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BOUNDARY DETERMINATION OF CONDUCTIVITIES AND RIEMANNIAN METRICS VIA LOCAL DIRICHLET-TO-NEUMANN OPERATOR*

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Abstract. We consider the inverse problem to identify an anisotropic conductivity from the Dirichlet-to-Neumann (DtN) map. We first find an explicit reconstruction of the boundary value of less regular anisotropic (transversally isotropic) conductivities and their derivatives. Based on the reconstruction formula, we prove Hölder stability, up to isometry, of the inverse problem using a local DtN map.

Key words. inverse boundary value problem, Dirichlet-to-Neumann map, anisotropic conductivity, boundary determination

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1. Introduction and statements of results.

The results of this paper are twofold. We first find an explicit reconstruction of boundary values of anisotropic conductivities and their derivatives. We then derive Hölder stability estimates for the inverse problem to identify Riemannian metrics (up to isometry) on the boundary via the local Dirichlet-to-Neumann (DtN) map using the same ideas and methods.

Boundary reconstruction. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with the smooth boundary. If we recover the conductivity up to mth derivatives, then it is enough to assume that $\partial\Omega$ is C^{m+2}-smooth. We consider the inverse problem of identifying the positive definite symmetric matrix $\gamma = (\gamma^{ij})$ entering the equation

$$L_\gamma u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega \tag{1.1}$$

by the DtN map. The DtN map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is defined to be

$$\langle \Lambda_\gamma f, h\rangle = \int_\Omega (\gamma\nabla u)\cdot\nabla v dx, \quad f, h \in H^{1/2}(\partial\Omega),$$

where $u \in H^1(\Omega)$ is the solution to (1.1) with the Dirichlet data $u|_{\partial\Omega} = f$, and $v \in H^1(\Omega)$ is such that $v|_{\partial\Omega} = h$. Here $\langle\ ,\ \rangle$ denotes the $H^{-1/2}(\partial\Omega)$-$H^{1/2}(\partial\Omega)$ pairing.

In this paper we are first concerned with an explicit reconstruction of the conductivity at the boundary. Quite recently, Nakamura and Tanuma [13, 14] obtained an explicit formula to reconstruct conductivity and its normal derivatives at the boundary. We will discuss more about their formula since the first main result of this paper is an improvement of it.

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Let us suppose that $\partial\Omega$ is flat around $x_0 = 0 \in \partial\Omega$, namely, there exists $\delta > 0$ such that

$$\Omega \cap B_\delta(0) = \{x = (x', x_n) \in B_\delta(0) \mid x_n > 0\}, \tag{1.2}$$

and in $\overline{\Omega} \cap B_\delta(0)$, γ is given by

$$\gamma = \begin{pmatrix} & & & 0 \\ & \gamma^{ij} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \gamma^{nn} \end{pmatrix}. \tag{1.3}$$

In fact, using the boundary normal coordinates [9], we can locally transform the general conductivity γ to one of the form (1.2). One can even take $\gamma^{nn} = 1$. Here we keep γ^{nn} in order to see how much we can recover.

Let $t' \in \mathbb{R}^{n-1}$, i.e., $(t', 0)$ is a tangent vector to $\partial\Omega$ at x_0. Let $\eta(x') \in C_0^\infty(\mathbb{R}^{n-1})$ be such that

$$0 \le \eta \le 1, \quad \|\eta\|_{L^2} = 1, \quad \text{supp } \eta \subset \{|x'| < 1\}.$$

For each large positive integer N, let

$$\phi_N(x') = \exp(iNx' \cdot t')\eta(N^{1/2}x'), \tag{1.4}$$

and for $z \in \partial\Omega \cap B_\delta(0)$, let

$$\phi_N^z(x') = \phi_N(x' - z'). \tag{1.5}$$

The function ϕ_N plays the role of Dirichlet data and test functions. Observe that ϕ_N oscillates rapidly as N becomes large. Kohn and Vogelius first used rapidly oscillating boundary data in their proof of uniqueness of the boundary determination [6]. The use of explicit functions such as ϕ_N for boundary reconstruction is due to Brown [3] and Nakamura and Tanuma [13].

Let $\gamma^k \in C^m(\overline{\Omega})$ be an anisotropic conductivity such that

$$\gamma^k(x) := \gamma(x', 0) + \partial_n\gamma(x', 0)x_n + \cdots + \frac{1}{(k-1)!}\partial_n^{k-1}\gamma(x', 0)x_n^{k-1} \tag{1.6}$$

near $\partial\Omega \cap B_\delta(0)$. Let

$$C_\gamma(z) := \sqrt{\gamma^{nn}(z)^{-1} \sum_{i,j=1}^{n-1} \gamma^{ij}(z)t_i t_j}. \tag{1.7}$$

Nakamura and Tanuma proved that for $k \le \frac{m}{2}$,

$$\begin{aligned} &\lim_{N\to\infty} N^{\frac{n-3}{2}+k}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_N^z, \overline{\phi_N^z}\rangle \\ &\quad = C_k C_\gamma(z)^{-a_n-1}\left(\sum_{i,j=1}^{n-1} \partial_{x_n}^k\gamma^{ij}(z)t_i t_j + C_\gamma(z)^2\partial_{x_n}^k\gamma^{nn}(z)\right) \end{aligned} \tag{1.8}$$

for some explicit constant C_k. (Even if they wrote the formula only for the isotropic γ, the proof gives (1.8).) If γ is isotropic and $|t'| = 1$, then $C_\gamma \equiv 1$, and hence the formula (1.8) reads

$$\lim_{N\to\infty} N^{\frac{n-3}{2}+k}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_N^z, \overline{\phi_N^z}\rangle = 2C_k\partial_{x_n}^k\gamma(z). \tag{1.9}$$

However, the reconstruction formula (1.8) is valid only for $k \le \frac{m}{2}$. Moreover, in the inductive reconstruction (1.8) or (1.9), it is required to know $\partial_{x_n}^{k-1}\gamma(x)$ for all $x \in \partial\Omega \cap B_\delta(0)$ in order to recover $\partial_{x_n}^k\gamma(z)$. It is also worth noting that in the formula (1.8) or (1.9), stable recovery of the tangential derivatives does not seem possible: If we take $N^{\frac{n-3}{2}+k}\langle(\Lambda_\gamma - \Lambda_{\gamma_k})\phi_N^z, \overline{\phi_N^z}\rangle$ as an approximation of $2C_k\partial_{x_n}^k\gamma(z)$ in (1.9), then its tangential derivatives do not seem to be good approximations of those of $2C_k\partial_{x_n}^k\gamma(z)$. It is our intention to improve these points.

