ϵ is the distance between two conductors. Yun [14, 15] has extended this blow-up result for the case of two adjacent perfect conductors of a sufficiently general shape in two dimensions. In Bao, Li and Yin's paper [7], it has been also investigated the blow-up rate in higher dimensional spaces, also see [2, 13]. They have also done a natural follow-up in [5, 8] that the blow-up rate known for a pair of fibers is also valid for the multiple inclusions in any dimensions. Accordingly, the explicit dependency of the blow-up with respect to the distance ϵ has been successfully obtained in the previous works.

In this paper, another factor, as well as the distance, strongly affecting the blow-up is presented. As has been mentioned above, a small geometrical change in the narrow region between inclusions can accelerate significantly the growth of the gradient and the explicit dependency of the gradient with respect to the geometric change is further established. To understand the effect of a geometrical change in the blow-up rate, we consider four cases where a small inclusion exists between a pair of inclusions: In the *Case A* and *C*, the small inclusion overlaps with the right one, and in the *Case B* and *D*, three inclusions are assumed to be disjoint. In particular, all inclusions of the *Case A* and *B* have circular shapes, so that the estimates in terms of the radii are provided.



FIG. 1.1. *Case A and Case B*



FIG. 1.2. *Case C and Case D*

Now we introduce the governing equation with the infinity conductivity. Let L be the number of conductors and for $l = 1, \dots, L$, D_l is assumed to be disjoint conducting inclusions in \mathbb{R}^2 . Then, under the action of the applied field H , the electric potential u satisfies the following conductivity equation:

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\cup_{l=1}^L D_l}, \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \text{as } |\mathbf{x}| \rightarrow \infty, \\ u|_{\partial D_l} = C_l \text{ (constant)}, & \text{for } l = 1, \dots, L, \\ \int_{\partial D_l} \partial_\nu u \, dS = 0, & \text{for } l = 1, \dots, L, \end{cases} \quad (1.1)$$

where $\mathbf{x} = (x_1, x_2)$, and H is an entire harmonic function in \mathbb{R}^2 , and ν is the outward unit normal vector of $\mathbb{R}^2 \setminus \overline{\cup_{l=1}^L D_l}$, i.e., directed inward of D_l . In this paper, we consider only the cases of $L = 2, 3$.

1.1. Case A and B: Circular inclusions. The detail description of *Case A* and *B* related to circular inclusions is given as follows, also see Figure 1.1:

- (A) One disk and a pair of partially overlapping disks: there is a portion of disk protruding from one of circular inclusions, i.e., $L = 2$, and D_1 and D_2 are ϵ -distanced domains defined as

$$D_1 = B_{r_1}(\mathbf{c}_1), \text{ and } D_2 = B_{r_2}(\mathbf{c}_2) \cup B_{r_3}(\mathbf{c}_3), \quad (1.2)$$

where $\mathbf{c}_1 = (-r_1 - \frac{\epsilon}{2}, 0)$, $\mathbf{c}_2 = (r_2 + \frac{\epsilon}{2}, 0)$, and $\mathbf{c}_3 = (r_3 + a + \frac{\epsilon}{2}, 0)$. Here, $B_{r_l}(\mathbf{c}_l)$ is the disk centered at \mathbf{c}_l with the radius r_l . We assume $B_{r_2}(\mathbf{c}_2) \cap B_{r_3}(\mathbf{c}_3) \neq \emptyset$, i.e., $0 < a < 2r_2$,

$$\text{dist}(D_1, B_{r_3}(\mathbf{c}_3)) \simeq r_2 \text{ and } 0 < \epsilon \ll r_2 \ll r_1 \simeq r_3.$$

- (B) Three disjoint disks: a small disk is disjointly embedded into the in-between area of two disks, i.e., $L = 3$, and

$$D_l = B_{r_l}(\mathbf{c}_l), \quad l = 1, 2, 3, \quad (1.3)$$

where $\mathbf{c}_1 = (-r_1 - \frac{\epsilon_1}{2}, 0)$, $\mathbf{c}_2 = (r_2 + \frac{\epsilon_1}{2}, 0)$ and $\mathbf{c}_3 = (r_3 + 2r_2 + \frac{\epsilon_1}{2} + \epsilon_2, 0)$. The distance between D_1 and D_2 is ϵ_1 , and the distance between D_2 and D_3 is ϵ_2 . We assume that

$$0 < \epsilon_i \ll r_2 \ll r_1 \simeq r_3, \quad \text{for } i = 1, 2.$$

In both cases, three disk centers are lined up in one straight line. The optimal blow-up rates remarkably increased due to the additional small fiber are given by the following theorems.

THEOREM 1.1 (Case A). *Let D_1 and D_2 be defined as (1.2). Then*

- (a) *Optimal lower bound: There is a positive constant C independent of ϵ , r_1 , r_2 and r_3 such that*

$$u|_{\partial D_2} - u|_{\partial D_1} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon},$$

where u is the solution to (1.1) with $H(x_1, x_2) = x_1$. As a result, by the Mean Value Theorem, there is a point \mathbf{x}_0 in the narrow region between D_1 and D_2 such that

$$|\nabla u(\mathbf{x}_0)| \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}.$$

(b) Optimal upper bound: For any entire harmonic function H , let u be the solution to (1.1) with H . Then, there is a positive constant C independent of ϵ , r_1 , r_2 and r_3 such that

$$|\nabla u(\mathbf{x})| \leq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}$$

for \mathbf{x} in the narrow region between D_1 and D_2 .

THEOREM 1.2 (Case B). Let D_i , $i = 1, 2, 3$, be the balls defined as (1.3). Then

(a) Optimal lower bound: There is a positive constant C independent of ϵ_1 , ϵ_2 , r_1 , r_2 and r_3 such that

$$u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1},$$

and

$$u \Big|_{\partial D_3} - u \Big|_{\partial D_2} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_2},$$

where u is the solution to (1.1) with $H(x_1, x_2) = x_1$. As a result, by the Mean Value Theorem, there exists points; \mathbf{x}_1 in the narrow region between D_1 and D_2 ; \mathbf{x}_2 in the narrow region between D_2 and D_3 , which satisfy that

$$|\nabla u(\mathbf{x}_i)| \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}} \quad \text{for } i = 1, 2.$$

(b) Optimal upper bound: For any entire harmonic function H , let u be the solution to (1.1) with H . Then, there is a positive constant C independent of ϵ_1 , ϵ_2 , r_1 , r_2 and r_3 such that

$$|\nabla u(\mathbf{x})| \leq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}} \quad \text{for } i = 1, 2.$$

for \mathbf{x} in the narrow regions between D_1 and D_2 , and between D_2 and D_3 , respectively.

1.2. Case C and D: More generalized shaped inclusions. Case C and D are designed for general-shaped conductors. The estimates presented in Theorem 1.1 and 1.2 can be extended to hold for these cases. We consider three domains D_{left} , D_{center} and D_{right} in \mathbb{R}^2 whose complementary sets are the image of complementary sets of the unit disk under conformal mappings φ_{left} , φ_{center} , and φ_{right} . More precisely, $\varphi_{\text{center}} : \mathbb{C} \setminus B_1(0) \rightarrow \mathbb{R}^2 \setminus D_{\text{center}}$ is a conformal mapping such that

$$\varphi_{\text{center}} \in C^2(\mathbb{C} \setminus B_1(0)).$$

Here, we do not distinguish \mathbb{R}^2 from \mathbb{C} . Likewise, φ_{left} , and φ_{right} are C^2 -conformal mappings. Moreover, we assume

$$\varphi'_{\text{left}}(z) \neq 0 \quad \text{and} \quad \varphi'_{\text{right}}(z) \neq 0, \quad \text{for } z \in \partial B_1(0).$$

The C^2 -regularity condition of these conformal mappings does not allow non-smooth inclusions such as polygons, however Riemann mapping theorem yields a sufficiently general class of shapes, refer to Ahlfors [1]. We formulate Case C and D analogously to A and B as follows, see Figure 1.2.

