IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **25** (2009) 045003 (7pp) doi:10.1088/0266-5611/25/4/045003

On the stability of an inverse problem for the wave equation

Gang Bao and Kihyun Yun

Department of Mathematics, Michigan State University East Lansing, MI 48824, USA

E-mail: bao@math.msu.edu and kyun@math.msu.edu

Received 19 September 2008, in final form 13 January 2009
Published 13 February 2009
Online at stacks.iop.org/IP/25/045003

Abstract
Consider the inverse problem of determining the potential q from the Neumann to Dirichlet map Λ_q of the wave equation $u_{tt} - \Delta u + qu = 0$ in $\Omega \times (0, T)$ with $u(x, 0) = u_t(x, 0) = 0$. In this paper, a nearly Lipschitz-type stability estimate is established for the inverse problem: for any small $\epsilon > 0$, there exists $\beta_0 > 0$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{1-\epsilon}$$

when $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M$ for some $\beta > \beta_0$. Here, $\|\cdot\|_*$ represents the operator norm.

1. Main result

Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ $(n \geqslant 2)$, and Γ be $\partial\Omega \times (0, T)$. Consider the following wave equation:

$$\begin{cases} u_{tt} - \Delta u + qu = 0 & (x, t) \in \Omega \times (0, T) \\ u = u_t = 0 & x \in \Omega, \quad t = 0 \\ \dfrac{\partial u}{\partial \nu} = f & (x, t) \in \Gamma, \end{cases} \tag{1.1}$$

where $q = q(x)$ is a potential function and $T >$ diameter(Ω). Define the Neumann to Dirichlet map Λ_q associated with (1.1) as follows:

$$\Lambda_q : f \to u|_\Gamma.$$

This paper is concerned with the inverse problem, i.e., to invert the map

$$q \mapsto \Lambda_q.$$

This inverse problem was initially studied by Rakesh and Symes [3]. Inspired by the Sylvester and Uhlmann complex geometric optics technique on inverse conductivity problems [5], Rakesh and Symes showed that q can be uniquely determined by Λ_q. Subsequently, Sun [4] modified their method for obtaining the following Hölder-type stability result for the inverse problem with an exponent $1/3 - \epsilon$: for any small $\epsilon > 0$, there is $\beta_0 > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C\|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\frac{1}{3}-\epsilon}$$

when $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M$ for some $\beta > \beta_0$. Throughout, assume that $q_1 - q_2 = 0$ in Ω^c, $\|\cdot\|_*$ represents the operator norm $\|\cdot\|_{B(L^2(\Gamma),H^1(\Gamma))}$, and $H^\beta(\mathbb{R}^n)$ stands for the standard Sobolev space of order β with the norm $\|f\|_{H^\beta(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}(1+|\eta|^2)^\beta|\widehat{f}(\eta)|^2\,\mathrm{d}\eta\right)^{1/2}$. A similar Hölder continuity result can also be found in Alessandrini and Sylvester [2].

Our main objective of this paper is to establish a new Hölder stability result with exponent $1 - \epsilon$ for the inverse problem under essentially the same assumptions as in Sun [4].

Theorem 1.1. *Suppose that* $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M^\beta$ *for some* $\beta > \frac{n}{2} + 1$ *and* $\|q_i\|_{L^\infty(\mathbb{R}^n)} \leqslant M$ $(i = 1, 2)$ *for some positive number M. Then there exists a constant* C_* *independent of* β *such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{(1-\varepsilon_\beta)/(1+2\varepsilon_\beta)} \tag{1.2}$$

and

$$\varepsilon_\beta = \frac{n+3}{2\beta+4+n}$$

for $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_* \leqslant C_*^{-1}$.

Note that the stability above holds only for small $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*$ due to the independence of the constant C_* on β. The stability without the smallness constraint is proved in the following theorem.

Theorem 1.2. *Suppose that* $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M$ *for some* $\beta > \frac{n}{2} + 1$ *and* $\|q_i\|_{L^\infty(\mathbb{R}^n)} \leqslant M$ $(i = 1, 2)$ *for some positive number* M. *There exists a constant* C^* *dependent on* β *such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C^*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{(1-\varepsilon_\beta)/(1+2\varepsilon_\beta)}, \tag{1.3}$$

where ε_β *is defined in theorem 1.1.*

Proof. We first show that theorem 1.2 follows from theorem 1.1 and the following arguments.