The reason for the above-mentioned drawbacks in the formula (1.8) is that $N^{1/2}$ is used in the definition (1.4). We use instead the following boundary data:

$$\phi_N(x') = \exp(iNx'\cdot t')\eta(N^{\alpha_1}x_1, \dots, N^{\alpha_{n-1}}x_{n-1}), \tag{1.10}$$

where α_j's are specified shortly. The definition (1.10) amounts to assigning each partial differential operator $\frac{\partial}{\partial x_j}$ with the weight α_j $(j = 1, \dots, n)$ so that we can distinguish each direction x_j. The numbers α_j are chosen as follows: throughout this paper conductivities under consideration are $C^{m,p}$-smooth $(m \ge 0,\ p > 0)$, and $C^{m,p}$, m nonnegative integer and $0 \le p \le 1$, denotes the usual Hölder space. Choose λ so that $\lambda = \frac{1}{l}$ for some integer l and satisfies the following: if $m \ge 1$, then

$$\lambda < p, \quad (1 - m^{n-1}\lambda)(m + p) \ge m + \lambda. \tag{1.11}$$

We then define a multi-index α by

$$\alpha = (\alpha_1, \dots, \alpha_n) := (1 - m^{n-1}\lambda, 1 - m^{n-2}\lambda, \dots, 1 - m\lambda, 1). \tag{1.12}$$

If $m = 0$, choose λ so that $\lambda < p$ and define α by

$$\alpha = (\alpha_1, \dots, \alpha_n) := (1 - (n-1)\lambda, 1 - (n-2)\lambda, \dots, 1). \tag{1.13}$$

We choose α and λ in this way so that they possess the following properties: $|\alpha_i - \alpha_j| \ge \lambda$ if $i \ne j$. If a and b are multi-indices with $|a| \le m$ and $|b| \le m$, then $a\cdot\alpha \ne b\cdot\alpha$ if and only if $a \ne b$. Thanks to these properties, we can define a linear ordering of multi-indices: Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two multi-indices. We define $a < b$ if $a\cdot\alpha < b\cdot\alpha$. Using the linear ordering, we are able to recover γ and its derivatives inductively.

For the function η in the definition (1.10), we further assume that for each $a' = (a_1, \dots, a_{n-1})$ with $|a'| \le m$

$$\int_{|y'|\le 1} (y')^{a'}\eta(y')^2 dy' \ne 0,$$

and we define $C(a)$ for a multi-index $a = (a', a_n)$ to be

$$C(a) := \frac{1}{a!}\int_0^\infty y_n^{a_n}e^{-2y_n}dy_n \int_{|y'|\le 1}(y')^{a'}\eta(y')^2 dy'. \tag{1.14}$$

For a given anisotropic conductivity γ and a multi-index a with $|a| \leq m$, define $\gamma^{a,z}$ to be a positive definite matrix-valued smooth function on Ω such that

$$\gamma^{a,z}(x) := \sum_{b<a} \frac{\partial^b \gamma(z)}{b!}(x-z)^b \quad \text{near } z. \tag{1.15}$$

Then the DtN map $\Lambda_{\gamma^{a,z}}$ corresponding to $\gamma^{a,z}$ is well defined. If $a = 0$, let $\Lambda_{\gamma^0} = 0$. Here and throughout this paper $\partial^b \gamma$ denotes $\partial^b \gamma = \partial_{x_1}^{b_1} \cdots \partial_{x_n}^{b_n} \gamma$.

Then we have the following reconstruction formula.

THEOREM 1.1. *Suppose that* $\gamma \in C^{m,p}(\overline{\Omega} \cap B_\delta(0))$. *For* $z = (z', 0) \in \partial\Omega \cap B_\delta(0)$, *a multi-index* $a = (a', a_n)$, *and* $k \leq m$, *we have*

$$\begin{aligned} &N^{-2+|\alpha|+a\cdot\alpha} \langle (\Lambda_\gamma - \Lambda_{\gamma^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle \\ &\qquad = C(a) C_\gamma(z)^{-a_n-1} \left(\sum_{i,j=1}^{n-1} \partial^a \gamma^{ij}(z) t_i t_j + C_\gamma(z)^2 \partial^a \gamma^{nn}(z) \right) + O(N^{-\lambda}), \end{aligned} \tag{1.16}$$

where $O(N^{-\lambda})$ *is independent of* z. *If* $a = 0$, *then* $C(0) = \frac{1}{2}$, *and hence we have*

$$N^{-2+|\alpha|} \langle \Lambda_\gamma \phi_N^z, \overline{\phi_N^z} \rangle = \sqrt{\gamma^{nn}(z) \sum_{i,j=1}^{n-1} \gamma^{ij}(z) t_i t_j} + O(N^{-\lambda}). \tag{1.17}$$

The formula (1.16) says that the boundary values of γ and its derivatives up to order m can be recovered in a stable way (modulo γ^{nn}-terms).

In particular, if γ is isotropic, namely, $\gamma = \gamma(\delta_{ij})$, and $|t'| = 1$, then $C_z \equiv 1$, and hence we have the following corollary.

COROLLARY 1.2. *If* $\gamma \in C^{m,p}(\overline{\Omega} \cap B_\delta(0))$ *is an isotropic conductivity, then for all multi-index* a *with* $|a| \leq m$, *we have*

$$N^{-2+|\alpha|+a\cdot\alpha} \langle (\Lambda_\gamma - \Lambda_{\gamma^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle = C(a) \partial^a \gamma(z) + O(N^{-\lambda}). \tag{1.18}$$

We note that Brown proved a reconstruction formula for γ and the normal derivative on $\partial\Omega$ [3]. In [2], Alessandrini and Gaburro considered reconstruction of special types of anisotropic conductivity.

It turns out that a slight variance of the reconstruction (1.18) gives an interesting stability result, which is the second subject of this paper.

Boundary determination of Riemannian metrics-stability. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with the smooth boundary. We consider the inverse problem of identifying a Riemannian metric or an anisotropic conductivity at the boundary $\partial\Omega$ via the (local) DtN map. Let (g_{ij}) be a Riemannian metric on $\overline{\Omega}$ and $g = (g^{ij}) := (g_{ij})^{-1}$. Then the corresponding DtN map $\Lambda_g : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is defined to be

$$\langle \Lambda_g f, h \rangle = \int_\Omega (|g|^{-1/2} g \nabla u) \cdot \nabla v dx, \quad f, h \in H^{1/2}(\partial\Omega), \tag{1.19}$$

where $u \in H^1(\Omega)$ is the solution to the problem

$$\begin{aligned} \Delta_g u := |g|^{1/2} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(|g|^{-1/2} g^{ij} \frac{\partial u}{\partial x_j} \right) &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega, \end{aligned}$$

and $v \in H^1(\Omega)$ is such that $v|_{\partial\Omega} = h$. Here $|g|$ denotes the determinant of g.

There is a well-known obstacle in identifying g: Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a C^1 diffeomorphism which is the identity on $\partial\Omega$. Then it is well known that

$$\Lambda_{\Psi^* g} = \Lambda_g,$$

where $\Psi^* g$ is the pull-back of g. So the general conjecture of the uniqueness in three dimensions is that if $\Lambda_{g_1} = \Lambda_{g_2}$, then there exists a diffeomorphism Ψ on $\overline{\Omega}$ such that $\Psi|_{\partial\Omega}$ is the identity on $\partial\Omega$ and

$$\Psi^* g_2 = g_1.$$

Lee and Uhlmann proved the conjecture for three dimensions under some restrictions when conductivities are real analytic in $\overline{\Omega}$ [9]. Recently, Lassas and Uhlmann extended the result to the case when g is real analytic up to a portion of $\partial\Omega$ and removed the restrictions [8]. There are also similar kinds of uniqueness theorems for two dimensions. See [15, 10, 8]. When γ is a scalar function, i.e., γ is isotropic, the inverse problem has been extensively studied [4, 5, 6, 7, 11, 12, 16, 17].

