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THE REFLECTION OF SOLUTIONS OF HELMHOLTZ EQUATION AND AN APPLICATION

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Abstract. It is the purpose of this paper to study the reflection of solutions of Helmholtz equation with Neumann boundary data. In detail let u be a solution of Helmholtz equation in the exterior of a ball in $\mathbb{R}^3$ with exterior Neumann data $\partial_\nu u = 0$ on the boundary of the ball. We prove that u can be extended to $\mathbb{R}^3$ except the center of the ball. As a corollary, we prove that a sound hard ball can be identified by the scattering amplitude corresponding to a single incident direction and a single frequency.

1. Introduction

Let D be a bounded simply connected smooth domain in $\mathbb{R}^3$. The Helmholtz equation corresponding to a positive frequency k is

$$\triangle u + k^2 u = 0 \quad \text{in} \quad D_e := \mathbb{R}^3 \backslash \overline{D}. \tag{1.1}$$

When D is a sound soft ball with center x_0, Colton [2] proved that all the solutions of the Helmholtz equation can be extended to $\mathbb{R}^3 \backslash \{x_0\}$ as a solution, where the sound soft obstacle means that the solution u satisfies the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial D. \tag{1.2}$$

The proof in the Colton [2] is based on the solution of the Goursat problem for wave equation.

When D is a sound hard ball, in other words, the boundary condition (1.2) is replaced by a Neumann boundary data

$$\partial_\nu u = 0 \quad \text{on} \quad \partial D \tag{1.3}$$

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where ν is the outward normal to ∂D, then Goursat problem mentioned above is not solved and there is a difficulty in applying the argument in Colton [2]. It is the purpose of this paper to prove the following theorem.

THEOREM 1. *Let $D = B(x_0, \rho)$ the ball of radius ρ with center x_0 and if u is a solution of the Helmholtz equation* (1.1) *with the exterior Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on ∂D, then u can be extended to a solution of Helmholtz equation in $\mathbb{R}^3 \setminus \{x_0\}$.*

We prove this theorem by careful estimating the radius of convergence of the spherical harmonic expansion of the solution.

As a consequence of Theorem 1, we prove that the uniqueness for the inverse scattering problem within the class of sound soft balls. In scattering theory, the function u is assumed to be the sum of the incident wave u^i that is an entire solution of Helmholtz equation and a scattered wave u^s satisfies the Sommerfeld radiation condition

$$\hat{x} \cdot \nabla u^s(x) - iku^s(x) = o\left(\frac{1}{r}\right) \text{ as } r \text{ goes to } \infty \tag{1.4}$$

where $r = |x|$ and $\hat{x} = \frac{x}{r}$. In other words u^s is a *radiating solution.*

It is well known (see the book of Colton and Kress [3]) that any u^s admits the representation of

$$r^{-1} \exp(ikr) u_\infty(\hat{x}) + O\left(\frac{1}{r^2}\right). \tag{1.5}$$

The function u_∞ is called the scattering amplitude (or the scattering pattern) and can reconstruct u^s in an exterior of a bounded domain (see p. 35 in [4]). In particular when $u^i(x)$ is the *incident plane wave* $\exp(ikd \cdot x)$ for some incident direction $d \in S^2$, u^s and u_∞ admit the representation of $u^s(;d)$ and $u_\infty(;d)$.

An inverse problem is to recover the obstacle from the knowledge of its scattering amplitude u_∞. In the case of soft obstacles, based on eigenvalue properties, Schiffer proved [4] that if D_1 and D_2 are two sound-soft obstacles such that the scattering amplitudes coincide for incident plain waves with infinite directions and one fixed wave number k, then $D_1 = D_2$. Moreover Colton and Sleeman found a finite number of directions to identify the sound soft obstacle in a bounded domain by incident plain waves with one fixed wave number k(see [4]). Based on Colton's theorem [2], Liu proved [9] that if D_1 and D_2 are two sound-soft balls such that the scattering amplitudes coincide for an incident plane wave of one direction and one fixed wave number k, then $D_1 = D_2$.