- (C) One domain and a pair of partially overlapping domains: there is a portion of a small conductor protruding from one of two inclusions, i.e., $L = 2$, and D_1 and D_2 are ϵ -distanced domains defined as

$$D_1 = D_{\text{left}} \text{ and } D_2 = (r_2 D_{\text{center}}) \cup D_{\text{right}}, \quad (1.4)$$

where $r_2 D_{\text{center}}$ is the r_2 -scaled domain of D_{center} . We assume that

$$D_1 \subset \mathbb{R}_- \times \mathbb{R} \text{ and } D_2 \subset \mathbb{R}_+ \times \mathbb{R},$$

and D_2 is a connected domain such that

$$\text{dist}(D_1, D_2) = \text{dist}(D_1, r_2 D_{\text{center}}), \quad \text{dist}(D_1, D_{\text{right}}) \simeq r_2,$$

In addition, we also assume that r_2 is small enough, and

$$0 < \epsilon \ll r_2.$$

Moreover, the boundaries ∂D_1 and ∂D_2 are strictly convex in the narrow region between D_1 and D_2 , and $\partial D_{\text{right}}$ is also strictly convex in the small region around D_2 .

- (D) Three disjoint domains: a small inclusion D_2 is disjointly embedded into the in-between area of two other domains, i.e., $L = 3$, and

$$D_1 = D_{\text{left}}, \quad D_2 = r_2 D_{\text{center}} \text{ and } D_3 = D_{\text{right}} \quad (1.5)$$

where $r_2 D_{\text{center}}$ is the r_2 -scaled domain of D_{center} , and r_2 is sufficiently small. We assume that

$$D_1 \subset \mathbb{R}_- \times \mathbb{R} \text{ and } D_2 \cup D_3 \subset \mathbb{R}_+ \times \mathbb{R},$$

and D_1 and D_2 are ϵ_1 apart, D_2 and D_3 are ϵ_2 apart, and

$$0 < \epsilon_i \ll r_2 \text{ for } i = 1, 2,$$

Moreover, ∂D_1 and ∂D_2 are strictly convex in the narrow region between D_1 and D_2 , and ∂D_3 is also strictly convex in the narrow region between D_2 and D_3 .

THEOREM 1.3 (Case C). *Let D_1 and D_2 be defined as (1.4). Then*

- (a) *Optimal lower bound: There is a positive constant C independent of ϵ and r_2 such that*

$$u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon},$$

where u is the solution to (1.1) with $H(x_1, x_2) = x_1$. As a result, by the Mean Value Theorem, there is a point \mathbf{x}_0 in the narrow region between D_1 and D_2 such that

$$|\nabla u(\mathbf{x}_0)| \geq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}.$$

- (b) Optimal upper bound: For any entire harmonic function H , let u be the solution to (1.1) with H . Then, there is a positive constant C independent of r_2 and ϵ such that

$$|\nabla u(\mathbf{x})| \leq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon}}$$

for \mathbf{x} in the narrow region between D_1 and D_2 .

THEOREM 1.4 (Case D). Let D_i , $i = 1, 2, 3$, be defined as (1.5). Then

- (a) Optimal lower bound: There is a positive constant C independent of r_2 , ϵ_1 and ϵ_2 such that

$$u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1},$$

and

$$u \Big|_{\partial D_3} - u \Big|_{\partial D_2} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_2},$$

where u is the solution to (1.1) with $H(x_1, x_2) = x_1$. As a result, by the Mean Value Theorem, there exist the points; \mathbf{x}_1 in the narrow region between D_1 and D_2 ; \mathbf{x}_2 in the narrow region between D_2 and D_3 , which satisfy that

$$|\nabla u(\mathbf{x}_i)| \geq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}}, \quad \text{for } i = 1, 2.$$

- (b) Optimal upper bound: For any entire harmonic function H , let u be the solution to (1.1) with H . Then, there is a positive constant C independent of r_2 , ϵ_1 and ϵ_2 such that

$$|\nabla u(\mathbf{x}_i)| \leq C \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_i}}, \quad \text{for } i = 1, 2.$$

for \mathbf{x} in the narrow regions between D_1 and D_2 , and D_2 and D_3 , respectively.

Figure 1.3 explains the growth of the gradient increased by a small geometric change. The gradient can be estimated by the potential difference between conductors. According to the previous gradient estimates [14, 15], the rate of the difference is $\sqrt{\epsilon}$ when the distance is ϵ . It follows from the Mean Value Theorem that the gradient increases at the rate $1/\sqrt{\epsilon}$. This means that the difference decreases slower than the distance ϵ , as ϵ approaches zero. In our case when a small conductor is added between them, Theorem 1.3 and 1.4 witness that the difference decreases much slower due to the multiplication factor $1/\sqrt{r_2}$ while the distance is reduced. For example, in the right Figure 1.3, the number of equipotential lines in the narrow region are preserved whereas the distance between them is reduced due to a protruding small disk. This increases the gradient concentration more.

This paper is organized as follows: In the Section 2, we explain the method to calculate the difference of the electric potential value in conductors. Based on this, we then derive the optimal lower bound in the *Case A* and *C* in the Section 3, and the *Case B* and *D* in the Section 4 to support the blow-up. The optimal upper bounds of the gradients are derived in the Section 4.5.



FIG. 1.3. Equipotential lines under the action of an electric potential $H = x_1$ on the boundary. The left figure is the case of two circular inclusions, and the right figure is the case when there are protruding small disks on one of two inclusions.

2. Preliminary. We explain the main idea to calculate the difference of potential between two adjacent conductors.

2.1. Calculation of the potential difference. We begin by introducing some notations to consider the difference of potential.

DEFINITION 2.1. Let w be the solution to (1.1) where differently from (1.1), D_i , $i = 1, 2$, is possibly the union of two disjoint domains in the definition, it is assigned one constant value throughout D_i even when D_i is disconnected, and h is assumed to be the solution to

$$\begin{cases} \Delta h = 0, & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ h = O(|\mathbf{x}|^{-1}), & \text{as } |\mathbf{x}| \rightarrow \infty, \\ h|_{\partial D_i} = k_i \text{ (constant)}, & \text{for } i = 1, 2, \\ \int_{\partial D_i} \partial_\nu h \, dS = (-1)^i, & \text{for } i = 1, 2, \end{cases} \quad (2.1)$$

where ν is the outward unit normal vector of $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$, directed inward of D_i . To indicate the dependence of w and h on D_1 and D_2 , we denote them as

$$w = \Phi[D_1, D_2], \quad (2.2)$$

$$h = \Psi[D_1, D_2]. \quad (2.3)$$

LEMMA 2.2 ([14]). Consider two conductors D_1 and D_2 embedded in \mathbb{R}^2 , i.e., $L = 2$. Then the potential difference of the solution u to (1.1) is represented in terms of $h = \Psi[D_1, D_2]$ as follows:

$$u|_{\partial D_2} - u|_{\partial D_1} = \int_{\partial D_1} H \partial_\nu h \, dS + \int_{\partial D_2} H \partial_\nu h \, dS. \quad (2.4)$$

The lemma above can be derived by the divergence theorem, see [14].

LEMMA 2.3. For three conductors D_i , $i = 1, 2, 3$, embedded in \mathbb{R}^2 , i.e., $L = 3$, the solution u to (1.1) satisfies

$$\begin{aligned} \int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 \, dS &= \left(1 - \int_{\partial D_3} \partial_\nu h_1 \, dS\right) (u|_{\partial D_2} - u|_{\partial D_1}) \\ &\quad + \left(\int_{\partial D_3} \partial_\nu h_1 \, dS\right) (u|_{\partial D_3} - u|_{\partial D_1}), \end{aligned}$$

where $h_1 = \Psi[D_1, (D_2 \cup D_3)]$.

Proof. Applying the (exterior) divergence theorem, we have

$$\int_{\cup_{i=1}^3 \partial D_i} (u - H) \partial_\nu h_1 \, dS = 0.$$

Since u is constant on each ∂D_i , $\int_{\partial D_1} \partial_\nu h_1 dS = -1$, and $\int_{\partial D_2 \cup \partial D_3} \partial_\nu h_1 dS = 1$, we prove the lemma. \square

We rely on the monotonic property of Ψ to derive the optimal bounds of ∇u of the solution to (1.1).