In this paper we prove the following Hölder stability estimates for the boundary determination: Let Γ be an open connected subset of $\partial\Omega$. Define the localized DtN map Λ_g^Γ by

$$\Lambda_g^\Gamma(f) := \Lambda_g(f)|_\Gamma, \quad f \in H^{1/2}(\partial\Omega), \quad \operatorname{supp}(f) \subset \Gamma.$$

We will use the following notation: Let f be a C^k function in a neighborhood of a compact set K. Then $\|f\|_{C_E^k(K)} := \sum_{|\alpha|=0}^k \sup_{x\in K} |\partial^\alpha f(x)|$. So, $C_E^k(\partial\Omega)$-norm involves not only the tangential derivatives but also the normal derivative.

THEOREM 1.3. *Suppose that g_1 and g_2 are Riemannian metrics on a domain Ω such that they are $C^{m,p}$ $(m \geq 1, p > 0)$ in a neighborhood of Γ and the $C^{m,p}$-norms are bounded by M and*

$$g_j\xi \cdot \xi \geq A|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \quad (j = 1, 2). \tag{1.20}$$

Let K be a compact subset of Γ. Then there are a neighborhood U of Γ, a C^m diffeomorphism Ψ in $U \cap \overline{\Omega}$ with $\Psi|_\Gamma$ = Identity, and a positive constant $C = C(m, p, \Gamma, K, A, M)$ such that for $k = 0, 1, \ldots, m$,

$$\|g_2 - \Psi^* g_1\|_{C_E^k(K)} \leq C \|\Lambda_{g_1}^\Gamma - \Lambda_{g_2}^\Gamma\|^{2^{-k/\lambda}}. \tag{1.21}$$

The norm on the right-hand side of (1.21) *is the operator norm from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.*

Thus the Riemannian metrics can be recovered at the boundary in a stable way via the local DtN map.

Using the boundary normal coordinates, we may assume that $\partial\Omega$ is flat around $x_0 = 0 \in \partial\Omega$, and g is given by

$$g = \begin{pmatrix} & & & 0 \\ & g^{ij} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{1.22}$$

We prove stability estimates for the Riemannian metrics of the form (1.22) by using methods similar to Theorem 1.1. Then Theorem 1.3 follows.

As was observed in [9], if we take

$$g_{ij} := |\gamma|^{1/(n-2)}\gamma^{-1} \quad (n \geq 3),$$

then $\gamma = |g|^{-1/2}g$ $(g = (g_{ij})^{-1})$ and $\Lambda_g = \Lambda_\gamma$. Hence, we have the same stability estimates for anisotropic conductivities.

A similar proof yields the following stability for the boundary determination of isotropic conductivities.

Theorem 1.4. *Suppose that* γ_1 *and* γ_2 *are isotropic conductivities. Let* m, p, Γ, K, A, M $(m \geq 0)$ *be as before. Then there exists a positive constant* $C = C(m, \Gamma, K, A, M)$ *such that for* $k = 0, 1, \dots, m$,

$$\|\gamma_2 - \gamma_1\|_{C^k_E(K)} \leq C\|\Lambda^\Gamma_{\gamma_1} - \Lambda^\Gamma_{\gamma_2}\|^{2^{-k/\lambda}}. \tag{1.23}$$

Alessandrini proved the following stability for isotropic conductivities using singular solutions [1] (see also [18]):

$$\|\gamma_2 - \gamma_1\|_{C^k_E(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\frac{1}{2k+1}}.$$

This stability estimate is better than (1.23). However, the stability estimate (1.23) uses the *local* DtN map.

This paper is organized as follows: In section 2, we construct approximate solutions of $L_\gamma u = 0$ with $u|_{\partial\Omega} = \phi^z_N$ on which the proof of Theorem 1.1 is based. The proof of Theorem 1.1 is given in section 3. Theorem 1.3 is proved in section 4.

2. Approximate solutions. Suppose that Ω and γ are of the forms (1.2) and (1.3) and that $\gamma \in C^{m,p}(\overline{\Omega} \cap B_\delta(0))$ for some integer $m \geq 0$. Let

$$\Omega_N := \left\{ x \mid |x_j| < N^{-\alpha_j} \ (j = 1, \dots, n-1),\ 0 \leq x_n < \frac{1}{\sqrt{N}} \right\}.$$

The following lemma and its proof are based on an idea in [13].

Lemma 2.1. *For each integer* N *there is an approximate solution* Φ_N *of the form*

$$\Phi_N(x) = \exp(iNx' \cdot t') \exp(-C_\gamma(z)Nx_n) \sum_{k=0}^{m/\lambda} N^{-k\lambda} v_k(y) \tag{2.1}$$

$$(y_j = N^{\alpha_j} x_j, \quad j = 1, \dots, n), \tag{2.2}$$

where $v_0(y', y_n) = \eta(y')$, *and* $v_l(y', y_n)$ *are polynomials of* y_n *whose coefficients are* C^∞ *functions of* y' *compactly supported in* $\{|y'| < 1\}$, *so that* Φ_N *satisfies*

$$\Phi_N|_{\partial\Omega} = \phi^z_N$$

and

$$|\nabla \cdot (\gamma \nabla \Phi_N)(x)| \leq CN^{(2-m)-\lambda} p(y_n) e^{-C_\gamma(z) y_n} \quad \text{for all } x \in \Omega_N \tag{2.3}$$

for some constant $C = C(m)$. *Here* $p(y_n)$ *is a polynomial with positive coefficients.*

Proof. Without loss of generality, assume that $z = 0$. Put $C_0 := C_\gamma(0)$. We seek a solution $\Phi_N(x)$ of the form

$$\Phi_N(x) = \exp(iNx' \cdot t') V(N^{\alpha_1} x_1, \dots, N^{\alpha_n} x_n).$$

Then straightforward computations show that

$$\begin{aligned}\nabla_x(\gamma(\nabla_x\Phi_N)) &= \sum_{i,j=1}^{n}\partial_{x_i}(\gamma^{ij}\partial_{x_j}\Phi_N)\\ &= \left[\sum_{i,j=1}^{n-1}\gamma^{ij}\partial_{x_i}\partial_{x_j}+\gamma^{nn}\partial_{x_n}^2+\sum_{i,j=1}^{n-1}\partial_{x_i}\gamma^{ij}\partial_{x_j}+\partial_{x_n}\gamma^{nn}\partial_{x_n}\right]\Phi_N\\ &= \exp\left(iNx'\cdot t'\right)\Bigg[-N^2\sum_{i,j=1}^{n-1}\gamma^{ij}t_it_j+\sqrt{-1}N\sum_{i,j=1}^{n-1}\gamma^{ij}(t_i\partial_{x_j}+t_j\partial_{x_i})\\ &\quad+\sum_{i,j=1}^{n-1}\gamma^{ij}\partial_{x_i}\partial_{x_j}+\gamma^{nn}\partial_{x_n}^2+\sqrt{-1}N\sum_{i,j=1}^{n-1}(\partial_{x_i}\gamma^{ij})t_j\\ &\quad+\sum_{i,j=1}^{n-1}(\partial_{x_i}\gamma^{ij})\partial_{x_j}+(\partial_{x_n}\gamma^{nn})\partial_{x_n}\Bigg]V(N^{\alpha_1}x_1,\dots,N^{\alpha_n}x_n).\end{aligned}$$

After the scaling (2.2), we have $\partial_{x_j}=N^{\alpha_j}\partial_{y_j}$ $(i=1,2,\dots,n)$, and hence

$$\begin{aligned}\nabla_x(\gamma(\nabla_x\Phi_N)) = \exp(iNx'\cdot t')\Bigg[&N^2\left(\gamma^{nn}\partial_{y_n}^2-\sum_{i,j=1}^{n-1}\gamma^{ij}t_it_j\right)\\ &+2\sum_{j=1}^{n-1}N^{1+\alpha_j}\left(\sum_{i=1}^{n-1}\gamma^{ij}t_i\right)\partial_{y_j}\\ &+\sum_{i,j=1}^{n-1}N^{\alpha_i+\alpha_j}\gamma^{ij}\partial_{y_i}\partial_{y_j}\\ &+N\left(\sqrt{-1}\sum_{i,j=1}^{n-1}(\partial_{x_i}\gamma^{ij})t_j+(\partial_{x_n}\gamma^{nn})\partial_{y_n}\right)\\ &+\sum_{j=1}^{n-1}N^{\alpha_j}\left(\sum_{i=1}^{n-1}\partial_{x_i}\gamma^{ij}\right)\partial_{y_j}\Bigg]V(y',y_n).\end{aligned}\tag{2.4}$$

Note that all the powers of N in the formula (2.4) are of the form $2-k\lambda$ for some integer k with $0\le k\le 2/\lambda$.