The main result in Sun [4] provides a stability without the smallness condition for $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*$. Taking $\alpha = n/2 + 1$ in theorem 1 of [4] yields

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C\left(\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_1} + \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_2} + \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_3}\right)$$

for a constant $C > 0$, where α_1, α_2 and α_3 are positive numbers satisfying

$$\frac{2n+10}{3n+10} < \alpha_1 < 1, \qquad \alpha_2 = \frac{1}{3} + \frac{3n+10}{3n}(\alpha_1 - 1) \qquad \text{and} \qquad \alpha_3 = \frac{1}{n}(1-\alpha_1)\left(\frac{n}{2}+1\right).$$

Note that the exponents α_1, α_2 and α_3 are less than 1, since $n \geqslant 2$.

To derive a bound of $\|q_1 - q_2\|_{L^\infty(\Omega)}$ from the inequality above, we use the interpolation inequality in Sobolev spaces [1],

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\|q_1 - q_2\|_{L^2(\Omega)}^\delta\|q_1 - q_2\|_{H^\beta(\Omega)}^{1-\delta},$$

where

$$0 < \delta < 1 - \frac{n}{2\beta}.$$

Hence, we have a stability without the smallness condition as follows: there is a constant $C_\delta > 0$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_\delta\big(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\delta_1} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\delta_2} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\delta_3}\big)$$

and $\delta_i = \delta\alpha_i$ for $i = 1, 2, 3$, since $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M$. Here, we select δ smaller than $(1-\varepsilon_\beta)/(1+2\varepsilon_\beta)$ so that $\max\{\delta_1, \delta_2, \delta_3\} \leqslant (1-\varepsilon_\beta)/(1+2\varepsilon_\beta)$. Thus, we can choose a constant $C^* \geqslant C_*$ such that

$$C_\delta \sum_{i=1}^{3} \|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\delta_i} \leqslant C^* \|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{(1-\varepsilon_\beta)/(1+2\varepsilon_\beta)} \tag{1.4}$$

for $\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \geqslant C_*^{-1}$, where C_* is the constant in theorem 1.1. On the other hand, without any loss of generality, we can assume that $M \geqslant 1$, that is, $M \leqslant M^\beta$. Therefore, the desirable stability (1.3) without any smallness constraint follows from a combination of the inequalities (1.2) and (1.4). □

Remark 1.1. We conjecture that the Lipschitz stability $\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_*\|\Lambda_{q_1} - \Lambda_{q_2}\|_*$ for the inverse problem, which would be an extension of the result above. However, it is not evident that theorem 1.1 could be applied to the Lipschitz case ($\beta = \infty$). It appears that new techniques must be developed to investigate this interesting open problem.

Remark 1.2. This research topic arises in our recent study of the stability and ill-posedness for the inverse boundary value problems. An important future direction is to investigate the inverse problem for the wave equation

$$\frac{1}{c^2(x)} u_{tt} - \Delta u = 0,$$

where the to-be-determined coefficient function $c(x)$ is in the principal part of the operator. Currently, little is known for the inverse problem.

2. Proof of theorem 1.1

The following result plays an important role in the proof of our theorem.

Lemma 2.1 (Z. Sun)**.** *Assume that* $\|q_i\|_{L^\infty(\mathbb{R}^n)} \leqslant M$ $(i = 1, 2)$ *for some positive number M. Then, there is a constant C independent of* $\sigma > 0$, *such that*

$$|\widehat{q_1 - q_2}(\eta)| \leqslant C\left((1+\sigma^2)\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + \left(\frac{1}{\sigma} + \frac{1}{\sigma^2}\right)\|q_1 - q_2\|_{L^\infty(\Omega)}\right)(1+|\eta|^2). \tag{2.1}$$

The reader is referred to Sun [4] for a proof of lemma 2.1.

This lemma was also crucial for proving the main Hölder stability with the exponent $1/3 - \epsilon$ for $\|q_1 - q_2\|_{L^2(\Omega)}$ in Sun [4]. Since the right-hand side bound (2.1) does not belong to $L^2(\Omega)$ due to the $(1+|\eta|^2)$ term, it is only used for small η with $|\eta| < l$. The regularity condition $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M$ yields another upper bound for large $|\eta| \geqslant l$, i.e., $\int_{|\eta|\geqslant l} |\widehat{q_1 - q_2}(\eta)|\, d\eta \leqslant M^2/l^{2\beta}$. These two bounds give rise to a bound of $\|q_1 - q_2\|_{L^2(\Omega)}$ in terms of l and σ. Therefore, the appropriate l and σ may be found to minimize the upper bound in [4].