LEMMA 2.4 (Monotonicity of Ψ). *Let D_1 , D_2 , \tilde{D}_1 and \tilde{D}_2 be conductors, which are possibly the union of two domains, satisfying that*

$$D_1 \subseteq \tilde{D}_1 \text{ and } D_2 \subseteq \tilde{D}_2.$$

Let

$$M = \frac{\Psi[\tilde{D}_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_2} - \Psi[\tilde{D}_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_1}}{\Psi[D_1, D_2] \Big|_{\partial D_2} - \Psi[D_1, D_2] \Big|_{\partial D_1}},$$

then we have

$$0 < M \leq 1. \quad (2.5)$$

Moreover, when $D_1 = \tilde{D}_1$,

$$0 < \partial_\nu \Psi[D_1, \tilde{D}_2] \leq M \partial_\nu \Psi[D_1, D_2], \quad \text{on } \partial D_2 \cap \partial \tilde{D}_2, \quad (2.6)$$

and, when $D_2 = \tilde{D}_2$,

$$0 > \partial_\nu \Psi[\tilde{D}_1, D_2] \geq M \partial_\nu \Psi[D_1, D_2], \quad \text{on } \partial D_1 \cap \partial \tilde{D}_1. \quad (2.7)$$

Proof. For $\mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup \tilde{D}_2)$, define

$$G_1(\mathbf{x}) = \left(\Psi[D_1, \tilde{D}_2](\mathbf{x}) - \Psi[D_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_2} \right) - M_1 \left(\Psi[D_1, D_2](\mathbf{x}) - \Psi[D_1, D_2] \Big|_{\partial D_2} \right),$$

where

$$M_1 = \frac{\Psi[D_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_2} - \Psi[D_1, \tilde{D}_2] \Big|_{\partial D_1}}{\Psi[D_1, D_2] \Big|_{\partial D_2} - \Psi[D_1, D_2] \Big|_{\partial D_1}}.$$

Since $G_1 \Big|_{\partial \tilde{D}_2} \geq G_1 \Big|_{\partial D_1} = 0$, by the Hopf's lemma,

$$\partial_\nu G_1 \leq 0 \text{ on } \partial D_1.$$

By integrating $\partial_\nu G_1$ on ∂D_1 , we have $M_1 \leq 1$. Similarly,

$$M_2 = \frac{\Psi[\tilde{D}_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_2} - \Psi[\tilde{D}_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_1}}{\Psi[D_1, \tilde{D}_2] \Big|_{\partial \tilde{D}_2} - \Psi[D_1, \tilde{D}_2] \Big|_{\partial D_1}} \leq 1.$$

This proves (2.5).

From definition, $G_1 \Big|_{\partial D_2 \cap \partial \tilde{D}_2} = 0$ and $G_1 \geq 0$ in $\mathbb{R}^2 \setminus (D_1 \cup \tilde{D}_2)$. Hence, by the Hopf's lemma,

$$\partial_\nu G_1 \leq 0 \text{ on } \partial D_2 \cap \partial \tilde{D}_2.$$

This means (2.6). Similarly, we also have (2.7). \square

2.2. Two disks in \mathbb{R}^2 . Using Lemma 2.2, we can easily calculate the potential difference $u|_{D_2} - u|_{D_1}$ of the solution u to (1.1) when

$$D_1 = B_{r_1}(\mathbf{c}_1) \text{ and } D_2 = B_{r_2}(\mathbf{c}_2), \quad (2.8)$$

where $\mathbf{c}_1 = (-r_1 - \frac{\epsilon}{2}, 0)$ and $\mathbf{c}_2 = (r_2 + \frac{\epsilon}{2}, 0)$.

Let R_i be the reflection with respect to D_i , in other words,

$$R_i(\mathbf{x}) = \frac{r_i^2(\mathbf{x} - \mathbf{c}_i)}{|\mathbf{x} - \mathbf{c}_i|^2} + \mathbf{c}_i, \quad i = 1, 2,$$

and $\mathbf{p}_1 \in D_1$ be the fixed point of $R_1 \circ R_2$, then $R_2(\mathbf{p}_1)(=: \mathbf{p}_2)$ is the fixed point of $R_2 \circ R_1$, and

$$\mathbf{p}_1 = \left(-\sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon), 0 \right) \text{ and } \mathbf{p}_2 = \left(\sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon), 0 \right).$$

Moreover, we can easily show that

$$\Psi[D_1, D_2](\mathbf{x}) = \frac{1}{2\pi} (\log|\mathbf{x} - \mathbf{p}_1| - \log|\mathbf{x} - \mathbf{p}_2|). \quad (2.9)$$

By an elementary calculation, it can be shown that the middle point $\frac{\mathbf{p}_1 + \mathbf{p}_2}{2}$ exists between two approaching points $(-\frac{\epsilon}{2}, 0)$ and $(\frac{\epsilon}{2}, 0)$. Applying the middle point property to estimate for $\Psi[D_1, D_2](\pm\frac{\epsilon}{2}, 0)$, we can get the following lemma.

LEMMA 2.5. *There is a constant $C > 0$ independent of ϵ , r_1 and r_2 such that*

$$\begin{aligned} \frac{1}{C} \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon} &\leq \Psi[D_1, D_2]|_{\partial D_2} - \Psi[D_1, D_2]|_{\partial D_1} \\ &\leq C \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon}, \end{aligned} \quad (2.10)$$

for small $\epsilon > 0$. From Lemma 2.2, we calculate the potential difference of u .

LEMMA 2.6. *Let $H(x_1, x_2)$ be an entire harmonic function. The solution u to (1.1) where $L = 2$ and D_l , $l = 1, 2$, are given as (2.8) satisfies*

$$\begin{aligned} u|_{\partial D_2} - u|_{\partial D_1} &= H(\mathbf{p}_2) - H(-\mathbf{p}_1) \\ &= 2\sqrt{2} \partial_{x_1} H(0, 0) \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon). \end{aligned} \quad (2.11)$$

REMARK 2.7. *Referring to the mean value theorem, there exists a point \mathbf{x}_2 between ∂D_1 and ∂D_2 such that*

$$|\nabla u(\mathbf{x}_2)| \geq 2\sqrt{2} |\partial_{x_1} H(0, 0)| \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \frac{1}{\sqrt{\epsilon}}. \quad (2.12)$$

for sufficiently small $\epsilon > 0$. Moreover, as a result in [4], there is a constant C independent of ϵ , r_1 and r_2 such that

$$\|\nabla u\|_{L^\infty(\Omega \setminus (D_1 \cup D_2))} \leq C \|\nabla H\|_{L^\infty(\Omega)} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \frac{1}{\sqrt{\epsilon}}$$

where $\Omega = B_{4(r_1 + r_2)}(0, 0)$.

3. Case A and C : Partially overlapping inclusions. In this section, we prove Theorem 1.1 and 1.3 that share common threads.

For this reason, we first consider *Case A* where all inclusions are of circular shape, i.e., D_1 and D_2 are given as (1.2), and define

$$h = \Psi[D_1, D_2] \text{ and } h_j = \Psi[D_1, B_{r_j}(\mathbf{c}_j)], \quad j = 2, 3, \quad (3.1)$$

where Ψ defined in section 2.1.

3.1. Properties of h and h_j . We derive the properties to prove the theorems.

LEMMA 3.1. *We have*

$$\partial_\nu h(x) = O(\sqrt{\epsilon}), \quad x \in \partial D_2 \setminus B_{r_2}(\mathbf{c}_2). \quad (3.2)$$

Proof. We choose a smooth domain $\tilde{\Omega}$ as follows:

$$\begin{aligned} B_{r_3}(\mathbf{c}_3) &\subset \tilde{\Omega} \subset D_2, \\ \partial\tilde{\Omega} \setminus B_{r_2}(\mathbf{c}_2) &= \partial B_{r_3}(\mathbf{c}_3) \setminus B_{r_2}(\mathbf{c}_2), \\ (\partial\tilde{\Omega} \cap \partial B_{r_2}(\mathbf{c}_2)) \setminus \partial B_{r_3}(\mathbf{c}_3) &= (\epsilon/2, 0) \end{aligned}$$

Then, define

$$\tilde{h} := \Psi[D_1, \tilde{\Omega}].$$

From the monotonicity of Ψ , Lemma 2.4, we have

$$\partial_\nu h \leq \partial_\nu \tilde{h}, \quad \text{on } \partial\tilde{\Omega} \setminus B_{r_2}(\mathbf{c}_2) (= \partial B_{r_3}(\mathbf{c}_3) \setminus B_{r_2}(\mathbf{c}_2)).$$

The domain $\tilde{\Omega}$ is smooth so that we can use the method presented by Yun [14, 15]. Then, up to a conformal mapping to a circle, $\partial_\nu h$ is bounded by constant times the Poisson Kernel with respect to a interior point $\sqrt{\epsilon}$ distanced from the boundary (refer to the inequality (9) in [15]). Note that $\partial B_{r_3}(\mathbf{c}_3) \setminus B_{r_2}(\mathbf{c}_2)$ distances enough from $(\epsilon, 0)$, since $r_2 \gg \epsilon$. Thus, we have

$$\partial_\nu h \leq \partial_\nu \tilde{h}|_{\partial\tilde{\Omega}} \leq C\sqrt{\epsilon}, \quad \text{on } \partial B_{r_3}(\mathbf{c}_3) \setminus B_{r_2}(\mathbf{c}_2).$$

Therefore, we have completed the proof of the lemma. \square

The monotonicity of Ψ in Lemma 2.4 also allows the following lemma.