We now expand γ in Taylor series in Ω_N:

$$\gamma(x)=\sum_{|a|\le m}\frac{1}{a!}\partial^a\gamma(0)x^a+O(|x|^{m+p}).$$

By the condition (1.11) imposed on λ, we have

$$\alpha_1(m+p)\ge m+\lambda.\tag{2.5}$$

Thus, after the scaling (2.2), we have

$$\gamma(x)=\sum_{|a|\le m}\frac{1}{a!}\partial^a\gamma(0)N^{-\alpha\cdot a}y^a+E_1(y),\tag{2.6}$$

where

$$|E_1(y)| \le CN^{-\alpha_1(m+p)} \sum_{k=0}^{m+1} y_n^k \le CN^{-m-\lambda} \sum_{k=0}^{m+1} y_n^k. \tag{2.7}$$

Similarly, we have, for $j = 1, 2, \ldots, n$,

$$\partial_{x_j}\gamma(x) = \sum_{|a| \le m-1} \frac{1}{a!} \partial^a \partial_{x_j}\gamma(0) N^{-\alpha \cdot a} y^a + E_2(y), \tag{2.8}$$

where

$$|E_2(y)| \le CN^{-m+1-\lambda} \sum_{k=0}^{m} y_n^k. \tag{2.9}$$

Note that this expansion holds uniformly for all $x \in \Omega_N$ and hence for all $y \in \{|y'| < 1,\ 0 \le y_n \le N^{\frac{1}{2}}\}$. Note also that the powers of N in the expansions in (2.6) and (2.8) are of the form $-k\lambda$ for some integer k.

It then follows from (2.4), (2.6), and (2.8) that

$$\nabla_x(\gamma(\nabla_x \Phi_N)) = \exp(iNx' \cdot t') \left[\sum_{k=0}^{m/\lambda} N^{2-k\lambda} L_k + L_R \right] V(y', y_n), \tag{2.10}$$

where L_k are at most second order differential operators in y' and y_n whose coefficients are polynomials in y' and y_n, and L_R is also a second order differential operator in y' and y_n whose coefficients are of the form $O(N^{2-m-\lambda}) \times$ polynomial in y_n with positive coefficients, and

$$L_0 = \gamma^{nn}(0)\partial_{y_n}^2 - \sum_{i,j=1}^{n-1} \gamma^{ij}(0) t_i t_j. \tag{2.11}$$

We look for $V(y', y_n)$ of the form

$$V(y', y_n) = \sum_{k=0}^{m/\lambda} N^{-k\lambda} V_k.$$

We have from (2.10) that

$$\begin{aligned} \nabla_x(\gamma(\nabla_x \Phi_N)) &= \exp(iNx' \cdot t') \left[\left(\sum_{k=0}^{m/\lambda} N^{2-k\lambda} L_k \right) \left(\sum_{j=0}^{m/\lambda} N^{-j\lambda} V_j \right) + L_R V \right] \\ &= \exp(iNx' \cdot t') \left[\sum_{l=0}^{m/\lambda} N^{2-l\lambda} \sum_{k+j=l} L_k V_j + E \right], \end{aligned} \tag{2.12}$$

where

$$E := \sum_{l=m/\lambda+1}^{2m/\lambda} N^{2-l\lambda} \sum_{k+j=l} L_k V_j + L_R V. \tag{2.13}$$

We solve the system of differential equations

$$\sum_{k+j=l} L_k V_j = 0 \quad (l = 0, 1, 2, \dots, m/\lambda), \tag{2.14}$$

namely,

$$\begin{aligned} L_0 V_0 &= 0, \\ L_0 V_1 + L_1 V_0 &= 0, \\ &\cdots \\ L_0 V_{m/\lambda} + \cdots + L_{m/\lambda} V_0 &= 0, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} V_0|_{y_n=0} &= \eta_N(x') = \eta(y'), \\ V_l|_{y_n=0} &= 0 \quad (l = 1, \dots, m/\lambda). \end{aligned}$$

We remark that this boundary value problem is underdetermined. Because of (2.11), this system of equations can be solved iteratively from top to bottom:

$$\begin{aligned} V_0(y', y_n) &= \eta(y') \exp(-C_0 y_n), \\ V_1(y', y_n) &= \sum_{k=0}^{1} P_1^k(y') y_n^k \exp(-C_0 y_n), \\ &\cdots \\ V_{m/\lambda}(y', y_n) &= \sum_{k=1}^{m/\lambda} P_{m/\lambda}^k(y') y_n^k \exp(-C_0 y_n) \end{aligned}$$

for some N_j $(j = 1, \dots, m/\lambda)$, where $P_j^k(y')$ are C^∞ functions supported in $\{|y'| < 1\}$. It then follows from (2.12) that

$$\nabla_x(\gamma(\nabla_x \Phi_N)) = \exp(iNx' \cdot t')E.$$

Recall that the coefficients of L_R are of the form $O(N^{2-m-\lambda}) \times$ polynomial in y_n. Thus there exists $C = C(m)$ such that

$$|E| \le CN^{2-m-\lambda} p(y_n) \exp(-C_0 y_n)$$

for some polynomial p. This completes the proof. ∎

The following lemma can be proved by straightforward computations. Recall that ϕ_N is defined in (1.10).