Since the results of Schiffer, Colton and Sleeman are based on eigenvalue properties, there is no analogue of them in the case of hard obstacles. Kirsch and Kress proved [8] that if D_1 and D_2 are two sound-hard obstacles such that the scattering amplitudes coincide for incident plain waves with all directions in S^2 and one fixed wave number, then $D_1 = D_2$. We prove the uniqueness within the class of sound hard balls by a scattering amplitude corresponding to a single incident direction and one fixed wave number k on Corollary 3. First of all, we propose the results of general incident waves on Corollary 2. As it mentioned before, this proof is based on Theorem 1 instead of Colton by (1.3). Here we use the definitions and the notations of spherical harmonic functions in Colton and Kress(see [4]).

Corollary 2. *Let D_1 and D_2 be balls. If the nonzero scattered waves of them is the same for some incident wave u^i, then*

(i) *they must coincide. Moreover u^i has an expansion with respect to spherical wave functions of the form*

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m j_n(k\,|x|) Y_n^m(\hat{x}),$$

where the origin is the center of them.

(ii) *If the sequence $\langle a_n^m \rangle$ are nonzero on infinite terms, then $D_1 = D_2$.*

As a consequence of Corollary 2, we prove the uniqueness within the class of sound hard balls.

Corollary 3. *A sound hard ball can be uniquely identified by scattering amplitude corresponding to a single incident direction with a single frequency.*

The following example show it impossible to identify a sound hard all by the zero scattering wave.

Example 4. Let $u^i(x)$ be $\frac{\sin(k|x|)}{|x|}$ denoted by $j_0(|x|)$. j_0' has infinitely many zero points. Hence we can find many sound hard balls with zero scattered wave. Thus it is impossible to identify a sound hard ball by the zero scattering wave in this case. Indeed an incident wave u^i with infinite nonzero Y_n^m terms has nonzero scattered wave by the proof of (ii) of corollary 2.

2. The proofs of theorem and corollary

We start our proofs by introducing the basic properties of spherical Bessel functions. For a more detailed analysis we refer to Colton and Kress [4] and Lebedev [Leb].

We look for solution to the Helmholtz equation of the form

$$u(x) = f(k\,|x|)Y_n(\hat{x})$$

where Y_n is a spherical harmonic of order n. From the differential equation for the spherical harmonics, it follows that u solves the Helmholtz equation provided f is a solution of the spherical Bessel differential equation

$$t^2 f''(t) + 2tf'(t) + [t^2 - n(n+1)]f(t) = 0.$$

By direct calculations, we see that for $n = 0, 1, 2, \cdots$ the functions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)}$$

and

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p+1)}$$

represent solutions to the spherical Bessel differential equation. The functions j_n and y_n are called *spherical Bessel functions* and *spherical Neumann functions* of order n, respectively, and the linear combination

$$h_n^{(1)} := j_n + iy_n$$

are known as *spherical Hankel functions* of the first kind of order n.

It is well known that

$$u_n = j_n(k\,|x|)Y_n(\hat{x})$$

is an entire solution to Helmholtz equation and

$$v_n(x) = h_n^{(1)}(k\,|x|)Y_n(\hat{x})$$

is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \backslash \{0\}$.

From the series representation of the spherical Bessel and Neumann functions, it is obvious that

$$j_n(t) = \frac{t^n}{1 \cdot 3 \cdots (2n+1)} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty, \tag{2.1}$$

uniformly on compact subsets of $\mathbb{R}$ and

$$h_n^{(1)}(t) = \frac{1\cdot 3\cdots(2n-1)}{it^n}\left(1+O\left(\frac{1}{n}\right)\right) \text{ as } n\to\infty. \tag{2.2}$$

uniformly on compact subsets of $(0,\infty)$. It is readily verified that both $f_n = j_n$ and $f_n = h_n^{(1)}$ satisfy the differentiation formula

$$f_{n+1}(t) = -t^n\frac{d}{dt}\{t^{-n}f_n(t)\},\ n = 1,2,\cdots. \tag{2.3}$$

Lemma 5. *$h_n^{(1)\prime}(t)$ has no zero point.*

Proof. Assume that $h_n^{(1)\prime}(\rho) = 0$ for some point $\rho > 0$. Consider $u(x) = h_n^{(1)\prime}(|x|)Y_n(\hat{x})$ where $Y_n(\hat{x})$ is a spherical harmonic function of order n. Then $u(x)$ is a solution of $\triangle u + u = 0$ in $\mathbb{R}^3\backslash B(0,\rho)$ with $\partial_\nu u = 0$ on $\partial B(0,\rho)$ and holds the Sommerfeld radiation condition of $k = 1$. By Rellich's lemma, $u = 0$ in $\mathbb{R}^3\backslash B(0,\rho)$. This is contradiction (See Lemma 6.1 in [6]). □