LEMMA 3.2.

$$\partial_\nu h(x) \leq M \partial_\nu h_3(x), \quad x \in \partial D_1, \quad (3.3)$$

where

$$M = \frac{h|_{\partial D_2} - h|_{\partial D_1}}{h_3|_{\partial B_{r_3}(\mathbf{c}_3)} - h_3|_{\partial D_1}}. \quad (3.4)$$

LEMMA 3.3.

$$h|_{\partial D_2} - h|_{\partial D_1} = h_2|_{\partial B_{r_2}(\mathbf{c}_2)} - h_2|_{\partial D_1} + O(\epsilon).$$

Proof. Note that

$$\int_{\partial D_1} \partial_\nu(h - h_2) dS = 0,$$

and

$$\begin{aligned} \int_{\partial D_2} \partial_\nu(h - h_2) dS &= \int_{\partial D_2} \partial_\nu h dS - \int_{\partial B_{r_2}(\mathbf{c}_2)} \partial_\nu h_2 dS - \int_{\partial(D_2 \setminus B_{r_2}(\mathbf{c}_2))} \partial_\nu h_2 dS \\ &= 1 - 1 - 0 = 0. \end{aligned}$$

With the fact that $h|_{\partial D_1}$ and $h|_{\partial D_2}$ are constants and the (exterior) Divergence Theorem, we have that

$$\begin{aligned} 0 &= \int_{\partial D_1} \partial_\nu(h - h_2)h dS + \int_{\partial D_2} \partial_\nu(h - h_2)h dS \\ &= \int_{\partial D_1} (h - h_2)\partial_\nu h dS + \int_{\partial D_2} (h - h_2)\partial_\nu h dS. \end{aligned}$$

Hence,

$$\begin{aligned} h|_{\partial D_1} - h|_{\partial D_2} &= \int_{\partial D_1} h\partial_\nu h dS + \int_{\partial D_2} h\partial_\nu h dS \\ &= \int_{\partial D_1} h_2\partial_\nu h dS + \int_{\partial D_2} h_2\partial_\nu h dS \\ &= h_2|_{\partial D_1} - h_2|_{\partial B_{r_2}(\mathbf{c}_2)} + \int_{\partial D_2} (h_2 - h_2|_{\partial B_{r_2}(\mathbf{c}_2)})\partial_\nu h dS. \end{aligned}$$

By (2.9), there is a constant C independent of ϵ such that

$$\left| (h_2 - h_2|_{\partial B_{r_2}(\mathbf{c}_2)})(x) \right| \leq C\sqrt{\epsilon}, \quad \text{for all } x \in \partial D_2 \setminus B_{r_2}(\mathbf{c}_2).$$

Therefore, with (3.2) as well, we prove the lemma. \square

3.2. Proof of Theorem 1.1 (a). We supposed that $H(x_1, x_2) = x_1$. Note that $\partial_\nu h|_{\partial D_2} < 0$, $H < 0$ on ∂D_1 and $H > 0$ on ∂D_2 , and, as a result, from Lemma 2.2, we have

$$\begin{aligned} u|_{\partial D_2} - u|_{\partial D_1} &= \int_{\partial D_1} (\partial_\nu h)H dS + \int_{\partial D_2} (\partial_\nu h)H dS \\ &\geq \int_{\partial D_1} H\partial_\nu h dS. \end{aligned} \tag{3.5}$$

Applying Lemma 3.2, 3.3 and Lemma 2.6, (3.5) becomes

$$\begin{aligned} u|_{\partial D_2} - u|_{\partial D_1} &\geq \frac{h|_{\partial D_2} - h|_{\partial D_1}}{h_3|_{\partial B_{r_3}(\mathbf{c}_3)} - h_3|_{\partial D_1}} \int_{\partial D_1} H\partial_\nu h_3 dS \\ &\geq \frac{h_2|_{\partial B_{r_2}(\mathbf{c}_2)} - h_2|_{\partial D_1} + O(\epsilon)}{h_3|_{\partial B_{r_3}(\mathbf{c}_3)} - h_3|_{\partial D_1}} \sqrt{2} \sqrt{\frac{r_1 r_3}{r_1 + r_3}} r_2. \end{aligned}$$

It follows from Lemma 2.5 that

$$h_2 \Big|_{\partial B_{r_2}(\mathbf{c}_2)} - h_2 \Big|_{\partial D_1} \geq C \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon} + O(\epsilon)$$

and

$$h_3 \Big|_{\partial B_{r_3}(\mathbf{c}_3)} - h_3 \Big|_{\partial D_1} \leq C \sqrt{\frac{r_1 + r_3}{r_1 r_3}} \sqrt{r_2} + O(r_2).$$

Therefore,

$$u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon}.$$

This proves Theorem 1.1 (a). \square

3.3. Proof of Theorem 1.3 (a). We consider the solution u to (1.1) where the conductors are given as the general shaped domains: D_{left} , $r_2 D_{\text{center}}$ and D_{right} instead of $B_{r_i}(\mathbf{c}_i)$. Similarly to Section 3.2, we obtain

$$\begin{aligned} u \Big|_{\partial r_2 D_{\text{center}}} - u \Big|_{\partial D_{\text{left}}} &\geq \int_{\partial D_{\text{left}}} H \partial_\nu h \, dS \\ &\simeq \frac{h_2 \Big|_{\partial(r_2 D_{\text{center}})} - h_2 \Big|_{\partial D_{\text{left}}} + O(\epsilon)}{h_3 \Big|_{\partial D_{\text{right}}} - h_3 \Big|_{\partial D_{\text{left}}}} \sqrt{r_2} \end{aligned}$$

when $H(x_1, x_2) = x_1$. Here, $h_2 = \Psi[D_{\text{left}}, r_2 D_{\text{center}}]$ and $h_3 = \Psi[D_{\text{left}}, D_{\text{right}}]$.

It follows that from Yun [14, 15] that

$$h_3 \Big|_{\partial D_{\text{right}}} - h_3 \Big|_{\partial D_{\text{left}}} \simeq \sqrt{r_2}.$$

To estimate $h_2 \Big|_{\partial r_2 D_{\text{center}}} - h_2 \Big|_{\partial D_{\text{left}}}$, we choose two disks B_{left} and B_{center} containing D_{left} and D_{center} such that the distance between B_{left} and $r_2 B_{\text{center}}$ is ϵ . Using Lemma 2.4 and 2.5, we have

$$h_2 \Big|_{\partial(r_2 D_{\text{center}})} - h_2 \Big|_{\partial D_{\text{left}}} \gtrsim \sqrt{\frac{\epsilon}{r_2}}.$$

Therefore,

$$u \Big|_{\partial r_2 D_{\text{center}}} - u \Big|_{\partial D_{\text{left}}} \geq C \frac{1}{\sqrt{r_2}} \sqrt{\epsilon}.$$

This proves Theorem 1.3 (a). \square

4. Case B and D: Three disjoint inclusions. We prove Theorem 1.2 (a) and 1.4 (a) in this section. Note that D_1 and D_2 are closely spaced with the distance ϵ_1 , and D_2 and D_3 are closely spaced with ϵ_2 , but D_1 and D_3 are not close. Without loss of generality, we assumed that

$$\left(-\frac{\epsilon_1}{2}, 0\right) \in \partial D_1, \quad \left(\frac{\epsilon_1}{2}, 0\right) \in \partial D_2.$$

4.1. Representation of the solution u . Let H^c be a harmonic function outside of $\cup_{i=1}^3 D_i$ and have the same constant value in $\cup_{i=1}^3 D_i$ satisfying that

$$\begin{cases} \Delta H^c = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\cup_{i=1}^3 D_i}, \\ H^c(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \text{as } |\mathbf{x}| \rightarrow \infty, \\ H^c|_{\cup_{i=1}^3 \partial D_i} = C_H \text{ (constant)}. \end{cases} \quad (4.1)$$

Since $H^c - H$ is harmonic at infinity, $H^c - H$ attains maximum/minimum only at the boundary points of D_i , $i = 1, 2, 3$. To make $H^c - H$ attains zero at infinity, C_H should satisfy

$$-\|H\|_{L^\infty(\cup_{i=1}^3 D_i)} \leq C_H \leq \|H\|_{L^\infty(\cup_{i=1}^3 D_i)}. \quad (4.2)$$

Moreover, H^c satisfies $\sum_{i=1}^3 \int_{\partial D_i} \partial_\nu H^c dS = 0$.