LEMMA 2.2. *For each $s \ge 0$, there exists a constant C_s such that*

$$\|\phi_N\|_{H^s(\partial\Omega)} + \|\Phi_N\|_{H^{s+1/2}(\Omega_N)} \le C_s N^{s+\frac{1}{2}-\frac{|\alpha|}{2}}. \tag{2.15}$$

For each multi-index a, there exists a constant C_a such that

$$\|x^a \nabla \Phi_N\|_{L^2(\Omega_N)} \le C_a N^{1-a\cdot\alpha-\frac{|\alpha|}{2}}. \tag{2.16}$$

3. Proof of Theorem 1.1. In this section we prove Theorem 1.1. Our proof is parallel to that of [13].

Without loss of generality we assume that $z = 0$. Put $C_0 := C_\gamma(0)$ for convenience. Let $\zeta(x_n) \in C^\infty([0,\infty))$ be such that $\zeta(x_n) = 1$ for $0 \le x_n \le 1/2$ and 0 for $1 \le x_n$. Put

$$\zeta_N(x_n) = \zeta(\sqrt{N} x_n).$$

Let a be a multi-index such that $|a| \le m$. Let $u_N \in H^1(\Omega)$ be the solution of

$$\begin{aligned} \nabla_x(\gamma \nabla_x u_N) = 0 \quad \text{in } \Omega, \\ u_N|_{\partial\Omega} = \phi_N, \end{aligned}$$

and let $v_N \in H^1(\Omega)$ be the solution of

$$\begin{aligned} \nabla_x(\gamma^a \nabla_x v_N) = 0 \quad \text{in } \Omega, \\ v_N|_{\partial\Omega} = \phi_N. \end{aligned}$$

Here $\gamma^a = \gamma^{a,0}$. Let Φ_N and Ψ_N be the extensions of ϕ_N given in Lemma 2.1 corresponding to γ and γ^a, respectively. Note that since $\langle \Lambda_{\gamma^a}\phi_N, \overline{\phi_N}\rangle$ is real, we have

$$\langle \Lambda_{\gamma^a}\phi_N, \overline{\phi_N}\rangle = \langle \overline{\Lambda_{\gamma^a}\phi_N}, \phi_N\rangle,$$

and hence

$$\begin{aligned} &\langle (\Lambda_\gamma - \Lambda_{\gamma^a})\phi_N, \overline{\phi_N}\rangle \\ &= \int_\Omega (\gamma\nabla_x u_N)\cdot\nabla_x\overline{(\zeta_N\Psi_N)}dx - \int_\Omega (\gamma^a\nabla_x\overline{v_N})\cdot\nabla_x(\zeta_N\Phi_N)dx. \end{aligned}$$

Put

$$u_N := \Phi_N + s_N \quad \text{and} \quad v_N := \Psi_N + r_N. \tag{3.1}$$

Then we have

$$\begin{aligned} &\langle (\Lambda_\gamma - \Lambda_{\gamma^a})\phi_N, \overline{\phi_N}\rangle \\ &= \int_\Omega \Big[(\gamma\nabla_x\Phi_N)\cdot\nabla_x\overline{(\zeta_N\Psi_N)} - (\gamma^a\nabla_x\overline{\Psi_N})\cdot\nabla_x(\zeta_N\Phi_N)\Big]dx \\ &\quad + \int_\Omega (\gamma\nabla_x s_N)\cdot\nabla_x\overline{(\zeta_N\Psi_N)}dx - \int_\Omega (\gamma^a\nabla_x\overline{r_N})\cdot\nabla_x(\zeta_N\Phi_N)dx \\ &:= I + II + III. \end{aligned} \tag{3.2}$$

We estimate I, II, and III separately.

Estimates of *I*. Put

$$\Omega_N' := \left\{x:\ |x'| \le N^{-\alpha_j}\ (j = 1,\dots,n-1),\ \frac{1}{2\sqrt{N}} \le x_n \le \frac{1}{\sqrt{N}}\right\}.$$

Since $\zeta_N = 1$ on $0 \le x_n \le \frac{1}{2\sqrt{N}}$, we can rewrite I as

$$\begin{aligned} I &= \int_{\Omega_N\setminus\Omega_N'} (\gamma - \gamma^a)\nabla_x\Phi_N\cdot\nabla_x\overline{\Psi_N}dx \\ &\quad + \int_{\Omega_N'} \Big[(\gamma\nabla_x\Phi_N)\cdot\nabla_x\overline{(\zeta_N\Psi_N)} - (\gamma^a\nabla_x\overline{\Psi_N})\cdot\nabla_x(\zeta_N\Phi_N)\Big]dx \\ &:= I_1 + I_2. \end{aligned}$$

By (2.1), there exists a constant C such that

$$|\Psi_N| + |\Phi_N| \le C\exp(-C_0 N x_n).$$

Therefore it is easy to see that

$$|I_2| \le C\exp\left(-\frac{C_0}{2}N^{\frac{1}{2}}\right). \tag{3.3}$$

From (2.1), we get

$$\nabla_x \Phi_N = \left[N\begin{pmatrix} it' \\ -C_0\end{pmatrix}\exp\left(iNx'\cdot t'\right)\eta_N(x') + O(N^{1-\lambda})\right]\exp\left(-C_0 N x_n\right).$$

Likewise, we have

$$\nabla_x \Psi_N = \left[N\begin{pmatrix} it' \\ -C_0\end{pmatrix}\exp\left(iNx'\cdot t'\right)\eta_N(x') + O(N^{1-\lambda})\right]\exp\left(-C_0 N x_n\right).$$

It then follows that

$$\begin{aligned} I_1 = N^2 \int_0^{\frac{1}{2\sqrt{N}}} \int_{|x'|\le\frac{1}{\sqrt{N}}} & \left((\gamma(x)-\gamma^a(x))\begin{pmatrix} it' \\ -C_0\end{pmatrix}\right)\cdot\begin{pmatrix} -it' \\ -C_0\end{pmatrix} \\ & \times e^{-2C_0Nx_n}\eta_N(x')^2dx'dx_n \\ & + O(N^{2-\lambda})\int_0^{\frac{1}{2\sqrt{N}}}\int_{|x'|\le\frac{1}{\sqrt{N}}} e^{-2C_0Nx_n}|\gamma(x)-\gamma^a(x)|dx'dx_n. \end{aligned}$$

Note that if two multi-indices a and b satisfy $a<b$, then

$$a\cdot\alpha + \lambda \le b\cdot\alpha.$$

Thus, applying the change of variables $y_j = N^{\alpha_j}x_j$, $j = 1,\dots,n$, we obtain

$$\begin{aligned} \gamma(x)-\gamma^a(x) &= \frac{\partial^a\gamma(0)}{a!}x^a + \sum_{b>a,|b|\le m}\frac{\partial^b\gamma(0)}{b!}x^b + O(|x|^{m+1}) \\ &= \frac{\partial^a\gamma(0)}{a!}N^{-a\cdot\alpha}y^a + O(N^{-a\cdot\alpha-\lambda})\sum_{k=0}^{m+1}y_n^k. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_1 &= \frac{N^{2-a\cdot\alpha-|\alpha|}}{a!}\int_0^{\frac{\sqrt{N}}{2}}\int_{|y'|\le1} y^a\partial^a\gamma(0)\begin{pmatrix} it' \\ -C_0\end{pmatrix}\cdot\begin{pmatrix} -it' \\ -C_0\end{pmatrix}e^{-2C_0y_n}\eta(y')^2dy'dy_n \\ &\quad + O(N^{2-a\cdot\alpha-\lambda-|\alpha|})\int_0^{\frac{\sqrt{N}}{2}}\int_{|y'|\le1} y^a e^{-2C_0y_n}[\eta(y')^2+1]\sum_{k=0}^{m+1}y_n^k dy'dy_n \\ &= N^{2-a\cdot\alpha-|\alpha|}\left(\sum_{i,j=1}^{n-1}\partial^a\gamma^{ij}(0)t_it_j + C_0^2\partial^a\gamma^{nn}(0)\right) \\ &\qquad \times\frac{1}{a!}\int_0^{\infty}\int_{|y'|\le1} y^a e^{-2C_0y_n}\eta(y')^2dy'dy_n + O(N^{2-a\cdot\alpha-\lambda-|\alpha|}). \end{aligned}$$

From these estimates and (3.3), we obtain

$$N^{-2+a\cdot\alpha+|\alpha|}I = C(a)C_0^{-a_n-1}\left(\sum_{i,j=1}^{n-1}\partial^a\gamma^{ij}(0)t_it_j + C_0^2\partial^a\gamma^{nn}(0)\right) + O(N^{-\lambda}), \tag{3.4}$$

where $C(a)$ is the quantity defined in (1.14).