Proof of Theorem 1. Without loss of generality, we may assume that the center of D is the origin. In other words $D = B(0,\rho)$. u has an expansion with respect to spherical wave functions of the form

$$u = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\{a_n^m j_n(k\,|x|) + b_n^m h_n^{(1)}(k\,|x|)\}Y_n^m(\hat{x}) \text{ for } x \in D_e.$$

First of all, we will prove that

$$u^i := \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m j_n(k\,|x|)Y_n^m(\hat{x}) \tag{2.4}$$

is entire. Define the sequence $u_l^i = \sum_{n=0}^{l}\sum_{m=-n}^{n} a_n^m j_n(k\,|x|)Y_n^m(\hat{x})$ for $l = 1,2,\cdots$. Let $R_0 > 0$. Then by (2.1) and (2.2) there is some integer n_0 such that $O\left(\frac{1}{n}\right)$ term of j_n is less then $\frac{1}{2}$ in the union of intervals $[0,kR_0]\cup[k(\rho+R_0),k(\rho+2R_0)]\cup[10k(\rho+R_0),10k(\rho+2R_0)]$ and $O\left(\frac{1}{n}\right)$ term of $h_n^{(1)}$ is less then $\frac{1}{2}$ in the union of intervals $[k(\rho+R_0),k(\rho+2R_0)]\cup[10k(\rho+R_0),10k(\rho+2R_0)]$ for $n > n_0$. Then by (2.1) and (2.2), we have

$$|\ a_n^m j_n(kt)\ | \le 10\left|\ a_n^m j_n(k(R_0+t+\rho)) + b_n^m h_n^{(1)}(k(R_0+t+\rho))\ \right|$$

or

$$|\ a_n^m j_n(kt)\ | \leq 10 \left|\ a_n^m j_n(10k(R_0+t+\rho)) + b_n^m h_n^{(1)}(10k(R_0+t+\rho))\ \right|$$

for $t \in [0, R_0]$ and $n > n_0$. Since the spherical harmonics Y_n^m for $m = -n, \cdots, n \ \ n = 0, 1, 2, \cdots$ form a complete orthonormal system in $L^2(S^2)$, the sequence $\langle u_l^i \rangle$ converges to u^i with respect to $L^2(B(0, R_0))$. Hence we have

$$\left\| u_{n_1}^i - u_{n_2}^i \right\|_{H^2\left(B_{\frac{R_0}{2}}\right)} \leq C \left\| u_{n_1}^i - u_{n_2}^i \right\|_{L^2\left(B_{R_0}\right)}, \tag{2.5}$$

by Sobolev inequality, we have

$$\left\| u_{n_1}^i - u_{n_2}^i \right\|_{C^{0,\frac{1}{2}}\left(B_{\frac{R_0}{2}}\right)} \leq C \left\| u_{n_1}^i - u_{n_2}^i \right\|_{H^2\left(B_{\frac{R_0}{2}}\right)} \tag{2.6}$$

and since $u_{n_1}^i - u_{n_2}^i \in C^{2,\frac{1}{2}}(B_{R_0})$,

$$\left\| u_{n_1}^i - u_{n_2}^i \right\|_{C^{2,\frac{1}{2}}\left(B_{\frac{R_0}{3}}\right)} \leq C \left\| u_{n_1}^i - u_{n_2}^i \right\|_{C^{0,\frac{1}{2}}\left(B_{\frac{R_0}{2}}\right)}, \tag{2.7}$$

where Cs depend only on R_0 and B_{R_0} is the ball of radius R_0 with the center origin. Hence u^i is an entire solution of Helmholtz equation.

By the Neumann boundary data of (1.3), we have

$$b_n^m = -a_n^m \frac{{j_n}'(k\rho)}{h_n^{(1)}{}'(k\rho)}$$

where ${j_n}'(k\rho) = \frac{d}{dt} j_n(k\rho)$ and $h_n^{(1)}{}'(k\rho) = \frac{d}{dt} h_n^{(1)}(k\rho)$.