The solution u to (1.1) is represented as

$$u(\mathbf{x}) = H^c(\mathbf{x}) + c_1 h_1(\mathbf{x}) + c_2 h_2(\mathbf{x}), \quad (4.3)$$

where

$$h_1 = \Psi[D_1, (D_2 \cup D_3)], \quad h_2 = \Psi[(D_1 \cup D_2), D_3], \quad (4.4)$$

and

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} -1 & \int_{\partial D_1} \partial_\nu h_2 dS \\ \int_{\partial D_2} \partial_\nu h_1 dS & \int_{\partial D_2} \partial_\nu h_2 dS \end{pmatrix}^{-1} \begin{pmatrix} \int_{\partial D_1} \partial_\nu H^c dS \\ \int_{\partial D_2} \partial_\nu H^c dS \end{pmatrix}, \quad (4.5)$$

where Ψ is defined as (2.3). The equality (4.5) is from the integration of $\partial_\nu u$ on ∂D_1 and ∂D_2 .

Applying the upper bound on the gradient of solution without the potential difference among the boundaries to conductivity equation derived in Bao et al. [7], we can show that ∇H^c does not blow-up (also refer to [14]). Using Lemma 4.2 in the following section, we have $\int_{\partial D_2} \partial_\nu h_1 dS = 1 + O(\sqrt{\epsilon_1})$. This implies that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \approx - \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \int_{\partial D_1} \partial_\nu H^c dS \\ \int_{\partial D_2} \partial_\nu H^c dS \end{pmatrix}.$$

Thus, the coefficient c_i , $i = 1, 2$, is bounded independently of ϵ_1 and ϵ_2 . Therefore, the blow-up rate of ∇u essentially relies on ∇h_i . In this respect, we consider the properties of h_i in the following section.

4.2. Properties of h_1 and h_2 . We build the optimal bounds of u based on (4.3). To do so, it is essential to drive properties of h_1 and h_2 in the narrow regions between inclusions. Here, h_1 and h_2 are supposed to be as follows:

$$h_1 = \Psi[D_1, (D_2 \cup D_3)] \text{ and } h_2 = \Psi[(D_1 \cup D_2), D_3].$$

From the monotonicity of Ψ , Lemma 2.4, we have the following result.

LEMMA 4.1. *We have the following properties:*

(i)

$$0 < h_1|_{\partial D_2} - h_1|_{\partial D_1} \leq \Psi[D_1, D_2] \Big|_{\partial D_2} - \Psi[D_1, D_2] \Big|_{\partial D_1}.$$

(ii)

$$0 < \partial_\nu h_1 \leq \partial_\nu \Psi[D_1, D_2] \text{ on } \partial D_2.$$

LEMMA 4.2. *There is a constant C such that*

$$0 < |\partial_\nu h_1| \leq C\sqrt{\epsilon_1}, \quad \text{on } \partial D_3.$$

Proof. We use Lemma 2.4. Let

$$M = \frac{\Psi[D_1, (D_2 \cup D_3)] \Big|_{\partial D_1} - \Psi[D_1, (D_2 \cup D_3)] \Big|_{\partial D_2 \cup \partial D_3}}{\Psi[D_1, D_3] \Big|_{\partial D_1} - \Psi[D_1, D_3] \Big|_{\partial D_3}}.$$

Then, we have

$$0 < \partial_\nu h_1 \leq M \partial_\nu \Psi[D_1, D_3] \text{ on } \partial D_3,$$

where

$$\begin{aligned} M \Psi[D_1, D_3] \Big|_{\partial D_3} - M \Psi[D_1, D_3] \Big|_{\partial D_1} &= h_1 \Big|_{\partial D_2 \cup \partial D_3} - h_1 \Big|_{\partial D_1} \\ &\leq \Psi[D_1, D_2] \Big|_{\partial D_2} - \Psi[D_1, D_2] \Big|_{\partial D_1} \\ &\leq C\sqrt{\epsilon_1}. \end{aligned}$$

The last inequality above was proven by Yun in his paper [14, 15], since $\Psi[D_1, D_2]$ is only for two domains. Note that D_3 is not close to D_2 . Owing to the method in Bao et al. [7], we have

$$\|\partial_\nu M \Psi[D_1, D_3]\|_{L^\infty(\partial D_3)} \leq C\sqrt{\epsilon_1}.$$

Therefore, we can obtain the result. \square

LEMMA 4.3. *Let D_4 is a disk containing D_2 and D_3 with*

$$\text{dist}(D_1, D_4) = \text{dist}(D_1, D_2).$$

(i) *There is a positive constant C such that*

$$0 > \partial_\nu h_1 \geq C \partial_\nu \Psi[D_1, D_4] \text{ on } \partial D_1.$$

(ii)

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \geq \Psi[D_1, D_4] \Big|_{\partial D_4} - \Psi[D_1, D_4] \Big|_{\partial D_1}.$$

(iii)

$$h_1|_{\partial D_2 \cup \partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{\epsilon_1}.$$

Proof. To prove (i) and (ii), we use a similar way to Lemma 2.4. Let

$$M = \frac{\Psi[D_1, D_4] \Big|_{\partial D_1} - \Psi[D_1, D_4] \Big|_{\partial D_4}}{\Psi[D_1, (D_2 \cup D_3)] \Big|_{\partial D_1} - \Psi[D_1, (D_2 \cup D_3)] \Big|_{\partial D_2 \cup \partial D_3}}$$

and

$$G(\mathbf{x}) = \Psi[D_1, D_4](\mathbf{x}) - \Psi[D_1, D_4] \Big|_{\partial D_4} \\ - M \left(\Psi[D_1, (D_2 \cup D_3)](\mathbf{x}) - \Psi[D_1, (D_2 \cup D_3)] \Big|_{\partial D_2 \cup \partial D_3} \right).$$

Then $G \Big|_{\partial D_1} = 0$ and $G \geq 0$ on ∂D_4 . By Hopf's lemma, we have

$$\partial_\nu G < 0 \text{ on } \partial D_1.$$

By the integration on ∂D_1 , we have (ii) and $M < 1$. Therefore, the inequality $\partial_\nu G < 0$ can also yield (i).

From (i) of Lemma 4.1 and (ii) in this lemma, we have

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \geq \Psi[D_1, D_4] \Big|_{\partial D_4} - \Psi[D_1, D_4] \Big|_{\partial D_1}$$

and

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \leq \Psi[D_1, D_2] \Big|_{\partial D_2} - \Psi[D_1, D_2] \Big|_{\partial D_1}.$$

The potential $\Psi[D_1, D_i]$ ($i = 1, 4$) is only for two domains and thus, its difference between D_1 and D_i ($i = 1, 2$) was already estimated in Yun [14, 15] as follows: for $i = 1, 2$,

$$\Psi[D_1, D_i] \Big|_{\partial D_i} - \Psi[D_1, D_i] \Big|_{\partial D_1} \simeq \sqrt{\epsilon_1}.$$

Therefore, we have (iii), moreover, the constant C of (i) in this lemma is independent of ϵ_1 . \square

PROPOSITION 4.4. *There are the following estimates for h_1 and h_2 :*

(i) *In the narrow region between D_1 and D_2 , we have*

$$\nabla h_1 = O\left(\frac{1}{\sqrt{\epsilon_1}}\right) \text{ and } \nabla h_2 = O(\sqrt{\epsilon_2}).$$

(ii) *In the narrow region between D_2 and D_3 , we have*

$$\nabla h_1 = O(\sqrt{\epsilon_1}) \text{ and } \nabla h_2 = O\left(\frac{1}{\sqrt{\epsilon_2}}\right).$$

(iii)

$$h_1|_{\partial D_2 \cup \partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{\epsilon_1}$$

and

$$h_2|_{\partial D_3} - h_2|_{\partial D_1 \cup \partial D_2} \simeq \sqrt{\epsilon_2}.$$