Estimates of II and III. We prove that

$$|II| + |III| \le CN^{2-m-|\alpha|-\lambda}. \tag{3.5}$$

Once (3.5) is proved, then Theorem 1.1 follows from (3.2), (3.4), and (3.5).

We only give the proof of (3.5) for II. Equation (3.5) for III can be proved in the same way. Note that

$$\begin{aligned}
II &= \int_\Omega (\gamma\nabla_x s_N)\cdot\overline{\nabla(\zeta_N\Psi_N)}dx \\
&= \int_{\Omega_N\setminus\Omega_N'}(\gamma\nabla_x s_N)\cdot\nabla_x\overline{\Psi_N}dx + \int_{\Omega_N'}(\gamma\nabla_x s_N)\cdot\nabla_x(\overline{\zeta_N\Psi_N})dx \\
&= \int_{\Omega_N\setminus\Omega_N'}(\gamma\nabla_x s_N)\cdot\nabla_x\overline{\Phi_N}dx + \int_{\Omega_N\setminus\Omega_N'}(\gamma\nabla_x s_N)\cdot\nabla_x\overline{(\Psi_N-\Phi_N)}dx \\
&\quad + \int_{\Omega_N'}(\gamma\nabla_x s_N)\cdot\nabla_x(\overline{\zeta_N\Psi_N})dx \\
&:= II_1 + II_2 + II_3.
\end{aligned}$$

In the same way as for (3.3), one can show that

$$|II_3| \le C\exp\left(-\frac{C_0}{4}N^{\frac{1}{2}}\right). \tag{3.6}$$

We now estimate II_1. Set $D_N := \Omega_N\setminus\Omega_N'$ for convenience. By the definition (3.1) of s_N and Lemma 2.2, we have

$$\begin{aligned}
\|s_N\|_{H^1(D_N)} &\le \|\Phi_N\|_{H^1(D_N)} + \|u_N\|_{H^1(D_N)} \\
&\le \|\Phi_N\|_{H^1(D_N)} + \|\phi_N\|_{H^{1/2}(\partial\Omega)} \\
&\le CN^{1-\frac{|\alpha|}{2}}.
\end{aligned} \tag{3.7}$$

Put $\Gamma_1 := \{|x_j| = N^{-\alpha_j}$ for some j $(j=1,\dots,n-1),\ 0\le x_n\le\frac{1}{2\sqrt{N}}\}$ and $\Gamma_2 := \{|x_j|\le N^{-\alpha_j}$ $(j=1,\dots,n-1),\ x_n=\frac{1}{2\sqrt{N}}\}$. Then $\partial D_N = \Gamma\cup\Gamma_1\cup\Gamma_2$. Since

$$s_N\left(x',\frac{1}{2\sqrt{N}}\right) = \int_0^{\frac{1}{2\sqrt{N}}}\partial_{x_n}s_N(x',t)dt,$$

it follows from the Cauchy–Schwarz inequality and (3.7) that

$$\|s_N\|_{L^2(\Gamma_2)} \le \left(\frac{1}{2\sqrt{N}}\right)^{1/2}\|s_N\|_{H^1(D_N)} \le CN^{\frac{3}{4}-\frac{|\alpha|}{2}}. \tag{3.8}$$

Observe that $\nabla_x\Phi_N|_{\Gamma_1} = 0$ and $\nabla_x\Phi_N|_{\Gamma_2} = O(e^{-\frac{1}{2}C_0N^{1/2}})$. Since $s_N = 0$ on Γ, it follows from the divergence theorem that

$$\begin{aligned} II_1 &= \int_{D_N} (\gamma\nabla_x s_N)\cdot\nabla\overline{\Phi_N}dx \\ &= -\int_{D_N} s_N\nabla_x\left(\gamma\nabla_x\overline{\Phi_N}\right)dx + O(e^{-\frac{1}{4}C_0N^{1/2}}). \end{aligned}$$

Note that

$$\left|\int_{D_N} s_N\nabla_x\left(\gamma\nabla_x\overline{\Phi_N}\right)dx\right| \leq \|x_n\nabla_x(\gamma\nabla_x\overline{\Phi_N})\|_{L^2(D_N)}\|x_n^{-1}s_N\|_{L^2(D_N)}.$$

By the Hardy inequality and (3.7), we have

$$\|x_n^{-1}s_N\|_{L^2(D_N)} \leq C\|s_N\|_{H^1(D_N)} \leq CN^{1-\frac{|\alpha|}{2}}.$$

On the other hand, we obtain from (2.3) that

$$\begin{aligned} \|x_n\nabla_x(\gamma\nabla_x\overline{\Phi_N})\|_{L^2(D_N)} &\leq CN^{2-m-\lambda}\|x_n p(y_n)e^{-C_0y_n}\|_{L^2(D_N)} \\ &\leq CN^{1-m-\lambda-\frac{|\alpha|}{2}}. \end{aligned} \tag{3.9}$$

It thus follows that

$$|II_1| \leq CN^{2-m-|\alpha|-\lambda}.$$

We now estimate II_2. Note that

$$(\Phi_N - \Psi_N)|_{\Gamma\cup\Gamma_1} = 0, \qquad (\Phi_N - \Psi_N)|_{\Gamma_2} = O\left(e^{-\frac{1}{2}C_0N^{1/2}}\right).$$

Hence, an integration by parts yields

$$II_2 = -\int_{D_N}\nabla_x(\gamma\nabla_x s_N)(\Psi_N - \Phi_N)dx + O(e^{-\frac{1}{4}C_0N^{1/2}}).$$

Since $\nabla\cdot(\gamma\nabla u_N) = 0$, we have

$$II_2 = \int_{D_N}\nabla_x(\gamma\nabla_x\Phi_N)(\Psi_N - \Phi_N)dx + O(e^{-\frac{1}{4}C_0N^{1/2}}).$$

In the same way as for II_1, one can show that

$$|II_2| \leq CN^{2-m-|\alpha|-\lambda}.$$

This completes the proof of (3.5).

4. Proof of Theorem 1.3. In this section we prove Theorem 1.3. Suppose that Ω and g are of the form (1.2) and (1.22). For such a metric g and a multi-index a with $|a| \leq m$, define $g^{a,z}$ to be a positive definite symmetric matrix-valued smooth function on Ω such that

$$g^{a,z}(x) := \sum_{b<a}\frac{\partial^b g(z)}{b!}(x-z)^b \tag{4.1}$$

near z. We then define $\Lambda_{g^{a,z}}$ by

$$\langle \Lambda_{g^{a,z}} f, h\rangle = \int_\Omega (|g|^{-1/2} g^{a,z} \nabla u)\cdot \nabla v dx, \quad f, h \in H^{1/2}(\partial\Omega), \tag{4.2}$$

where $u \in H^1(\Omega)$ is the solution to the problem

$$\begin{aligned} \nabla \cdot (|g|^{-1/2} g^{a,z} \nabla u) = 0 \quad &\text{in } \Omega, \\ u = f \quad &\text{on } \partial\Omega, \end{aligned}$$

and $v \in H^1(\Omega)$ is such that $v|_{\partial\Omega} = h$. If $a = 0$, let $\Lambda_{g_0} = 0$. Note that $\Lambda_{g^{a,z}}$ is *not* a DtN map corresponding to an invariant Laplacian. Consider it as a DtN map corresponding to a divergence equation.