Now we will prove that

$$u^s := -\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \frac{{j_n}'(k\rho)}{h_n^{(1)}{}'(k\rho)} h_n^{(1)}(k\,|x|) Y_n^m(\hat{x})$$

can be extended to a radiation solution of Helmholtz equation in $\mathbb{R}^3 \backslash \{0\}$. Define the sequence $u_l^s = -\sum_{n=0}^{l} \sum_{m=-n}^{n} a_n^m \frac{{j_n}'(k\rho)}{h_n^{(1)}{}'(k\rho)} h_n^{(1)}(k\,|x|) Y_n^m(\hat{x})$ for $l = 1, 2, \cdots$. Then $u_l^i + u_l^s$ is a solution to Helmholtz (1.1) with the Neumann boundary condition (1.3). Let $R > 10$. By (2.1), (2.2) and

(2.3), we have

$$\begin{aligned}(2.8)\qquad & \frac{{j_n}'(k\rho)h_n^{(1)}(k|x|)}{h_n^{(1)}{}'(k\rho)} \\ = \ & {j_n}'(k\rho)\frac{k(\rho)^{n+2}(1+O(\frac{1}{n}))}{(-n-O(1))|x|^{n+1}} \\ = \ & \Bigg(\frac{n}{k\rho}\frac{(k\rho)^n}{1\cdot 3\cdots(2n+1)}\Big(1+O\Big(\frac{1}{n}\Big)\Big) \\ & -\frac{(k\rho)^{n+1}}{1\cdot 3\cdots(2n+3)}\Big(1+O\Big(\frac{1}{n+1}\Big)\Big)\Bigg)\end{aligned}$$

$$(2.9)\qquad \frac{k(\rho)^{n+2}(1+O(\frac{1}{n}))}{(-n-O(1))|x|^{n+1}}$$

uniformly on compact annulus with center 0.

For convenience, we assume that $n_1 > n_2$,

$$\begin{aligned} & \left\|u^i_{n_1}-u^i_{n_2}\right\|^2_{L^2\left(B_R\setminus B_{\frac{1}{R}}\right)} \\ = \ & \int_{\frac{1}{R}}^R\int_{S^2}\left|\sum_{n=n_1+1}^{n_2}\sum_{m=-n}^{n}a_n^m j_n(k|x|)Y_n^m(\hat{x})\right|^2|x|^2\,d\hat{x}d|x| \\ (2.10)\qquad = \ & \int_{\frac{1}{R}}^R\sum_{n=n_1+1}^{n_2}\sum_{m=-n}^{n}|a_n^m j_n(k|x|)|^2|x|^2\,d|x|\end{aligned}$$

and by (2.1) we know

$$(2.11)\qquad j_n(k|x|)=\frac{(k|x|)^n}{1\cdot 3\cdots(2n+1)}\left(1+O\left(\frac{1}{n}\right)\right)\ \text{as } n\to\infty.$$

On the other hand, let $R_1 > \rho + 10$.

$$\begin{aligned} & \left\|u^s_{n_1}-u^s_{n_2}\right\|^2_{L^2\left(B_{R_1}\setminus B_{\frac{1}{R_1}}\right)} \\ = & \int_{\frac{1}{R_1}}^{R_1}\int_{S^2}\left|\sum_{n=n_1+1}^{n_2}\sum_{m=-n}^{n}a_n^m\frac{{j_n}'(k\rho)h_n^{(1)}(k|x|)}{h_n^{(1)}{}'(k\rho)}Y_n^m(\hat{x})\right|^2|x|^2\,d\hat{x}d|x| \\ (2.12) = & \int_{\frac{1}{R_1}}^{R_1}\int_S^2\sum_{n=n_1+1}^{n_2}\sum_{m=-n}^{n}\left|a_n^m\frac{{j_n}'(k\rho)h_n^{(1)}(k|x|)}{h_n^{(1)}{}'(k\rho)}\right|^2|x|^2\,d|x|\end{aligned}$$

uniformly on compact annulus with center 0.