Proof. We consider ∇h_1 . By Lemma 4.1 and 4.3, we have

$$0 > \partial_\nu h_1 \geq C \partial_\nu \Psi[D_1, D_4] \text{ on } \partial D_1$$

and

$$0 < \partial_\nu h_1 \leq \partial_\nu \Psi[D_1, D_2] \text{ on } \partial D_2,$$

and by Lemma 4.2,

$$0 < |\partial_\nu h_1| \leq C \sqrt{\epsilon_1} \text{ on } \partial D_3,$$

where D_4 has been defined in Lemma 4.3. Without any loss of generality, we assume that

$$\left(-\frac{\epsilon_1}{2}, 0\right) \in \partial D_1, \quad \left(\frac{\epsilon_1}{2}, 0\right) \in \partial D_2 \text{ and } \text{dist}(D_1, D_2) = \epsilon_1.$$

Let

$$p(\mathbf{x}) = \log |\mathbf{x} - (\sqrt{\epsilon_1}, 0)| - \log |\mathbf{x} + (\sqrt{\epsilon_1}, 0)|.$$

Referring to the inequality (9) in [15], there is a constant C_1 such that

$$0 < |\nabla h_1| \leq C_1 |\nabla p| \text{ on } \partial(D_1 \cup D_2 \cup D_3).$$

Regarding (x_1, x_2) as a complex number $z = x_1 + x_2 i$, we consider

$$\rho(z) = \frac{\partial_1 h_1(z) - \partial_2 h_1(z) i}{C_1 (\partial_1 p(z) - \partial_2 p(z) i)}.$$

Then, $\rho(z)$ can be extended to ∞ as an analytic function. From definition, $|\rho(z)| < 1$ on $\partial D_1 \cup \partial D_2 \cup \partial D_3$. By the maximum principle,

$$|\rho(z)| < 1 \text{ in } \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3).$$

Thus, we have

$$|\nabla h_1| \leq C_1 |\nabla p| \text{ in } \mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3).$$

Therefore, $\nabla h_1 = O\left(\frac{1}{\sqrt{\epsilon_1}}\right)$ in the narrow region between D_1 and D_2 , and $\nabla h_1 = O(\sqrt{\epsilon_1})$ in the narrow region between D_2 and D_3 . Similarly, we have $\nabla h_2 = O\left(\frac{1}{\sqrt{\epsilon_2}}\right)$ in the narrow region between D_2 and D_3 , and $\nabla h_2 = O(\sqrt{\epsilon_2})$ in the narrow region between D_1 and D_2 . We have proven (i) and (ii).

The estimate (iii) is presented by Lemma 4.3. □

LEMMA 4.5. *Let H is an entire harmonic function. Then, we have*

$$\left| \int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 \, dS \right| \leq C \sqrt{\epsilon_1}.$$

Proof. Without loss of generality, we assume that

$$\left(-\frac{\epsilon_1}{2}, 0\right) \in \partial D_1, \quad \left(\frac{\epsilon_1}{2}, 0\right) \in \partial D_2, \quad \text{dist}(D_1, D_2) = \epsilon_1 \quad \text{and} \quad (-1, 0) \in D_2.$$

We consider \tilde{H} as follows:

$$\tilde{H} = H - \partial_2 H(0, 0) \frac{x_2}{|\mathbf{x} - (1, 0)|^2}.$$

It follows from the Divergence Theorem that

$$\int_{\partial D_1 \cup \partial D_2 \cup \partial D_3} \frac{x_2}{|\mathbf{x} - (1, 0)|^2} \partial_\nu h_1 dS = \int_{\partial D_1 \cup \partial D_2 \cup \partial D_3} \partial_\nu \left(\frac{x_2}{|\mathbf{x} - (1, 0)|^2} \right) h_1 ds = 0,$$

since $\frac{x_2}{|\mathbf{x} - (1, 0)|^2} = O(|x|^{-1})$ as $|x| \rightarrow \infty$. Hence, we have

$$\int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 dS = \int_{\partial D_1 \cup \partial D_2} \tilde{H} \partial_\nu h_1 dS + \int_{\partial D_3} \tilde{H} \partial_\nu h_1 dS.$$

We first consider $\int_{\partial D_1 \cup \partial D_2} \tilde{H} \partial_\nu h_1 dS$. By Lemma 4.1 and 4.3, we have

$$0 > \partial_\nu h_1 \geq C \partial_\nu \Psi[D_1, D_4] \quad \text{on} \quad \partial D_1$$

and

$$0 < \partial_\nu h_1 \leq \partial_\nu \Psi[D_1, D_2] \quad \text{on} \quad \partial D_2.$$

From definition, $\partial_2 \tilde{H} = 0$. Hence, we can use Lemma 3.2 in [15] so that

$$\left| \int_{\partial D_1} \tilde{H} \partial_\nu h_1 dS \right| \leq C \int_{\partial D_1} \left| \tilde{H} \Psi[D_1, D_4] \right| dS \leq C \sqrt{\epsilon_1}$$

and

$$\left| \int_{\partial D_2} \tilde{H} \partial_\nu h_1 dS \right| \leq \int_{\partial D_2} \left| \tilde{H} \Psi[D_1, D_2] \right| dS \leq C \sqrt{\epsilon_1}.$$

We second consider $\int_{\partial D_3} \tilde{H} \partial_\nu h_1 dS$. By Lemma 4.2, we can have

$$\left| \int_{\partial D_3} \tilde{H} \partial_\nu h_1 dS \right| \leq C \sqrt{\epsilon_1}.$$

Therefore, we have done it. \square

REMARK 4.6. We draw attention of readers to the independent work of Bao, Li and Yin in [5, 8]. Bao et al. have shown that the blow-up rate known for a pair of inclusion is still valid to the multiple inclusions cases. As a byproduct of our work, the blow-up rate of the gradient for three inclusions is established in the theorem follows.

THEOREM 4.7. Let D_1 , D_2 and D_3 be as assumed in the beginning of Section 4. Note that D_2 is not assumed to be smaller than the others.

Then, for any entire harmonic function $H(x_1, x_2)$, we have a constant C independent of ϵ such that

$$|\nabla u| \leq C \frac{1}{\sqrt{\epsilon_1}},$$

in the narrow region between D_1 and D_2 , and

$$|\nabla u| \leq C \frac{1}{\sqrt{\epsilon_2}},$$

in the narrow region between D_2 and D_3 ,

For $H(x_1, x_2) = x_1$, there exist \mathbf{x}_0 in the narrow region between D_1 and D_2 such that

$$|\nabla u(\mathbf{x}_0)| \geq C \frac{1}{\sqrt{\epsilon_1}}.$$

Similarly, there is a linear function $H(x_1, x_2)$ and a point \mathbf{y}_0 between D_2 and D_3 such that

$$|\nabla u(\mathbf{y}_0)| \geq C \frac{1}{\sqrt{\epsilon_2}}.$$

Proof. From (4.5) and the fact that the coefficient c_i , $i = 1, 2$, is bounded independently of ϵ_1 and ϵ_2 . Proposition 4.4 yields the upper bound.

Now, we derive the blow-up rate. From (4.2), there is a constant C independent of ϵ_i , $i = 1, 2$, such that

$$\|u\|_{L^\infty(\cup_{i=1}^3 \partial D_i)} \leq C \|H\|_{L^\infty(\cup_{i=1}^3 D_i)}.$$

From Lemma 2.3,

$$\begin{aligned} \int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 \, dS &= \left(1 - \int_{\partial D_3} \partial_\nu h_1 \, dS\right) (u|_{\partial D_2} - u|_{\partial D_1}) \\ &\quad + \left(\int_{\partial D_3} \partial_\nu h_1 \, dS\right) (u|_{\partial D_3} - u|_{\partial D_1}). \end{aligned} \quad (4.6)$$

It follows from Lemma 4.2 and 4.5 that

$$u|_{\partial D_2} - u|_{\partial D_1} \leq C\sqrt{\epsilon_1}, \quad (4.7)$$

where the constant C above depends on $\|H\|_{L^\infty(\cup_{i=1}^3 \partial D_i)}$. Similarly, we have

$$u|_{\partial D_3} - u|_{\partial D_2} \leq C\sqrt{\epsilon_2}. \quad (4.8)$$

Again, using (4.6) and the fact that $H > 0$ on $\partial D_2 \cup \partial D_3$, we have

$$\begin{aligned} u|_{\partial D_2} - u|_{\partial D_1} + O(\sqrt{\epsilon_1 \epsilon_2}) &= \int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 \, dS \\ &\geq \int_{\partial D_1} H \partial_\nu h_1 \, dS. \end{aligned} \quad (4.9)$$