In this paper, a novel iterative method for using the inequality (2.1) and the regularity condition is developed for a recursive argument to improve the stability estimate. The new method is based on the following lemma.

Lemma 2.2. *Suppose that*

$$\begin{cases} \int_0^\infty \rho(r)\,\mathrm{d}r \leqslant M_0 \\ 0 \leqslant \rho(r) \leqslant b(r). \end{cases}$$

Then

$$\int_0^\infty \frac{\rho(r)}{(1+r^2)^m}\,\mathrm{d}r \leqslant \int_0^l \frac{b(r)}{(1+r^2)^m}\,\mathrm{d}r \qquad \textit{for} \quad m > 0,$$

where

$$\int_0^l b(r)\,\mathrm{d}r \geqslant M_0. \tag{2.2}$$

Proof. From the definition, it is clear that $0 \leqslant \rho(r) \leqslant b(r)$ and $\frac{1}{(1+r^2)^m}$ is a decreasing function. Therefore,

$$\begin{aligned} \int_0^l \frac{b(r)}{(1+r^2)^m}\,\mathrm{d}r - \int_0^\infty \frac{\rho(r)}{(1+r^2)^m}\,\mathrm{d}r &\geqslant \int_0^l \frac{b(r)-\rho(r)}{(1+r^2)^m}\,\mathrm{d}r - \int_l^\infty \frac{\rho(r)}{(1+r^2)^m}\,\mathrm{d}r \\ &\geqslant \int_0^l \frac{b(r)-\rho(r)}{(1+l^2)^m}\,\mathrm{d}r - \int_l^\infty \frac{\rho(r)}{(1+l^2)^m}\,\mathrm{d}r \\ &\geqslant \frac{1}{(1+l^2)^m}\left(\int_0^l b(r)\,\mathrm{d}r - \int_0^\infty \rho(r)\,\mathrm{d}r\right) \geqslant 0. \qquad \square \end{aligned}$$

The next result is the key to the recursive formulations for the proof of theorem 1.1.

Lemma 2.3. *Let* α, β, M *and* B *be non-negative constants. Suppose that*

$$\begin{aligned} &M \geqslant 1, \qquad B \geqslant 1 \qquad 0 \leqslant \alpha \leqslant 1, \qquad \beta > \frac{n}{2} + 1 \\ &\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leqslant M^\beta \qquad \|q_i\|_{L^\infty(\Omega)} \leqslant M \quad (i = 1, 2) \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant B \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^\alpha \leqslant 1 && \text{for} \quad 0 < \alpha \leqslant 1 \\ &\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_* \leqslant 1 \qquad \textit{and} \qquad \|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant B = 2M && \text{for} \quad \alpha = 0 \end{aligned} \tag{2.4}$$

Then

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_0 B^{\frac{2}{3}} \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha'}$$

and

$$\alpha' = \frac{1+2\alpha}{3}\left(1 - \frac{3+n}{2\beta+4+n}\right),$$

where the constant $C_0 \geqslant 1$ *is independent of* β *and* α.

Proof of lemma 2.3. Let A be $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*$, and from hypotheses (2.3) and (2.4),

$$A \leqslant 1 \leqslant B.$$

Then, for $\alpha \in (0, 1]$,

$$\begin{aligned} \|q_1 - q_2\|_{L^\infty(\Omega)} &\leqslant B \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^\alpha \\ &= B \cdot A^\alpha. \end{aligned}$$

Let $\sigma = \left(\frac{B}{A^{1-\alpha}}\right)^{1/3} \geqslant 1$. It follows from lemma 2.1 that

$$\begin{aligned}
|\widehat{q_1 - q_2}(\eta)| &\leqslant C_1' \left((1+\sigma^2) \left\| \Lambda_{q_1} - \Lambda_{q_2} \right\|_* + \left(\frac{1}{\sigma} + \frac{1}{\sigma^2} \right) \|q_1 - q_2\|_{L^\infty(\Omega)} \right) (1+|\eta|^2) \\
&\leqslant C_1'' \left(\sigma^2 \left\| \Lambda_{q_1} - \Lambda_{q_2} \right\|_* + \frac{1}{\sigma} \|q_1 - q_2\|_{L^\infty(\Omega)} \right) (1+|\eta|^2) \\
&\leqslant C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}} (1+|\eta|^2).
\end{aligned} \tag{2.5}$$