Let $R = \max\left(10\rho^2 R_1, 10\frac{R_1}{\rho^2}\right)$. Then we have $B_{\rho^2 R_1} \backslash B_{\rho^2 \frac{1}{R_1}} \subset B_R \backslash B_{\frac{1}{R}}$. By (2.9) and (2.11) choose n_0 such that $O\left(\frac{1}{n}\right) < \frac{1}{10}$ and $O(1) < \frac{1}{2}n$ for $n_0 \leq n$ on the interval $[\frac{1}{R}, R]$. Indeed n_0 depends only on R_1. Then by (2.9), (2.10), (2.11), and (2.12), we obtain

$$\begin{aligned} &\left\| u^s_{n_1} - u^s_{n_2} \right\|^2_{L^2\left(B_{R_1} \backslash B_{\frac{1}{R_1}}\right)} \\ &\leq C \left\| u^i_{n_1} - u^i_{n_2} \right\|^2_{L^2\left(B_R \backslash B_{\frac{1}{R}}\right)} \text{ for } n_1 \text{ and } n_2 > n_0, \end{aligned}$$

where C depends only R, ρ and k. Similar to (2.5), (2.6), and (2.7), $\langle u^s_l \rangle$ converges to u^s in $C^2\left(B_{\frac{R_1}{3}} \backslash B_{\frac{3}{R_1}}\right)$. Since R_1 is arbitrary, u^s can be extended to a solution of Helmholtz equation in $\mathbb{R}^3 \backslash \{0\}$. □

This result is contained in the following corollary, indeed that is well-known argument in the uniqueness of obstacles of ball type (see [7]).

Proof of Corollary 2(i). If D_1 and D_2 have the different centers, then by the extensions of u^s_1 and u^s_2 and the uniqueness of continuation, u^s_1 can be extended in $\mathbb{R}^3$ with the Sommerfeld condition. But this is a contradiction for Rellich's lemma (See Lemma 6.1 in [6]). □

Proof of Corollary 2(ii). We know that D_1 and D_2 have the same center. Let $D_1 = B(0, \rho_1)$, $D_2 = B(0, \rho_2)$ and

$$u^s_j = -\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a^m_n \frac{{j_n}'(k\rho_j)}{{h^{(1)}_n}'(k\rho_j)} h^{(1)}_n(k\,|x|) Y^m_n(\hat{x}) \ for \ j = 1, 2.$$

Most of all, we will show that u^s_j is the scattered wave of D_j for $j = 1$ and 2. From the proof of theorem 1, we know that u^s_j is well defined in $\mathbb{R}^3 \backslash \{0\}$ and holds $\partial_\nu(u^i_j + u^s_j) = 0$ on ∂D_j for $j = 1$ and 2. Now we prove that u^s_1 holds the Sommerfeld radiation conditions. Let v be the radiating solution to helmholtz equation (1.1) with $\partial_\nu(u^i + v) = 0$ on ∂D_1. By the well-posedness of exterior Neumann problems, in other words, the stability of (3.54) on Theorem 3.34 in [3], there are a sequence $\langle \varphi_l \rangle$ and φ such that $\langle \varphi_l \rangle$ converges to φ in $C(\partial D_1)$, $u^s_l = D\varphi_l + iS\varphi_l$ and $v = D\varphi + iS\varphi$ when the sequence $\langle u^s_l \rangle$ is defined in the proof of theorem 1. Hence $\langle u^s_l \rangle$ converges to v in C^2 on a subdomain of

$B_{\frac{R_1}{3}} \backslash \overline{D}_1$. By the uniqueness of continuation, $u_1^s = v$ in $\mathbb{R}^3 \backslash D_1$. Hence u_j^s holds the Sommerfeld radiation condition for $j = 1$ and 2. At last, compare $\int_{S^2} u_1^s(x) Y_n^m(x) d\sigma(x)$ with $\int_{S^2} u_2^s(x) Y_n^m(x) d\sigma(x)$. By (2.9), we can conclude $\rho_1 = \rho_2$. □

Proof of Corollary 3. Let D_1 and D_2 be the sound hard balls $B(x_0, \rho_0)$ and $B(x_1, \rho_1)$. Assume D_1 and D_2 have the same scattering amplitude corresponding to a single direction $d \in S^2$. Then the scattering waves of D_1 and D_2 is the same in the exterior of $D_1 \cup D_2$. Let u^i be $\exp(ik(x) \cdot d)$, so called by *the time harmonic acoustic plane wave* for d. Then we have

$$u^i = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\exp(ikx_l \cdot d) i^n j_n(k\,|x - x_l|) \overline{Y_n^m(d)} \right) \times Y_n^m(\widehat{x - x_l}) \text{ for } l = 0, 1.$$

By Theorem 2.8 in [4], u^i has nonzero a_n^m at infinite terms in expansions with respect to spherical wave function for x_0 and x_1 and by (2.9) the scattering waves are nonzero. Hence by corollary 2 we have $D_1 = D_2$. □

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