THEOREM 4.1. *Suppose that* $g \in C^{m,p}(\overline{\Omega} \cap B_\delta(0))$. *For* $z = (z', 0) \in \partial\Omega \cap B_\delta(0)$, *let*

$$C_g(z) := \sqrt{\sum_{i,j=1}^{n-1} g^{ij}(z) t_i t_j}. \tag{4.3}$$

Then for a multi-index $a = (a', a_n)$ *and* $k \le m$, *we have*

$$\begin{aligned} &N^{-2+|\alpha|+a\cdot\alpha} \langle (\Lambda_g - \Lambda_{g^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle \\ &\quad = C(a) C_g(z)^{-a_n-1} |g(z)|^{-1/2} \sum_{i,j=1}^{n-1} \partial^a g^{ij}(z) t_i t_j + O(N^{-\lambda}), \end{aligned} \tag{4.4}$$

where $O(N^{-\lambda})$ *is independent of* z. *In particular, when* $a = 0$, $C(0) = \frac{1}{2}$, *and hence we have*

$$N^{-2+|\alpha|} \langle \Lambda_g \phi_N^z, \overline{\phi_N^z}\rangle = 2\sqrt{|g(z)|^{-1} \sum_{i,j=1}^{n-1} g^{ij}(z) t_i t_j} + O(N^{-\lambda}). \tag{4.5}$$

Despite a slight difference between Theorem 4.1 and Theorem 1.1, it can be proved in the same way, and so we omit the proof. We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose first that g_1 and g_2 are of the forms (1.22) ($j = 1, 2$) and Ω is of the form (1.2). Let K be a compact subset of Γ such that $\text{dist}(K, \partial\Gamma) > \delta_0$ for some $\delta_0 > 0$. If N is large enough so that $\text{supp}\phi_N^z \subset \Gamma$ for all $z \in K$, then we have from (2.15) that

$$\begin{aligned} N^{-2+|\alpha|} |\langle (\Lambda_1 - \Lambda_2)\phi_N^z, \overline{\phi_N^z}\rangle| &\le N^{-2+|\alpha|} \|\Lambda_1 - \Lambda_2\| \|\phi_N^z\|^2_{H^{1/2}(\partial\Omega)} \\ &\le \|\Lambda_1 - \Lambda_2\|, \end{aligned}$$

where the norm in the last term is the operator norm from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. It follows from (4.5) that

$$\left| \sqrt{|g_1(z)|^{-1} \sum_{i,j=1}^{n-1} g_1^{ij}(z) t_i t_j} - \sqrt{|g_2(z)|^{-1} \sum_{i,j=1}^{n-1} g_2^{ij}(z) t_i t_j} \right| \le C\|\Lambda_1 - \Lambda_2\| + O(N^{-\lambda}).$$

Since t' is arbitrary and g_j satisfies (1.20), we have

$$||g_1(z)|^{-1}g_1(z) - |g_2(z)|^{-1}g_2(z)| \le C\|\Lambda_1 - \Lambda_2\| + O(N^{-\lambda}). \tag{4.6}$$

Then, by taking determinants, we have

$$||g_1(z)|^{2-n} - |g_2(z)|^{2-n}| \le C\|\Lambda_1 - \Lambda_2\| + O(N^{-\lambda}),$$

and hence

$$||g_1(z)| - |g_2(z)|| \le C\|\Lambda_1 - \Lambda_2\| + O(N^{-\lambda}). \tag{4.7}$$

It then follows from (4.6) and (4.7) that

$$|g_1(z) - g_2(z)| \le C\|\Lambda_1 - \Lambda_2\|. \tag{4.8}$$

Suppose now that a is a multi-index and $|a| > 0$. Note that

$$\begin{aligned}\langle(\Lambda_1 - \Lambda_2)\phi_N^z, \overline{\phi_N^z}\rangle &= \langle(\Lambda_1 - \Lambda_{g_1^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle - \langle(\Lambda_2 - \Lambda_{g_2^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle \\ &\quad + \langle(\Lambda_{g_1^{a,z}} - \Lambda_{g_2^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle.\end{aligned}$$

Thus it follows from (4.4) that

$$\begin{aligned}&\left|\frac{\sum_{i,j=1}^{n-1}\partial^a g_1^{ij}(z)t_i t_j}{C_{g_1}(z)^{a_n+1}|g_1(z)|^{1/2}} - \frac{\sum_{i,j=1}^{n-1}\partial^a g_2^{ij}(z)t_i t_j}{C_{g_2}(z)^{a_n+1}|g_2(z)|^{1/2}}\right| \\ &\le CN^{-2+|\alpha|+a\cdot\alpha}\|\Lambda_1 - \Lambda_2\|\|\phi_N^z\|^2_{H^{1/2}(\partial\Omega)} \\ &\quad + CN^{-2+|\alpha|+a\cdot\alpha}|\langle(\Lambda_{g_1^{a,z}} - \Lambda_{g_2^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle| + CN^{-\lambda}.\end{aligned} \tag{4.9}$$

Let Φ_N^j be approximate solutions of $\nabla\cdot(|g_j|^{-1/2}g_j^{a,z}\nabla u) = 0$ with the boundary value ϕ_N^z on $\partial\Omega$. Then in the same way as the proof of Theorem 1.1, we can show that

$$\langle(\Lambda_{g_1^{a,z}} - \Lambda_{g_2^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle = \int_{D_N}|g|^{-1/2}(g_1^{a,z} - g_2^{a,z})\nabla\Phi_N^1\cdot\nabla\overline{\Phi_N^2}dx + O(N^{2-m-|\alpha|-\lambda}).$$

Since

$$g_1^{a,z} - g_2^{a,z} = |g_1|^{-1/2}\sum_{b<a}\frac{\partial^b g_1(z)}{b!}(x-z)^b - |g_2|^{-1/2}\sum_{b<a}\frac{\partial^b g_2(z)}{b!}(x-z)^b,$$

we have

$$\begin{aligned}&\left|\int_{D_N}(g_1^{a,z} - g_2^{a,z})\nabla\Phi_N^1\cdot\nabla\overline{\Phi_N^2}dx\right| \\ &\le C\left||g_1|^{-1/2} - |g_2|^{-1/2}\right|\|\nabla\Phi_N^1\|_{L^2(D_N)}\|\nabla\Phi_N^2\|_{L^2(D_N)} \\ &\quad + C\sum_{b<a}\left|\partial^b g_1(z) - \partial^b g_2(z)\right|\left|\int_{D_N}(x-z)^b\nabla\Phi_N^1\cdot\nabla\overline{\Phi_N^2}dx\right|.\end{aligned}$$

By (2.15) and (2.16), we have

$$\begin{aligned}|\langle(\Lambda_{g_1^{a,z}} - \Lambda_{g_2^{a,z}})\phi_N^z, \overline{\phi_N^z}\rangle| &\le C(\|\Lambda_1 - \Lambda_2\| + O(N^{-\lambda}))N^{2-|\alpha|} \\ &\quad + C\sum_{b<a}|\partial^b g_1(z) - \partial^b g_2(z)|N^{2-|\alpha|-b\cdot\alpha} + CN^{2-m-|\alpha|-\lambda}.\end{aligned}$$