Next, in order to apply lemma 2.2, we choose

$$\begin{aligned}
\rho(r) &= \int_{|\eta|=r} (1+|\eta|^2)^\beta |\widehat{q_1 - q_2}(\eta)|^2 \, \mathrm{d}s(\eta), \\
b(r) &= \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 \int_{|\eta|=r} (1+|\eta|^2)^{\beta+2} \, \mathrm{d}s(\eta), \\
m &= \beta - 1 - \frac{n}{2} \qquad \text{and} \qquad M_0 = M^{2\beta},
\end{aligned}$$

where $\int_{|\eta|=r} f(\eta) \, \mathrm{d}s(\eta)$ is the integration of $f(\eta)$ on the sphere (or circle) $|\eta| = r$. From (2.5), the assumption of lemma 2.2 is satisfied.

An application of lemma 2.2 yields

$$\begin{aligned}
\|q_1 - q_2\|^2_{H^{\frac{n}{2}+1}(\Omega)} &= \int_0^\infty (1+r^2)^{-m} \left(\int_{|\eta|=r} (1+|\eta|^2)^\beta |\widehat{q_1 - q_2}(\eta)|^2 \, \mathrm{d}s(\eta) \right) \mathrm{d}r \\
&\leqslant \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 \int_{|\eta| \leqslant l} (1+|\eta|^2)^2 (1+|\eta|^2)^{\frac{n}{2}+1} \, \mathrm{d}\eta \\
&\leqslant C_2' \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 (1+l^2)^{3+\frac{n}{2}} l^n \\
&\leqslant C_2 \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 l^{6+2n},
\end{aligned} \tag{2.6}$$

provided that the constant l is properly chosen to satisfy: $l \geqslant 1$ and

$$M^{2\beta} = M_0 \leqslant \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 \int_{|n|<l} (1+|\eta|^2)^{\beta+2} \, \mathrm{d}\eta. \tag{2.7}$$

Here, the second condition (2.7) is from (2.2) of lemma 2.2.

In order to serve the above purposes, we first choose l as follows:

$$l := \left(\frac{M^{2\beta}(2\beta + 4 + n)}{C_3 \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2} \right)^{1/(2\beta+4+n)}.$$

Here, the constant C_3 is chosen to be small enough that $\frac{1}{C_3 (C_1)^2} \geqslant 2^{4/3}$ and C_3 is less than the surface area (or perimeter) of the unit sphere (or circle). Second, from the hypotheses on A, B, $A^\alpha B$ and β, we deduce that $l \geqslant 1$ and the condition (2.7) holds as follows:

$$\begin{aligned}
M^{2\beta} &= C_3 \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 l^{2\beta+4+n} (2\beta + 4 + n)^{-1} \\
&\leqslant \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 \int_0^l \int_{|\eta|=r} |\eta|^{2\beta+4} \, \mathrm{d}s(\eta) \, \mathrm{d}r \\
&\leqslant \left(C_1 A^{\frac{1+2\alpha}{3}} B^{\frac{2}{3}}\right)^2 \int_{|n|<l} (1+|\eta|^2)^{\beta+2} \, \mathrm{d}\eta.
\end{aligned}$$

It follows from (2.6) and $B \geqslant 1$ that

$$\|q_1 - q_2\|_{H^{\frac{n}{2}+1}(\Omega)} \leqslant C_1\sqrt{C_2}A^{\frac{1+2\alpha}{3}}B^{\frac{2}{3}}l^{3+n}$$
$$= C_1\sqrt{C_2}A^{\frac{1+2\alpha}{3}}B^{\frac{2}{3}}\left(\frac{M^{2\beta}(2\beta+4+n)}{C_3\left(C_1A^{\frac{1+2\alpha}{3}}B^{\frac{2}{3}}\right)^2}\right)^{(3+n)/(2\beta+4+n)}$$
$$\leqslant C_1\sqrt{C_2}C_4B^{\frac{2}{3}}A^{\frac{1+2\alpha}{3}(1-\frac{3+n}{2\beta+4+n})},$$

where the constant

$$C_4 \geqslant \sup_{\beta>n/2+1}\left(\frac{M^{2\beta}(2\beta+4+n)}{C_3C_1^2}\right)^{(3+n)/(2\beta+4+n)}.$$

Here, the right-hand side term $\left(\frac{M^{2\beta}(2\beta+4+n)}{C_3C_1^2}\right)^{(3+n)/(2\beta+4+n)}$ is bounded independent of $\beta > \frac{n}{2}+1$, since $\left(\frac{1}{C_3C_1^2}\right)^{(3+n)/(2\beta+4+n)}$, $(2\beta+4+n)^{1/(2\beta+4+n)}$ and $\frac{2\beta(3+n)}{(2\beta+4+n)}$ are bounded independent of $\beta > \frac{n}{2}+1$. Consequently, the constants C_1, C_2, C_3 and C_4 are all independent of β.