To get the lower bound from the inequality above for $H(x_1, x_2) = x_1$, we again take an advantage of a monotonic property similar to Ψ in Lemma 2.4, and please refer to the proof of Lemma 4.3:

$$\partial_\nu h \leq \left(\frac{h_1|_{\partial D_1} - h_1|_{\partial D_2 \cup \partial D_3}}{\Psi[D_1, D_2]|_{\partial D_1} - \Psi[D_1, D_2]|_{\partial D_2}} \right) \partial_\nu \Psi[D_1, D_2] < 0, \quad \text{on } \partial D_1.$$

From (i) in Proposition 4.4, we have

$$h_1 \Big|_{\partial D_1} - h_1 \Big|_{\partial D_2 \cup \partial D_3} \simeq \sqrt{\epsilon_1}$$

and

$$\Psi[D_1, D_2] \Big|_{\partial D_1} - \Psi[D_1, D_2] \Big|_{\partial D_2} \simeq \sqrt{\epsilon_1}.$$

The inequality (4.9) implies

$$u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \gtrsim \sqrt{\epsilon}.$$

By the Mean Value Theorem, we have the desirable lower bound in the narrow region between D_1 and D_2 . Similarly, we can also obtain the other lower bound. \square

4.3. Proof of Theorem 1.2 (a). We derive the optimal bounds of the gradient of the solution to (1.1), when there are adjacent three disks:

$$D_l = B_{r_l}(\mathbf{c}_l), \quad l = 1, 2, 3, \quad (4.10)$$

where $\mathbf{c}_1 = (-r_1 - \frac{\epsilon_1}{2}, 0)$, $\mathbf{c}_2 = (r_2 + \frac{\epsilon_1}{2}, 0)$ and $\mathbf{c}_3 = (r_3 + r_2 + \frac{\epsilon_1}{2} + \epsilon_2, 0)$. As defined before, $h_1 = \Psi[D_1, (D_2 \cup D_3)]$. Let $w_1 = \Psi[D_1, D_2]$.

We begin the proof by showing that

$$w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1} \simeq h_1 \Big|_{\partial D_2} - h_1 \Big|_{\partial D_1}. \quad (4.11)$$

By the monotonic property of Lemma 2.4, we have

$$h_1 \Big|_{\partial D_2} - h_1 \Big|_{\partial D_1} \leq w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1}.$$

Considering

$$h_1 - \left(\frac{h_1 \Big|_{\partial D_2} - h_1 \Big|_{\partial D_1}}{w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1}} \right) w_1,$$

we can obtain, from the Hopf's Lemma,

$$\int_{\partial D_2} \partial_\nu h_1 dS \leq \left(\frac{h_1 \Big|_{\partial D_2} - h_1 \Big|_{\partial D_1}}{w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1}} \right) \int_{\partial D_2} \partial_\nu w_1 dS.$$

By Lemma 4.2, we have

$$\int_{\partial D_3} \partial_\nu h_1 dS = O(\sqrt{\epsilon_1}).$$

Since $\int_{\partial D_2 \cup \partial D_3} \partial_\nu h_1 dS = 1$, we have

$$\left(w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1} \right) (1 + O(\sqrt{\epsilon_1})) \leq h_1 \Big|_{\partial D_2} - h_1 \Big|_{\partial D_1}.$$

Therefore, we can obtain (4.11). Owing to the estimate for $w_1 \Big|_{\partial D_2} - w_1 \Big|_{\partial D_1}$ in Lemma 2.5, we have

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \simeq \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon_1}. \quad (4.12)$$

Let $w_2 = \Psi[D_1, D_3]$. Considering

$$h_1 - \left(\frac{h_1|_{\partial D_3} - h_1|_{\partial D_1}}{w_2|_{\partial D_3} - w_2|_{\partial D_1}} \right) w_2,$$

from the Hopf's Lemma, we obtain

$$\partial_\nu h_1 \leq \left(\frac{h_1|_{\partial D_3} - h_1|_{\partial D_1}}{w_2|_{\partial D_3} - w_2|_{\partial D_1}} \right) \partial_\nu w_2 \leq 0 \text{ on } \partial D_1.$$

Here, we estimate the coefficient in the right hand side. Note that $h_1|_{\partial D_2} = h_1|_{\partial D_3}$. Thus, we have

$$h_1|_{\partial D_3} - h_1|_{\partial D_1} \simeq \sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon_1}.$$

Since $r_2 \ll r_1$ and $r_2 \ll r_3$, we also have

$$w_2|_{\partial D_3} - w_2|_{\partial D_1} \simeq \sqrt{\frac{r_1 + r_3}{r_1 r_3}} \sqrt{r_2}.$$

This implies that

$$\partial_\nu h_1 \lesssim \frac{\sqrt{\frac{r_1 + r_2}{r_1 r_2}} \sqrt{\epsilon_1}}{\sqrt{\frac{r_1 + r_3}{r_1 r_3}} \sqrt{r_2}} \partial_\nu w_2 \leq 0 \text{ on } \partial D_1.$$

Therefore, we have

$$\begin{aligned} \int_{\partial D_1} H \partial_\nu h_1 dS &\gtrsim \sqrt{\frac{r_1 + r_2}{r_1 + r_3}} \frac{\sqrt{r_3}}{r_2} \sqrt{\epsilon_1} \int_{\partial D_1} H \partial_\nu w_2 dS \\ &\gtrsim \sqrt{\frac{r_1 + r_2}{r_1 + r_3}} \frac{\sqrt{r_3}}{r_2} \sqrt{\epsilon_1} \sqrt{\frac{r_1 r_3}{r_1 + r_3}} \sqrt{r_2} \\ &\geq \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1} \geq 0. \end{aligned} \quad (4.13)$$

Owing to (4.6), (4.7) and (4.8), we have

$$\begin{aligned} \int_{\cup_{i=1}^3 \partial D_i} u \partial_\nu h_1 dS &= \left(1 - \int_{\partial D_3} \partial_\nu h_1 dS \right) (u|_{\partial D_2} - u|_{\partial D_1}) \\ &\quad + \left(\int_{\partial D_3} \partial_\nu h_1 dS \right) (u|_{\partial D_3} - u|_{\partial D_1}) \\ &= (1 - O(\sqrt{\epsilon_1})) (u|_{\partial D_2} - u|_{\partial D_1}) + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
u|_{\partial D_2} - u|_{\partial D_1} &\geq \frac{1}{2} \int_{\cup_{i=1}^3 \partial D_i} u \partial_\nu h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \\
&= \frac{1}{2} \int_{\cup_{i=1}^3 \partial D_i} H \partial_\nu h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \\
&\geq \frac{1}{2} \int_{\partial D_1} H \partial_\nu h_1 \, dS + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \\
&\geq C \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1} + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})).
\end{aligned}$$

Therefore, we have completed the proof. \square

4.4. Proof of Theorem 1.4(a). We pursue the proof of Theorem 1.2, taking an advantage of the monotonic property of Lemma 2.4. The domains D_1 , D_2 and D_3 are as assumed in Theorem 1.4. As assumed before, $h_1 = \Psi[D_1, (D_2 \cup D_3)]$. Let $w_1 = \Psi[D_1, D_2]$. By the same way as Theorem 1.2, we have

$$w_1|_{\partial D_2} - w_1|_{\partial D_1} \simeq h_1|_{\partial D_2} - h_1|_{\partial D_1}.$$

Here, we use the monotonic property of Lemma 2.4 to estimate the difference between domains. Choosing two pairs of proper disks containing D_1 and D_2 , and contained D_1 and D_2 , respectively, we can obtain

$$h_1|_{\partial D_2} - h_1|_{\partial D_1} \simeq \sqrt{\frac{\epsilon_1}{r_2}}$$

under the assumption that r_2 is small.