It then follows from (4.9) and the above estimates that

$$\left|\frac{\sum_{i,j=1}^{n-1}\partial^a g_1^{ij}(z)t_it_j}{C_{g_1}(z)^{a_n+1}|g_1(z)|^{1/2}}-\frac{\sum_{i,j=1}^{n-1}\partial^a g_2^{ij}(z)t_it_j}{C_{g_2}(z)^{a_n+1}|g_2(z)|^{1/2}}\right|$$
$$\leq C\left(\|\Lambda_1-\Lambda_2\|N^{a\cdot\alpha}+\sum_{b<a}|\partial^b g_1(z)-\partial^b g_2(z)|N^{(a-b)\cdot\alpha}+N^{-\lambda}\right).$$

It then follows from (4.8) that

$$\begin{aligned}(4.10)\qquad &|\partial^a g_1(z)-\partial^a g_2(z)|\\ &\leq C\left(\|\Lambda_1-\Lambda_2\|N^{a\cdot\alpha}+\sum_{b<a}|\partial^b g_1(z)-\partial^b g_2(z)|N^{(a-b)\cdot\alpha}+N^{-\lambda}\right).\end{aligned}$$

From (4.10), one can show that there exists $C=C(m,\lambda)$ such that

$$|\partial^a g_1(z)-\partial^a g_2(z)|\leq C\|\Lambda_1-\Lambda_2\|^{2^{-a\cdot\alpha/\lambda}}. \tag{4.11}$$

We will give a proof of (4.11) at the end of this paper.

If $|a|=k$, then $a\cdot\alpha\leq k$, and hence we have the following stability: If K is a subset of Γ such that $\mathrm{dist}(K,\Gamma)>\delta_0$ for some $\delta_0>0$, then we have

$$\|g_1-g_2\|_{C_E^k(K)}\leq C\|\Lambda_1-\Lambda_2\|^{2^{-k/\lambda}}. \tag{4.12}$$

We now deal with the general case. Suppose that Γ is an open portion of $\partial\Omega$ and K is a compact subset of Γ. For each $x\in K$, there exists an open neighborhood U_x of x and a diffeomorphism (boundary normal coordinates) $\Phi_{j,x}$ on $U_x\cap\overline{\Omega}_N$ such that $\Phi_{j,x}(U_x\cap\overline{\Omega}_N)$ is of the form (1.2) and $(\Phi_{j,x}^{-1})^*g_j$ is of the form (1.22). Moreover $\Phi_{1,x}(z)=\Phi_{2,x}(z)$ for all $z\in U_x\cap\partial\Omega$. Let K_x be a relatively compact subset of $U_x\cap\Gamma$. Then by (4.8) we have

$$\|(\Phi_{1,x}^{-1})^*g_1-(\Phi_{2,x}^{-1})^*g_2\|_{C_E^k(K_x)}\leq C\|\Lambda_{(\Phi_{1,x}^{-1})^*g_1}-\Lambda_{(\Phi_{2,x}^{-1})^*g_2}\|^{2^{-k/\lambda}}.$$

Put $\phi_x(z):=\Phi_{1,x}(z)=\Phi_{2,x}(z)$ for $z\in U_x\cap\partial\Omega$. Then

$$\Lambda_{(\Phi_{j,x}^{-1})^*g_j}=(\phi_x)_*\Lambda_{g_j}.$$

For the proof of this relation, see [15]. Therefore, we have

$$\|g_1-(\Phi_{2,x}^{-1}\circ\Phi_{1,x})^*g_2\|_{C_E^k(K_x)}\leq C\|(\phi_x)_*(\Lambda_{g_1}-\Lambda_{g_2})\|^{2^{-k/\lambda}}.$$

Put $\Psi_x:=\Phi_{2,x}^{-1}\circ\Phi_{1,x}$. Then we have

$$\|g_1-(\Psi_x)^*g_2\|_{C_E^k(K_x)}\leq C\|\Lambda_{g_1}-\Lambda_{g_2}\|^{2^{-k/\lambda}}.$$

By using a partition of unity, we have the theorem. Note that $\Psi_x=Id$ on $U_x\cap\partial\Omega$ for each x. This completes the proof. ∎

Derivation of (4.11). Since two multi-indices a,b satisfy $b<a$ if and only if $b\cdot\alpha<a\cdot\alpha$, and $b\cdot\alpha=k\lambda$ for some integer k, we may assign a one-to-one relation from multi-indices a with $|a|\leq m$ into the set $\{0,1,\dots,m/\lambda\}$. Let

$$a(k):=|\partial^b g_1(z)-\partial^b g_2(z)|$$

if $b \cdot \alpha = k\lambda$. Put $a(0) := \|\Lambda_1 - \Lambda_2\|$. Then (4.10) reads

$$a(l) \leq C \left(\sum_{k<l} a(k)(N^{\lambda})^{l-k} + N^{-\lambda} \right) \quad \text{for all large } N.$$

From this one can show inductively that

$$a(l) \leq C a(0)^{2^{-l}}.$$

Thus (4.11) is obtained. $\square$

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REFERENCES

[1] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, J. Differential Equations, 84 (1990), pp. 252–273.

[2] G. Alessandrini and R. Gaburro, *Determining conductivity with special anisotropy by boundary measurements*, SIAM J. Math. Anal., 33 (2001), pp. 153–171.

[3] R. M. Brown, *Recovering conductivity at the boundary from the Dirichlet to Neumann map: A pointwise result*, J. Inverse Ill-Posed Probl., 9 (2001), pp. 567–574.

[4] R. M. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. Partial Differential Equations, 22 (1997), pp. 1009–1027.

[5] V. Isakov, *On uniqueness of recovery of a discontinuous conductivity coefficient*, Comm. Pure Appl. Math., 41 (1988), pp. 865–877.

[6] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math., 37 (1984), pp. 281–298.

[7] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements* II, *Interior results*, Comm. Pure Appl. Math., 38 (1985), pp. 643–667.

[8] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup. (4), 34 (2001), pp. 771–787.

[9] J. Lee and G. Uhlmann, *Determining anistropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math., 42 (1989), pp. 1097–1112.

[10] W. R. B. Lionheart, *Conformal uniqueness results in anisotropic electrical impedance imaging*, Inverse Problems, 13 (1997), pp. 125–134

[11] A. Nachman, *The inverse reconstructions from boundary measurements*, Ann. of Math., 128 (1988), pp. 531–587.

[12] A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math., 143 (1996), pp. 71–96.

[13] G. Nakamura and K. Tanuma, *Local determination of conductivity at the boundary from Dirichlet to Neumann map*, Inverse Problems, 17 (2001), pp. 405–419.

[14] G. Nakamura and K. Tanuma, *Direct determination of the derivatives of conductivity at the boundary from the localized Dirichlet to Neumann map*, Second Japan-Korea Joint Seminar on Inverse Problems and Related Topics (Seoul, 2001), Commun. Korean Math. Soc., 16 (2001), pp. 415–425.

[15] J. Sylvester, *An anisotropic inverse boundary value problem*, Comm. Pure Appl. Math., 43 (1990), pp. 201–232.

[16] J. Sylvester and G. Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure Appl. Math., 39 (1986), pp. 91–112.

[17] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., 125 (1987), pp. 153–169.

[18] J. Sylvester and G. Uhlmann, *Inverse boundary value problems at the boundary-continuous dependence*, Comm. Pure Appl. Math., 41 (1988), pp. 197–221.