Therefore, by the Sobolev imbedding theorem [1], we obtain

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_0B^{\frac{2}{3}}A^{\frac{1+2\alpha}{3}(1-\frac{3+n}{2\beta+4+n})},$$

where C_0 is independent of β and $C_0 \geqslant 1$. □

We are now ready to prove theorem 1.1.

Proof of theorem 1.1. We apply a recursive argument to lemma 2.3. Thus, without loss of generalities, we consider the following sequences:

$$\alpha_1 = 0 \qquad \alpha_{i+1} = \frac{1+2\alpha_i}{3}\left(1 - \frac{3+n}{2\beta+4+n}\right) \qquad \text{for} \quad i = 1, 2, 3, \ldots \tag{2.8}$$

$$B_1 = 2M, \qquad B_{i+1} = C_0(B_i)^{2/3}, \tag{2.9}$$

where M is supposed to be larger than 1 and C_0 is the constant at lemma 2.3. The sequence $\{B_i\}$ converges to C_0^3. Thus, we can choose a constant $\widetilde{C}_*$ independent of β such that

$$\widetilde{C}_* \geqslant \sup_{i\in\mathbb{N}} B_i \geqslant 1,$$

and let

$$C_* = (\widetilde{C}_*)^{27} \geqslant 1.$$

On the other hand, it follows from a standard calculation that the sequence $\{\alpha_i\}$ is increasing and can be expressed as

$$\alpha_i = \left(1 - \frac{3+n}{2\beta+4+n}\right)\left(1+2\left(\frac{3+n}{2\beta+4+n}\right)\right)^{-1}\left(1 - \left\{\frac{2}{3}\left(1 - \frac{3+n}{2\beta+4+n}\right)\right\}^{i-1}\right),$$

and $\frac{1}{\alpha_2} \leqslant 27$.

The smallness condition $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_* \leqslant C_*^{-1} = \widetilde{C}_*^{-27}$ implies $\widetilde{C}_*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_2} \leqslant 1$. Hence from $B_i \leqslant \widetilde{C}_*, \alpha_2 \leqslant \alpha_i$ and $\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_* \leqslant 1$,

$$B_i\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_i} \leqslant 1$$

which is the condition (2.4) of lemma 2.3.

Therefore,

$$\begin{aligned}\|q_1 - q_2\|_{L^\infty(\Omega)} &\leqslant B_{i+1}\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_{i+1}} \\ &\leqslant \widetilde{C}_*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_{i+1}} \\ &\leqslant C_*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{\alpha_{i+1}} \qquad \text{for} \quad i = 2, 3, 4, 5, \ldots.\end{aligned}$$

Here, the constant C_* is independent of β and i.

Finally, since the sequence $\{\alpha_i\}$ is increasing and converges to $\left(1 - \frac{n+3}{2\beta+4+n}\right)\left(1 + \frac{2(n+3)}{2\beta+4+n}\right)^{-1}$, we conclude that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C_*\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_*^{(1-\frac{n+3}{2\beta+4+n})(1+\frac{2(n+3)}{2\beta+4+n})^{-1}}.$$

The proof of theorem 1.1 is now complete. □

Acknowledgments

The research was supported in part by the NSF and the ONR.

References

[1] Adams R 1975 *Sobolev Spaces* (New York: Academic)

[2] Alessandrini G and Sylvester J 1990 Stability for a multidimensional inverse spectral theorem *Comm. Partial Differential Equations* **15** pp 711–736

[3] Rakesh and Symes W 1988 Uniqueness for an inverse problem for the wave equation *Comm. Partial Differential Equations* **13** pp 87–96

[4] Sun Z 1990 On continuous dependence for an inverse initial boundary value problem for the wave equation *J. Math. Anal. Appl.* **150** pp 188–204

[5] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem *Ann. Math.* **125** pp 153–69