Let $w_2 = \Psi[D_1, D_3]$. Choosing two pairs of proper disks containing D_1 and D_3 , and contained D_1 and D_3 , respectively, Then, we have

$$w_2|_{\partial D_3} - w_2|_{\partial D_1} \simeq \sqrt{r_2}.$$

By the same argument as Theorem 1.2, we have

$$\int_{\partial D_1} H \partial_\nu h_1 \, dS \gtrsim \sqrt{\frac{\epsilon_1}{r_1}} \geq 0.$$

Note that $D_1 \subset \mathbb{R}_- \times \mathbb{R}$ and $D_2 \cup D_3 \subset \mathbb{R}_+ \times \mathbb{R}$. Continuing to follow the proof of Theorem 1.2, we can obtain

$$u|_{\partial D_2} - u|_{\partial D_1} \geq C \sqrt{\frac{\epsilon_1}{r_1}} + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})).$$

Therefore, we have done the proof. \square

4.5. Derivation for the optimal upper bounds. We consider the optimal upper bounds presented on (b) of Theorem 1.1, 1.2, 1.3 and 1.4. These proofs have essential thing in common. In this respect, we prove only the optimal upper bound presented in Theorem 1.2. As have assumed them before, we set

$$\begin{aligned}
h_1 &= \Psi[D_1, D_2 \cup D_3] \\
h_2 &= \Psi[D_1, D_2] \\
h_3 &= \Psi[D_1, D_3] \\
h_4 &= \Psi[D_1, D_4].
\end{aligned}$$

Here, the domain D_4 is given in Lemma 4.3, which is a disk containing D_2 and D_3 with

$$\text{dist}(D_1, D_4) = \text{dist}(D_1, D_2),$$

and the diameter of D_4 is in proportion as r_3 , because r_2 is sufficiently small. Then, we compare h_1 with h_2 , h_3 and h_4 . The proof of Lemma 2.4 contains

$$0 \leq \partial_\nu h_1 \leq \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_2|_{\partial D_2} - h_2|_{\partial D_1}} \right) \partial_\nu h_2 \text{ on } \partial D_2,$$

and

$$0 \leq \partial_\nu h_1 \leq \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_3|_{\partial D_3} - h_3|_{\partial D_1}} \right) \partial_\nu h_3 \text{ on } \partial D_3,$$

and the proof of Lemma 4.3 implies

$$0 \leq -\partial_\nu h_1 \leq - \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_4|_{\partial D_4} - h_4|_{\partial D_1}} \right) \partial_\nu h_4 \text{ on } \partial D_1.$$

In the same way as Lemma 4.5, we can consider \tilde{H} by choosing the point in D_3 . In this respect, without any loss of generality, we can assume that

$$\partial_{x_2} H(0, 0) = 0.$$

The reason why we assumed above is because the integration representation for the potential difference is not good enough, refer to [15]. The geometrical assumption of Case (B) implies that D_1 and $D_2 \cup D_3$ are separated by $x_1 = 0$ and they are approaching to $(0, 0)$.

Therefore, by the proof of Theorem 1.2 and Lemma 2.6, we have

$$\begin{aligned} & \left| u|_{\partial D_2} - u|_{\partial D_1} \right| + O(\sqrt{\epsilon_1}) (O(\sqrt{\epsilon_1}) + O(\sqrt{\epsilon_2})) \\ &= \left| \int_{\partial(\cup_{i=1}^3 D_i)} H \partial_\nu h_1 dS \right| \\ &\lesssim \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_2|_{\partial D_2} - h_2|_{\partial D_1}} \right) \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \epsilon_1 \\ &\quad + \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_3|_{\partial D_3} - h_3|_{\partial D_1}} \right) \sqrt{\frac{r_1 r_3}{r_1 + r_3}} r_2 \\ &\quad + \left(\frac{h_1|_{\partial(D_2 \cup D_3)} - h_1|_{\partial D_1}}{h_4|_{\partial D_4} - h_4|_{\partial D_1}} \right) \sqrt{\frac{r_1 r_3}{r_1 + r_3}} \epsilon_1 \end{aligned}$$

and

$$h_1 \Big|_{\partial(D_2 \cup D_3)} - h_1 \Big|_{\partial D_1} \approx h_2 \Big|_{\partial D_2} - h_2 \Big|_{\partial D_1}.$$

Here, note that the radius of D_4 can be chosen between $\frac{3}{2}r_3$ and $2r_3$. Lemma 2.5 implies that

$$\left| u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \right| \lesssim \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1}.$$

Therefore, we establish the optimal upper bound for $\left| u \Big|_{\partial D_2} - u \Big|_{\partial D_1} \right|$.

Based on this, the optimal upper bound on the gradient of u in the narrow region be obtained. Here, the main idea to get the gradient estimate from the potential difference has already been presented by Bao et al. (Theorem 1.3, Lemma 2.2 and 2.3 in [7]), and has been modified to fit our problem by Lim and Yun in [13]. Thus, we give a brief description on the method. We choose a large domain D_0 containing D_1 , D_2 and D_3 , where ∂D_0 is at a sufficient distance from D_1 , D_2 and D_3 . Then, u can be decomposed as follows:

$$u = C_0 + v_0 + C_1 v_1 + C_3 v_3$$

where for $i = 0, 1, 3$, v_i is a harmonic function in $D_0 \setminus (D_1 \cup D_2 \cup D_3)$ with the boundary data

$$v_i = \delta_{0j} \text{ on } \partial D_j \text{ for } i = 1, 3$$

and

$$v_0 = \delta_{ij} u \text{ on } \partial D_j$$

for any $j = 0, 1, 2, 3$. Thus, the constants C_1 and C_3 keep

$$|C_1| \lesssim \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_1}$$

and

$$|C_3| \lesssim \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \sqrt{\epsilon_2}.$$

To estimate ∇v_0 , we consider a harmonic function ρ in $D_0 \setminus (D_1 \cup D_2 \cup D_3)$ with the boundary data

$$\rho = \delta_{0j} \text{ on } \partial D_j$$

for any $j = 0, 1, 2, 3$. By comparing with the harmonic function ρ_i in $D_0 \setminus D_i$ with $\rho_i = 0$ on ∂D_i and $\rho_i = 1$ on ∂D_0 , the Hopf's Lemma yields

$$\begin{aligned} & \|\nabla \rho\|_{L^\infty(D_0 \setminus (D_1 \cup D_2 \cup D_3))} \\ & \leq \max\{\|\nabla \rho_1\|_{L^\infty(\partial D_1)}, \|\nabla \rho_2\|_{L^\infty(\partial D_2)}, \|\nabla \rho_3\|_{L^\infty(\partial D_3)}, \|\nabla \rho\|_{L^\infty(\partial D_0)}\} < C. \end{aligned}$$

Applying the Hopf's Lemma again, we can have that the gradient of v_0 is bounded independent of ϵ_1 , refer to Lemma 2.2 in [7].

We estimate $C_1 \nabla v_1$ in the narrow region between D_1 and D_2 . Since v_1 is constant on the boundaries and the boundaries is smooth enough in the narrow region, the proof of Lemma 4.3 implies that v_1 can be extended into the interior areas of D_1 and D_2 by the distance almost ϵ from the boundaries in the narrow region, independently of r_1 and r_2 . By the gradient estimate for harmonic functions allows

$$|C_1 \nabla v_1| \lesssim \frac{r_1 r_3}{r_1 + r_3} \frac{1}{\sqrt{r_2}} \frac{1}{\sqrt{\epsilon_1}}$$

in the narrow region between D_1 and D_2 . Note that the inequality above is a local property independent of choosing D_0 .

Now, we consider $C_3 \nabla v_3$ in the narrow region between D_1 and D_2 . Let $\tilde{\rho}$ be a harmonic function in $D_0 \setminus (D_2 \cup D_3)$ with the boundary data

$$\tilde{\rho} = 0 \text{ on } \partial(D_0 \cup D_2) \text{ and } \tilde{\rho} = 1 \text{ on } \partial D_3.$$

By the maximum principle, we have

$$0 \leq v_3 \leq \tilde{\rho} \text{ in } D_0 \setminus (D_1 \cup D_2 \cup D_3)$$

Considering the standard estimate for $\Psi[D_2, D_3]$, we can obtain

$$|C_3 v_3| \leq C \sqrt{\epsilon_2}.$$

Similarly to the estimate for $C_1 \nabla v_1$, the gradient estimate for harmonic functions yields

$$|C_3 \nabla v_3| \lesssim \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

in the narrow region between D_1 and D_2 .

Therefore, we can obtain the desirable upper bound. Here, it is noteworthy that the upper bound is dominated only by the estimate for $C_2 \nabla v_1$, which is independent of choosing D_0 . In this respect, the constant C of the upper bound in Theorem 1.2 (b) is independent of $r_1, r_2, r_3, \epsilon_1$ and ϵ_2 .

In the same way, we can also derive the upper bounds of Theorem 1.1, 1.3 and 1.4 (b). \square

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